1-2 Composition of Functions

OBJECTIVES
- Perform operations with functions.
- Find composite functions.
- Iterate functions using real numbers.

Real World Application

BUSINESS Each year, thousands of people visit Yellowstone National Park in Wyoming. Audiotapes for visitors include interviews with early settlers and information about the geology, wildlife, and activities of the park. The revenue $r(x)$ from the sale of x tapes is $r(x) = 9.5x$. Suppose that the function for the cost of manufacturing x tapes is $c(x) = 0.8x + 1940$. What function could be used to find the profit on x tapes? *This problem will be solved in Example 2.*

To solve the profit problem, you can subtract the cost function $c(x)$ from the revenue function $r(x)$. If you have two functions, you can form new functions by adding, subtracting, multiplying, or dividing the functions.

GRAPHING CALCULATOR EXPLORATION

Use a graphing calculator to explore the sum of two functions.

- Enter the functions $f(x) = 2x - 1$ and $f(x) = 3x + 2$ as **Y1** and **Y2**, respectively.
- Enter **Y1** + **Y2** as the function for **Y3**. To enter **Y1** and **Y2**, press VARS, then select **Y-VARS**. Then choose the equation name from the menu.
- Use **TABLE** to compare the function values for **Y1**, **Y2**, and **Y3**.

TRY THESE

Use the functions $f(x) = 2x - 1$ and $f(x) = 3x + 2$ as Y1 and Y2. Use TABLE to observe the results for each definition of Y3.

1. Y3 = Y1 − Y2 **2.** Y3 = Y1 · Y2

3. Y3 = Y1 ÷ Y2

WHAT DO YOU THINK?

4. Repeat the activity using functions $f(x) = x^2 - 1$ and $f(x) = 5 - x$ as **Y1** and **Y2**, respectively. What do you observe?

5. Make conjectures about the functions that are the sum, difference, product, and quotient of two functions.

The Graphing Calculator Exploration leads us to the following definitions of operations with functions.

Operations with Functions

Sum: $(f + g)(x) = f(x) + g(x)$
Difference: $(f - g)(x) = f(x) - g(x)$
Product: $(f \cdot g)(x) = f(x) \cdot g(x)$
Quotient: $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)},\ g(x) \neq 0$

For each new function, the domain consists of those values of x common to the domains of f and g. The domain of the quotient function is further restricted by excluding any values that make the denominator, $g(x)$, zero.

Example 1 **Given $f(x) = 3x^2 - 4$ and $g(x) = 4x + 5$, find each function.**

a. $(f + g)(x)$

$$\begin{aligned}(f + g)(x) &= f(x) + g(x)\\ &= 3x^2 - 4 + 4x + 5\\ &= 3x^2 + 4x + 1\end{aligned}$$

b. $(f - g)(x)$

$$\begin{aligned}(f - g)(x) &= f(x) - g(x)\\ &= 3x^2 - 4 - (4x + 5)\\ &= 3x^2 - 4x - 9\end{aligned}$$

c. $(f \cdot g)(x)$

$$\begin{aligned}(f \cdot g)(x) &= f(x) \cdot g(x)\\ &= (3x^2 - 4)(4x + 5)\\ &= 12x^3 + 15x^2 - 16x - 20\end{aligned}$$

d. $\left(\frac{f}{g}\right)(x)$

$$\begin{aligned}\left(\frac{f}{g}\right)(x) &= \frac{f(x)}{g(x)}\\ &= \frac{3x^2 - 4}{4x + 5},\ x \neq -\frac{5}{4}\end{aligned}$$

You can use the difference of two functions to solve the application problem presented at the beginning of the lesson.

Example 2 **BUSINESS** **Refer to the application at the beginning of the lesson.**

Real World Application

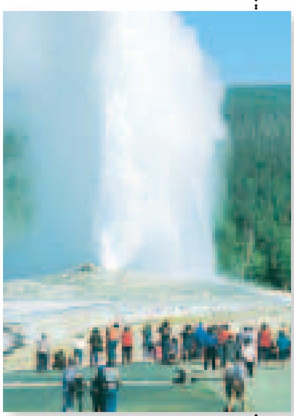
a. Write the profit function.

b. Find the profit on 500, 1000, and 5000 tapes.

a. Profit is revenue minus cost. Thus, the profit function $p(x)$ is $p(x) = r(x) - c(x)$.

The revenue function is $r(x) = 9.5x$. The cost function is $c(x) = 0.8x + 1940$.

$$\begin{aligned}p(x) &= r(x) - c(x)\\ &= 9.5x - (0.8x + 1940)\\ &= 8.7x - 1940\end{aligned}$$



b. To find the profit on 500, 1000, and 5000 tapes, evaluate $p(500)$, $p(1000)$, and $p(5000)$.

$p(500) = 8.7(500) - 1940$ or 2410

$p(1000) = 8.7(1000) - 1940$ or 6760

$p(5000) = 8.7(5000) - 1940$ or $41{,}560$

The profit on 500, 1000, and 5000 tapes is \$2410, \$6760, and \$41,560, respectively. *Check by finding the revenue and the cost for each number of tapes and subtracting to find profit.*

Functions can also be combined by using **composition.** In a composition, a function is performed, and then a second function is performed on the result of the first function. You can think of composition in terms of manufacturing a product. For example, fiber is first made into cloth. Then the cloth is made into a garment.

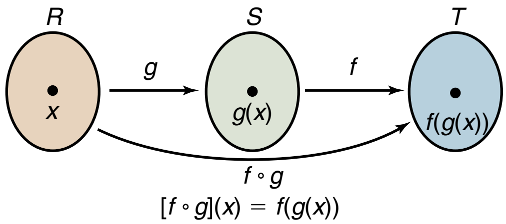
In composition, a function g maps the elements in set R to those in set S. Another function f maps the elements in set S to those in set T. Thus, the range of function g is the same as the domain of function f. A diagram is shown below.

R	S
x	$g(x) = \frac{1}{4}x$
4	1
8	2
12	3

S	T
x	$f(x) = 6 - 2x$
1	4
2	2
3	0

domain of $g(x)$ *The range of $g(x)$ is the domain of $f(x)$.* *range of $f(x)$*

The function formed by composing two functions f and g is called the **composite** of f and g. It is denoted by $f \circ g$, which is read as "*f composition g*" or "*f of g.*"



Composition of Functions

Given functions f and g, the composite function $f \circ g$ can be described by the following equation.

$$[f \circ g](x) = f[g(x)]$$

The domain of $f \circ g$ includes all of the elements x in the domain of g for which $g(x)$ is in the domain of f.

Example 3 **Find $[f \circ g](x)$ and $[g \circ f](x)$ for $f(x) = 2x^2 - 3x + 8$ and $g(x) = 5x - 6$.**

$[f \circ g](x) = f(g(x))$

$= f(5x - 6)$ *Substitute $5x - 6$ for $g(x)$.*

$= 2(5x - 6)^2 - 3(5x - 6) + 8$ *Substitute $5x - 6$ for x in $f(x)$.*

$= 2(25x^2 - 60x + 36) - 15x + 18 + 8$

$= 50x^2 - 135x + 98$

$[g \circ f](x) = g(f(x))$

$= g(2x^2 - 3x + 8)$ *Substitute $2x^2 - 3x + 8$ for $f(x)$.*

$= 5(2x^2 - 3x + 8) - 6$ *Substitute $2x^2 - 3x + 8$ for x in $g(x)$.*

$= 10x^2 - 15x + 34$

The domain of a composed function $[f \circ g](x)$ is determined by the domains of both $f(x)$ and $g(x)$.

Example 4 **State the domain of $[f \circ g](x)$ for $f(x) = \sqrt{x - 4}$ and $g(x) = \frac{1}{x^2}$.**

$f(x) = \sqrt{x - 4}$ *Domain: $x \geq 4$*

$g(x) = \frac{1}{x^2}$ *Domain: $x \neq 0$*

If $g(x)$ is undefined for a given value of x, then that value is excluded from the domain of $[f \circ g](x)$. Thus, 0 is excluded from the domain of $[f \circ g](x)$.

The domain of $f(x)$ is $x \geq 4$. So for x to be in the domain of $[f \circ g](x)$, it must be true that $g(x) \geq 4$.

$g(x) \geq 4$

$\frac{1}{x^2} \geq 4$ *$g(x) = \frac{1}{x^2}$*

$1 \geq 4x^2$ *Multiply each side by x^2.*

$\frac{1}{4} \geq x^2$ *Divide each side by 4.*

$\frac{1}{2} \geq |x|$ *Take the square root of each side.*

$-\frac{1}{2} \leq x \leq \frac{1}{2}$ *Rewrite the inequality.*

Therefore, the domain of $[f \circ g](x)$ is $-\frac{1}{2} \leq x \leq \frac{1}{2}$, $x \neq 0$.

The composition of a function and itself is called iteration. Each output of an iterated function is called an iterate. To iterate a function $f(x)$, find the function value $f(x_0)$, of the initial value x_0. The value $f(x_0)$ is the first iterate, x_1. The second iterate is the value of the function performed on the output; that is, $f(f(x_0))$ or $f(x_1)$. Each iterate is represented by x_n, where n is the iterate number. For example, the third iterate is x_3.

Example 5 **Find the first three iterates, x_1, x_2, and x_3, of the function $f(x) = 2x - 3$ for an initial value of $x_0 = 1$.**

To obtain the first iterate, find the value of the function for $x_0 = 1$.

$$x_1 = f(x_0) = f(1)$$
$$= 2(1) - 3 \text{ or } -1$$

To obtain the second iterate, x_2, substitute the function value for the first iterate, x_1, for x.

$$x_2 = f(x_1) = f(-1)$$
$$= 2(-1) - 3 \text{ or } -5$$

Now find the third iterate, x_3, by substituting x_2 for x.

$$x_3 = f(x_2) = f(-5)$$
$$= 2(-5) - 3 \text{ or } -13$$

Thus, the first three iterates of the function $f(x) = 2x - 3$ for an initial value of $x_0 = 1$ are -1, -5, and -13.

Check for Understanding

Communicating Mathematics

Read and study the lesson to answer each question.

1. **Write** two functions $f(x)$ and $g(x)$ for which $(f \cdot g)(x) = 2x^2 + 11x - 6$. Tell how you determined $f(x)$ and $g(x)$.
2. **Explain** how iteration is related to composition of functions.
3. **Determine** whether $[f \circ g](x)$ is always equal to $[g \circ f](x)$ for two functions $f(x)$ and $g(x)$. Explain your answer and include examples or counterexamples.
4. *Math Journal* Write an explanation of function composition. Include an everyday example of two composed functions and an example of a realworld problem that you would solve using composed functions.

Guided Practice

5. Given $f(x) = 3x^2 + 4x - 5$ and $g(x) = 2x + 9$, find $f(x) + g(x)$, $f(x) - g(x)$, $f(x) \cdot g(x)$, and $\left(\frac{f}{g}\right)(x)$.

Find $[f \circ g](x)$ and $[g \circ f](x)$ for each $f(x)$ and $g(x)$.

6. $f(x) = 2x + 5$
   $g(x) = 3 + x$
7. $f(x) = 2x - 3$
   $g(x) = x^2 - 2x$
8. State the domain of $[f \circ g](x)$ for $f(x) = \frac{1}{(x-1)^2}$ and $g(x) = x + 3$.
9. Find the first three iterates of the function $f(x) = 2x + 1$ using the initial value $x_0 = 2$.
10. **Measurement** In 1954, the Tenth General Conference on Weights and Measures adopted the kelvin K as the basic unit for measuring temperature for all international weights and measures. While the kelvin is the standard unit, degrees Fahrenheit and degrees Celsius are still in common use in the United States. The function $C(F) = \frac{5}{9}(F - 32)$ relates Celsius temperatures and Fahrenheit temperatures. The function $K(C) = C + 273.15$ relates Celsius temperatures and Kelvin temperatures.
    a. Use composition of functions to write a function to relate degrees Fahrenheit and kelvins.
    b. Write the temperatures $-40°$F, $-12°$F, $0°$F, $32°$F, and $212°$F in kelvins.

Exercises

Practice

Find $f(x) + g(x)$, $f(x) - g(x)$, $f(x) \cdot g(x)$, and $\left(\frac{f}{g}\right)(x)$ for each $f(x)$ and $g(x)$.

11. $f(x) = x^2 - 2x$
    $g(x) = x + 9$
12. $f(x) = \frac{x}{x+1}$
    $g(x) = x^2 - 1$
13. $f(x) = \frac{3}{x-7}$
    $g(x) = x^2 + 5x$
14. If $f(x) = x + 3$ and $g(x) = \frac{2x}{x-5}$, find $f(x) + g(x)$, $f(x) - g(x)$, $f(x) \cdot g(x)$, and $\left(\frac{f}{g}\right)(x)$.

Find $[f \circ g](x)$ and $[g \circ f](x)$ for each $f(x)$ and $g(x)$.

15. $f(x) = x^2 - 9$
$g(x) = x + 4$

16. $f(x) = \frac{1}{2}x - 7$
$g(x) = x + 6$

17. $f(x) = x - 4$
$g(x) = 3x^2$

18. $f(x) = x^2 - 1$
$g(x) = 5x^2$

19. $f(x) = 2x$
$g(x) = x^3 + x^2 + 1$

20. $f(x) = 1 + x$
$g(x) = x^2 + 5x + 6$

21. What are $[f \circ g](x)$ and $[g \circ f](x)$ for $f(x) = x + 1$ and $g(x) = \frac{1}{x-1}$?

State the domain of $[f \circ g](x)$ for each $f(x)$ and $g(x)$.

22. $f(x) = 5x$
$g(x) = x^3$

23. $f(x) = \frac{1}{x}$
$g(x) = 7 - x$

24. $f(x) = \sqrt{x - 2}$
$g(x) = \frac{1}{4x}$

Find the first three iterates of each function using the given initial value.

25. $f(x) = 9 - x;\ x_0 = 2$

26. $f(x) = x^2 + 1;\ x_0 = 1$

27. $f(x) = x(3 - x);\ x_0 = 1$

Applications and Problem Solving

Real World Application

28. Retail Sara Sung is shopping and finds several items that are on sale at 25% off the original price. The items that she wishes to buy are a sweater originally at \$43.98, a pair of jeans for \$38.59, and a blouse for \$31.99. She has \$100 that her grandmother gave her for her birthday. If the sales tax in San Mateo, California, where she lives is 8.25%, does Sara have enough money for all three items? Explain.

29. Critical Thinking Suppose the graphs of functions $f(x)$ and $g(x)$ are lines. Must it be true that the graph of $[f \circ g](x)$ is a line? Justify your answer.

30. Physics When a heavy box is being pushed on the floor, there are two different forces acting on the movement of the box. There is the force of the person pushing the box and the force of friction. If W is work in joules, F is force in newtons, and d is displacement of the box in meters, $W_p = F_p d$ describes the work of the person, and $W_f = F_f d$ describes the work created by friction. The increase in kinetic energy necessary to move the box is the difference between the work done by the person W_p and the work done by friction W_f.

a. Write a function in simplest form for net work.

b. Determine the net work expended when a person pushes a box 50 meters with a force of 95 newtons and friction exerts a force of 55 newtons.

31. Finance A sales representative for a cosmetics supplier is paid an annual salary plus a bonus of 3% of her sales *over* \$275,000. Let $f(x) = x - 275{,}000$ and $h(x) = 0.03x$.

a. If x is greater than \$275,000, is her bonus represented by $f[h(x)]$ or by $h[f(x)]$? Explain.

b. Find her bonus if her sales for the year are \$400,000.

32. Critical Thinking Find $f\left(\frac{1}{2}\right)$ if $[f \circ g](x) = \frac{x^4 + x^2}{1 + x^2}$ and $g(x) = 1 - x^2$.

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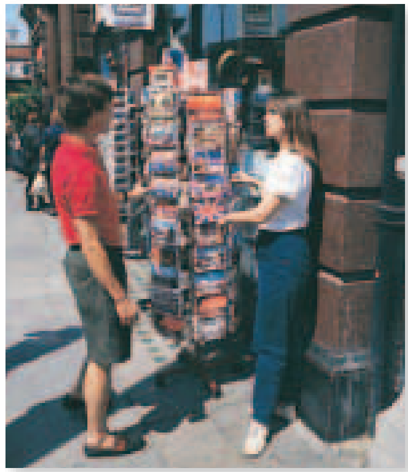
33. International Business Value-added tax, VAT, is a tax charged on goods and services in European countries. Many European countries offer refunds of some VAT to non-resident based businesses. VAT is included in a price that is quoted. That is, if an item is marked as costing \$10, that price includes the VAT.



a. Suppose an American company has operations in The Netherlands, where the VAT is 17.5%. Write a function for the VAT amount paid $v(p)$ if p represents the price including the VAT.

b. In The Netherlands, foreign businesses are entitled to a refund of 84% of the VAT on automobile rentals. Write a function for the refund an American company could expect $r(v)$ if v represents the VAT amount.

c. Write a function for the refund expected on an automobile rental $r(p)$ if the price including VAT is p.

d. Find the refunds due on automobile rental prices of \$423.18, \$225.64, and \$797.05.

Mixed Review

34. Finance The formula for the simple interest earned on an investment is $I = prt$, where I is the interest earned, p is the principal, r is the interest rate, and t is the time in years. Assume that \$5000 is invested at an annual interest rate of 8% and that interest is added to the principal at the end of each year. *(Lesson 1-1)*

a. Find the amount of interest that will be earned each year for five years.

b. State the domain and range of the relation.

c. Is this relation a function? Why or why not?

35. State the relation in the table as a set of ordered pairs. Then state the domain and range of the relation. *(Lesson 1-1)*

x	y
−1	8
0	4
2	−6
5	−9

36. What are the domain and the range of the relation {(1, 5), (2, 6), (3, 7), (4, 8)}? Is the relation a function? Explain. *(Lesson 1-1)*

37. Find $g(-4)$ if $g(x) = \frac{x^3 + 5}{4x}$. *(Lesson 1-1)*

38. Given that x is an integer, state the relation representing $y = |-3x|$ and $-2 \leq x \leq 3$ by making a table of values. Then graph the ordered pairs of the relation. *(Lesson 1-1)*

39. SAT/ACT Practice Find $f(n - 1)$ if $f(x) = 2x^2 - x + 9$.

A $2n^2 - n + 9$

B $2n^2 - n + 8$

C $2n^2 - 5n + 12$

D 9

E $2n^2 + 4n + 8$

Extra Practice See p. A26.

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