1-7

Piecewise Functions

OBJECTIVE

- Identify and graph piecewise functions including greatest integer, step, and absolute value functions.

Real World Application

ACCOUNTING The Internal Revenue Service estimates that taxpayers who itemize deductions and report interest and capital gains will need an average of almost 24 hours to prepare their returns. The amount that a single taxpayer owes depends upon his or her income. The table shows the tax brackets for different levels of income for a certain year.

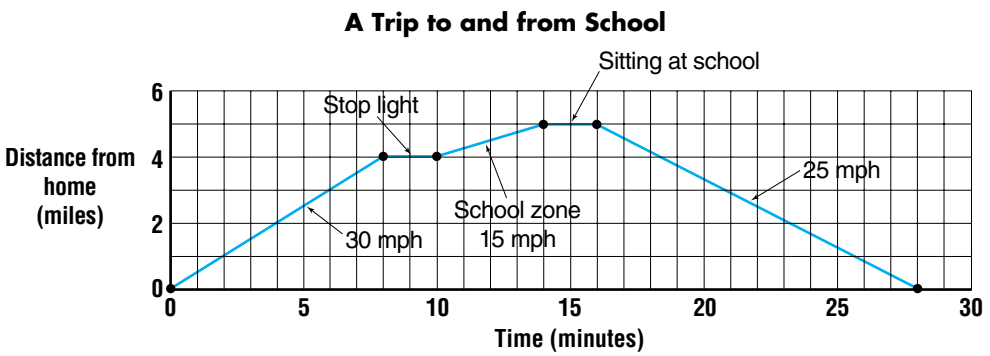
Single Individual Income Tax	
Limits of Taxable Income	**Tax Bracket**
$0 to $25,350	15%
$25,351 to $61,400	28%
$61,401 to $128,100	31%
$128,101 to $278,450	36%
over $278,450	39.6%

Source: *World Almanac*

A problem related to this will be solved in Example 3.

The tax table defines a special function called a **piecewise function.** For piecewise functions, different equations are used for different intervals of the domain. The graph below shows a piecewise function that models the number of miles from home as a function of time in minutes. Notice that the graph consists of several line segments, each of which is a part of a linear function.

Brittany traveled at a rate of 30 mph for 8 minutes. She stopped at a stoplight for 2 minutes. Then for 4 minutes she traveled 15 mph through the school zone. She sat at the school for 3 minutes while her brother got out of the car. Then she traveled home at 25 mph.



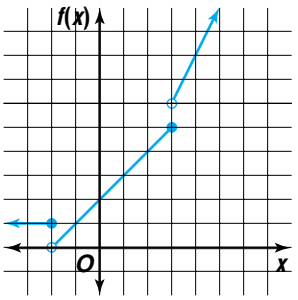
When graphing piecewise functions, the partial graphs over various intervals do not necessarily connect. The definition of the function on the intervals determines if the graph parts connect.

Example 1 Graph $f(x) = \begin{cases} 1 \text{ if } x \leq -2 \\ 2 + x \text{ if } -2 < x \leq 3. \\ 2x \text{ if } x > 3 \end{cases}$

First, graph the constant function $f(x) = 1$ for $x \leq -2$. This graph is part of a horizontal line. Because the point at $(-2, 1)$ is included in the graph, draw a closed circle at that point.

Second, graph the function $f(x) = 2 + x$ for $-2 < x \leq 3$. Because $x = -2$ is not included in this part of the domain, draw an open circle at $(-2, 0)$. $x = 3$ is included in the domain, so draw a closed circle at $(3, 5)$ since for $f(x) = 2 + x$, $f(3) = 5$.

Third, graph the line $y = 2x$ for $x > 3$. Draw an open circle at $(3, 6)$ since for $f(x) = 2x$, $f(3) = 6$.



Graphing Calculator Tip

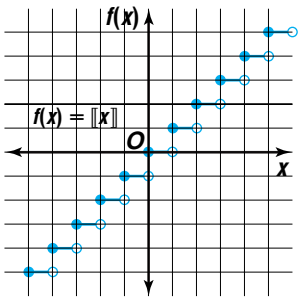
On a graphing calculator, **int(X)** indicates the greatest integer function.

A piecewise function where the graph looks like a set of stairs is called a **step function.** In a step function, there are breaks in the graph of the function. You cannot trace the graph of a step function without lifting your pencil. One type of step function is the **greatest integer function.** The symbol $[\![x]\!]$ means *the greatest integer not greater than x.* This does not mean to round or truncate the number. For example, $[\![8.9]\!] = 8$ because 8 is the greatest integer not greater than 8.9. Similarly, $[\![-3.9]\!] = -4$ because -3 is greater than -3.9. The greatest integer function is given by $f(x) = [\![x]\!]$.

Example 2 Graph $f(x) = [\![x]\!]$.

Make a table of values. The domain values will be intervals for which the greatest integer function will be evaluated.

x	$f(x)$
$-3 \leq x < -2$	-3
$-2 \leq x < -1$	-2
$-1 \leq x < 0$	-1
$0 \leq x < 1$	0
$1 \leq x < 2$	1
$2 \leq x < 3$	2
$3 \leq x < 4$	3
$4 \leq x < 5$	4



Notice that the domain for this greatest integer function is all real numbers and the range is integers.

The graphs of step functions are often used to model real-world problems such as fees for cellular telephones and the cost of shipping an item of a given weight.

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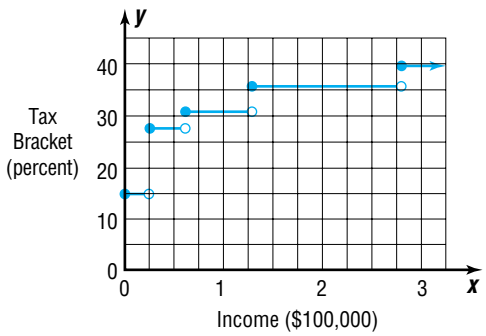
Example 3 **Refer to the application at the beginning of the lesson.**

a. Graph the tax brackets for the different incomes.

b. What is the tax bracket for a person who makes $70,000?

a.



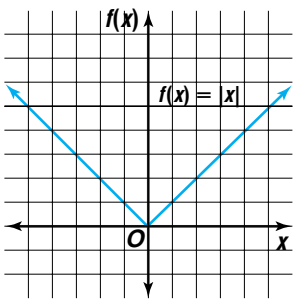
b. $70,000 falls in the interval $61,401 to $128,100. Thus, the tax bracket for $70,000 is 31%.

The **absolute value function** is another piecewise function. Consider $f(x) = |x|$. The absolute value of a number is always nonnegative. The table lists a specific domain and resulting range values for the absolute value function. Using these points, a graph of the absolute value function can be constructed. Notice that the domain of the graph includes all real numbers. However, the range includes only nonnegative real numbers.

table

$f(x) = \|x\|$	
x	$f(x)$
−3	3
−2.4	2.4
0	0
0.7	0.7
2	2
3.4	3.4

graph



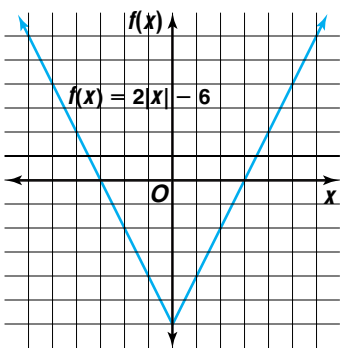
piecewise function

$$f(x) = \begin{cases} -x \text{ if } x < 0 \\ x \text{ if } x \geq 0 \end{cases}$$

Example 4 **Graph $f(x) = 2|x| - 6$.**

Use a table of values to determine points on the graph.

x	$2\|x\| - 6$	$(x, f(x))$
−6	$2\|-6\| - 6 = 6$	(−6, 6)
−3	$2\|-3\| - 6 = 0$	(−3, 0)
−1.5	$2\|-1.5\| - 6 = -3$	(−1.5, −3)
0	$2\|0\| - 6 = -6$	(0, −6)
1	$2\|1\| - 6 = -4$	(1, −4)
2	$2\|2\| - 6 = -2$	(2, −2)

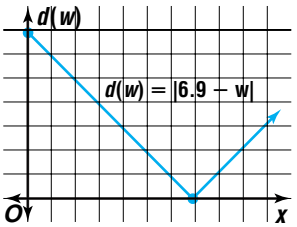


Many real-world situations can be modeled by a piecewise function.

Example 5 **Identify the type of function that models each situation. Then write a function for the situation.**

a. Manufacturing **The stated weight of a box of rice is 6.9 ounces. The company randomly chooses boxes to test to see whether their equipment is dispensing the right amount of product. If the discrepancy is more than 0.2 ounce, the production line is stopped for adjustments.**

The situation can be represented with an absolute value function. Let w represent the weight and $d(w)$ represent the discrepancy. Then $d(w) = |6.9 - w|$.

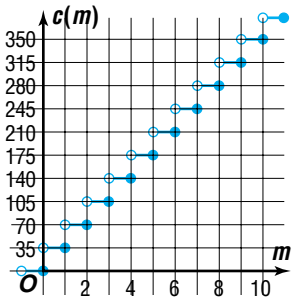


b. Business **On a certain telephone rate plan, the price of a cellular telephone call is 35¢ per minute or fraction thereof.**

This can be described by a greatest integer function.

Let m represent the number of minutes of the call and $c(m)$ represent the cost in cents.

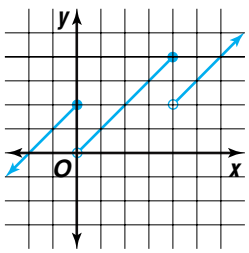
$$c(m) = \begin{cases} 35m \text{ if } [\![m]\!] = m \\ 35[\![m + 1]\!] \text{ if } [\![m]\!] < m \end{cases}$$



CHECK FOR UNDERSTANDING

Communicating Mathematics Read and study the lesson to answer each question.

1. **Write** $f(x) = |x|$ as a piecewise function.
2. **State** the domain and range of the function $f(x) = 2[\![x]\!]$.
3. **Write** the function that is represented by the graph.



4. **You Decide** Misae says that a step graph does not represent a function because the graph is not connected. Alex says that it does represent a function because there is only one y for every x. Who is correct and why?

Guided Practice Graph each function.

5. $f(x) = \begin{cases} 2x \text{ if } 0 \le x \le 4 \\ 8 \text{ if } 4 < x \le 7 \end{cases}$

6. $f(x) = \begin{cases} 6 \text{ if } x \le -6 \\ |x| \text{ if } -6 < x < 6 \\ 6 \text{ if } x > 6 \end{cases}$

7. $f(x) = -[\![x]\!]$

8. $f(x) = |x - 3|$

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9. **Business** Identify the type of function that models the labor cost for repairing a computer if the charge is \$50 per hour or fraction thereof. Then write and graph a function for the situation.

10. **Consumerism** Guillermo Lujan is flying from Denver to Dallas for a convention. He can park his car in the Denver airport long-term parking lot at the terminal or in the shuttle parking facility closeby. In the long-term lot, it costs \$1.00 per hour or any part of an hour with a maximum charge of \$6.00 per day. In shuttle facility, he has to pay \$4.00 for each day or part of a day. Which parking lot is less expensive if Mr. Lujan returns after 2 days and 3 hours?

EXERCISES

Practice

Graph each function.

11. $f(x) = \begin{cases} 2x + 1 \text{ if } x < 0 \\ 2x - 1 \text{ if } x \geq 0 \end{cases}$

12. $g(x) = |x - 5|$

13. $h(x) = [\![x]\!] + 2$

14. $g(x) = |2x + 3|$

15. $f(x) = [\![x - 1]\!]$

16. $h(x) = \begin{cases} 3 \text{ if } -1 \leq x \leq 1 \\ 4 \text{ if } 1 < x \leq 4 \\ x \text{ if } x > 4 \end{cases}$

17. $g(x) = 2|x - 3|$

18. $f(x) = [\![-3x]\!]$

19. $h(x) = \begin{cases} x + 3 \text{ if } x \leq 0 \\ 3 - x \text{ if } 1 < x \leq 3 \\ 3x \text{ if } x > 3 \end{cases}$

20. $f(x) = \begin{cases} -2x \text{ if } x < 1 \\ 3 \text{ if } x = 1 \\ 4x \text{ if } x > 1 \end{cases}$

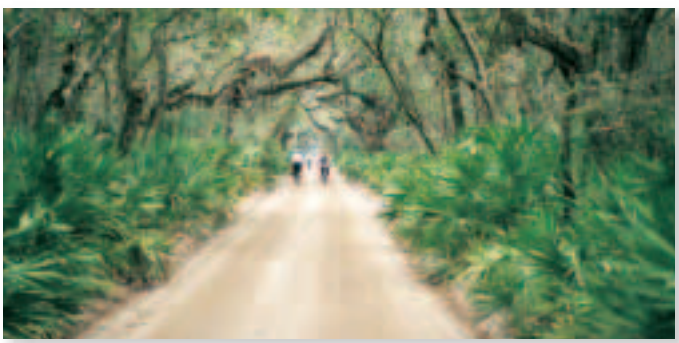
21. $j(x) = \dfrac{2}{[\![x]\!]}$

22. $g(x) = |9 - 3|x||$

Identify the type of function that models each situation. Then write and graph a function for the situation.

23. **Tourism** The table shows the charge for renting a bicycle from a rental shop on Cumberland Island, Georgia, for different amounts of time.

Island Rentals	
Time	**Price**
$\frac{1}{2}$ hour	\$6
1 hour	\$10
2 hours	\$16
Daily	\$24



24. **Postage** The cost of mailing a letter is \$0.33 for the first ounce and \$0.22 for each additional ounce or portion thereof.

25. **Manufacturing** A can of coffee is supposed to contain one pound of coffee. How does the actual weight of the coffee in the can compare to 1 pound?

Applications and Problem Solving

26. Retail Sales The table shows the shipping charges that apply to orders placed in a catalog.

a. What type of function is described?

b. Write the shipping charges as a function of the value of the order.

c. Graph the function.

Shipping to or within the United States	
Value of Order	**Shipping, Packing, and Handling Charge**
\$0.00–25.00	\$3.50
\$25.01–75.00	\$5.95
\$75.01–125.00	\$7.95
\$125.01 and up	\$9.95

27. Critical Thinking Describe the values of x and y which are solutions to $[\![x]\!] = [\![y]\!]$.

28. Engineering The degree day is used to measure the demand for heating or cooling. In the United States, 65°F is considered the desirable temperature for the inside of a building. The number of degree days recorded on a given date is equal to the difference between 65 and the mean temperature for that date. If the mean temperature is above 65°F, cooling degree days are recorded. Heating degree days are recorded if the mean temperature is below 65°F.

a. What type of function can be used to model degree days?

b. Write a function to model the number of degree days $d(t)$ for a mean temperature of t°F.

c. Graph the function.

d. The mean temperature is the mean of the high and low temperatures for a day. How many degree days are recorded for a day with a high of temperature of 63°F and a low temperature of 28°F? Are they heating degree days or cooling degree days?

29. Accounting The income tax brackets for the District of Columbia are listed in the tax table.

Income	Tax Bracket
up to \$10,000	6%
more than \$10,000, but no more than \$20,000	8%
more than \$20,000	9.5%

a. What type of function is described by the tax rates?

b. Write the function if x is income and $t(x)$ is the tax rate.

c. Graph the tax brackets for different taxable incomes.

d. Alicia Davis lives in the District of Columbia. In which tax bracket is Ms. Davis if she made \$36,000 last year?

30. Critical Thinking For $f(x) = [\![x]\!]$ and $g(x) = |x|$, are $[f \circ g](x)$ and $[g \circ f](x)$ equivalent? Justify your answer.

Mixed Review

31. Transportation The table shows the percent of workers in different cities who use public transportation to get to work. *(Lesson 1-6)*

a. Graph the data on a scatter plot.

b. Use two ordered pairs to write the equation of a best-fit line.

c. Use a graphing calculator to find an equation for the regression line for the data. What is the correlation value?

d. If the equation of the regression line shows a moderate or strong relationship, predict the percent of workers using public transportation in Baltimore, Maryland. Is the prediction reliable? Explain.

City	Workers 16 years and older	Percent who use Public Transportation
New York, NY	3,183,088	53.4
Los Angeles, CA	1,629,096	10.5
Chicago, IL	1,181,677	29.7
Houston, TX	772,957	6.5
Philadelphia, PA	640,577	28.7
San Diego, CA	560,913	4.2
Dallas, TX	500,566	6.7
Phoenix, AZ	473,966	3.3
San Jose, CA	400,932	3.5
San Antonio, TX	395,551	4.9
San Francisco, CA	382,309	33.5
Indianapolis, IN	362,777	3.3
Detroit, MI	325,054	10.7
Jacksonville, FL	312,958	2.7
Baltimore, MD	307,679	22.0

Source: U.S. Bureau of the Census

32. Write the standard form of the equation of the line that passes through the point at (4, 2) and is parallel to the line whose equation is $y = 2x - 4$. *(Lesson 1-5)*

33. Sports During a basketball game, the two highest-scoring players scored 29 and 15 points and played 39 and 32 minutes, respectively. *(Lesson 1-3)*

a. Write an ordered pair of the form (minutes played, points scored) to represent each player.

b. Find the slope of the line containing both points.

c. What does the slope of the line represent?

34. Business For a company, the revenue $r(x)$ in dollars, from selling x items is $r(x) = 400x - 0.2x^2$. The cost for making and selling x items is $c(x) = 0.1x + 200$. Write the profit function $p(x) = (r - c)(x)$. *(Lesson 1-2)*

35. Retail Winston bought a sweater that was on sale 25% off. The original price of the sweater was \$59.99. If sales tax in Winston's area is 6.5%, how much did the sweater cost including sale tax? *(Lesson 1-2)*

36. State the domain and range of the relation $\{(0, 2), (4, -2), (9, 3), (-7, 11), (-2, 0)\}$. Is the relation a function? Explain. *(Lesson 1-1)*

37. SAT Practice Which of the following expressions is *not* larger than 5×6^{12}?

A $5 + 6^{12}$

B 7×6^{12}

C 5×8^{12}

D 5×6^{14}

E 10^{13}

Extra Practice See p. A27.