12-3

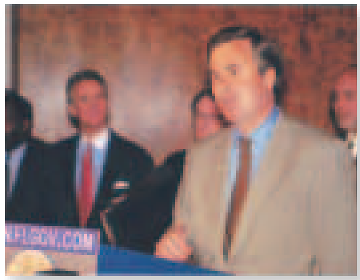
Infinite Sequences and Series

OBJECTIVES

- Find the limit of the terms of an infinite sequence.
- Find the sum of an infinite geometric series.

Real World Application

ECONOMICS On January 28, 1999, Florida governor Jeb Bush proposed a tax cut that would allow the average family to keep an additional $96. The *marginal propensity to consume (MPC)* is defined as the percentage of a dollar by which consumption increases when income rises by a dollar. Suppose the *MPC* for households and businesses in 1999 was 75%. What would be the total amount of money spent in the economy as a result of just one family's tax savings? *This problem will be solved in Example 5.*



Governor Jeb Bush

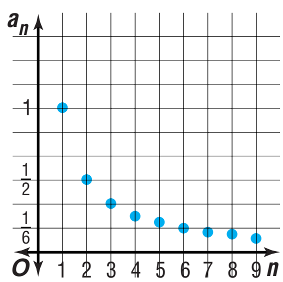
Transaction	Expenditure	Terms of Sequence
1	$96(0.75)^1$	72
2	$96(0.75)^2$	54
3	$96(0.75)^3$	40.50
4	$96(0.75)^4$	30.76
5	$96(0.75)^5$	22.78
⋮	⋮	⋮
10	$96(0.75)^{10}$	5.41
⋮	⋮	⋮
100	$96(0.75)^{100}$	3.08×10^{-11}
⋮	⋮	⋮
500	$96(0.75)^{500}$	3.26×10^{-61}
⋮	⋮	⋮
n	$96(0.75)^n$	ar^n

Study the table at the left. Transaction 1 represents the initial expenditure of \$96(0.75) or \$72 by a family. The businesses receiving this money, Transaction 2, would in turn spend 75%, and so on. We can write a geometric sequence to model this situation with $a_1 = 72$ and $r = 0.75$. Thus, the geometric sequence representing this situation is

$$72, 54, 40.50, 30.76, 22.78, \ldots.$$

In theory, the sequence above can have infinitely many terms. Thus, it is called an **infinite sequence.** As n increases, the terms of the sequence decrease and get closer and closer to zero. The terms of the modeling sequence will never actually become zero; however, the terms approach zero as n increases without bound.

Consider the infinite sequence $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots,$ whose nth term, a_n, is $\frac{1}{n}$. Several terms of this sequence are graphed at the right.



Notice that the terms approach a value of 0 as n increases. Zero is called the **limit** of the terms in this sequence.

This limit can be expressed as follows.

$$\lim_{n \to \infty} \frac{1}{n} = 0 \quad \text{∞ is the symbol for infinity.}$$

This is read "the limit of 1 over n, as n approaches infinity, equals zero."

In fact, when any positive power of n appears only in the denominator of a fraction and n approaches infinity, the limit equals zero.

$$\lim_{n \to \infty} \frac{1}{n^r} = 0, \text{ for } r > 0$$

If a general expression for the nth term of a sequence is known, the limit can usually be found by substituting large values for n. Consider the following infinite geometric sequence.

$$7, \frac{7}{4}, \frac{7}{16}, \frac{7}{64}, \frac{7}{256}, \ldots$$

This sequence can be defined by the general expression $a_n = 7\left(\frac{1}{4}\right)^{n-1}$.

$$a_{10} = 7\left(\frac{1}{4}\right)^{10-1} \approx 2.67 \times 10^{-5}$$

$$a_{50} = 7\left(\frac{1}{4}\right)^{50-1} \approx 2.21 \times 10^{-25}$$

$$a_{100} = 7\left(\frac{1}{4}\right)^{100-1} \approx 4.36 \times 10^{-60}$$

Notice that as the value of n increases, the value for a_n appears to approach 0, suggesting $\lim_{n \to \infty} 7\left(\frac{1}{4}\right)^{n-1} = 0$.

Example 1 **Estimate the limit of $\frac{9}{5}, \frac{16}{4}, \frac{65}{27}, \ldots, \frac{7n^2 + 2}{2n^2 + 3n}, \ldots$.**

The 50th term is $\frac{7(50)^2 + 2}{2(50)^2 + 3(50)}$, or about 3.398447.

The 100th term is $\frac{7(100)^2 + 2}{2(100)^2 + 3(100)}$, or about 3.448374.

The 500th term is $7\frac{(500)^2 + 2}{2(500)^2 + 3(500)}$, or about 3.489535.

The 1000th term is $\frac{7(1000)^2 + 2}{2(1000)^2 + 3(1000)}$, or 3.494759.

Notice that as $n \to \infty$, the values appear to approach 3.5, suggesting $\lim_{n \to \infty} \frac{7n^2 + 2}{2n^2 + 3n} = 3.5$.

For sequences with more complicated general forms, applications of the following limit theorems, which we will present without proof, can make the limit easier to find.

Theorems for Limits

If the $\lim_{n\to\infty} a_n$ exists, $\lim_{n\to\infty} b_n$ exists, and c is a constant, then the following theorems are true.

Limit of a Sum	$\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$
Limit of a Difference	$\lim_{n\to\infty} (a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n$
Limit of a Product	$\lim_{n\to\infty} a_n \cdot b_n = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$
Limit of a Quotient	$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$, where $\lim_{n\to\infty} b_n \neq 0$
Limit of a Constant	$\lim_{n\to\infty} c_n = c$, where $c_n = c$ for each n

The form of the expression for the nth term of a sequence can often be altered to make the limit easier to find.

Example 2 **Find each limit.**

a. $\lim_{n\to\infty} \frac{(1 + 3n^2)}{n^2}$

$$\lim_{n\to\infty} \frac{(1 + 3n^2)}{n^2} = \lim_{n\to\infty} \left(\frac{1}{n^2} + 3\right)$$ *Rewrite as the sum of two fractions and simplify.*

$$= \lim_{n\to\infty} \frac{1}{n^2} + \lim_{n\to\infty} 3$$ *Limit of a Sum*

$$= 0 + 3 \text{ or } 3$$ *$\lim_{n\to\infty} \frac{1}{n^2} = 0$ and $\lim_{n\to\infty} 3 = 3$*

Thus, the limit is 3.

Note that the Limit of a Sum theorem only applies here because $\lim_{n\to\infty} \frac{1}{n^2}$ and $\lim_{n\to\infty} 3$ each exist.

b. $\lim_{n\to\infty} \frac{5n^2 + n - 4}{n^2 + 1}$

The highest power of n in the expression is n^2. Divide each term in the numerator and the denominator by n^2. *Why does doing this produce an equivalent expression?*

$$\lim_{n\to\infty} \frac{5n^2 + n - 4}{n^2 + 1} = \lim_{n\to\infty} \frac{\frac{5n^2}{n^2} + \frac{n}{n^2} - \frac{4}{n^2}}{\frac{n^2}{n^2} + \frac{1}{n^2}}$$

$$= \lim_{n\to\infty} \frac{5 + \frac{1}{n} - \frac{4}{n^2}}{1 + \frac{1}{n^2}}$$ *Simplify.*

$$= \frac{\lim_{n\to\infty} 5 + \lim_{n\to\infty} \frac{1}{n} - \lim_{n\to\infty} 4 \cdot \lim_{n\to\infty} \frac{1}{n^2}}{\lim_{n\to\infty} 1 + \lim_{n\to\infty} \frac{1}{n^2}}$$ *Apply limit theorems.*

$$= \frac{5 + 0 - 4 \cdot 0}{1 + 0} \text{ or } 5$$ *$\lim_{n\to\infty} 5 = 5$, $\lim_{n\to\infty} \frac{1}{n} = 0$, $\lim_{n\to\infty} 4 = 4$, $\lim_{n\to\infty} 1 = 1$, and $\lim_{n\to\infty} \frac{1}{n^2} = 0a$*

Thus, the limit is 5.

Limits do not exist for all infinite sequences. If the absolute value of the terms of a sequence becomes arbitrarily great or if the terms do not approach a value, the sequence has no limit. Example 3 illustrates both of these cases.

Example 3 **Find each limit.**

a. $\lim\limits_{n\to\infty} \frac{2 + 5n + 4n^2}{2n}$

$$\lim_{n\to\infty} \frac{2 + 5n + 4n^2}{2n} = \lim_{n\to\infty} \left(\frac{1}{n} + \frac{5}{2} + 2n\right)$$ *Simplify.*

Note that $\lim\limits_{n\to\infty} \frac{1}{n} = 0$ and $\lim\limits_{n\to\infty} \frac{5}{2} = \frac{5}{2}$, but $2n$ becomes increasingly large as n approaches infinity. Therefore, the sequence has no limit.

b. $\lim\limits_{n\to\infty} \frac{(-1)^n n}{8n + 1}$

Begin by rewriting $\frac{(-1)^n n}{8n + 1}$ as $(-1)^n \cdot \frac{n}{8n + 1}$. Now find $\lim\limits_{n\to\infty} \frac{n}{8n + 1}$.

$$\lim_{n\to\infty} \frac{n}{8n + 1} = \lim_{n\to\infty} \frac{\frac{n}{n}}{\frac{8n}{n} + \frac{1}{n}}$$ *Divide the numerator and denominator by n.*

$$= \lim_{n\to\infty} \frac{1}{8 + \frac{1}{n}}$$ *Simplify.*

$$= \frac{\lim_{n\to\infty} 1}{\lim_{n\to\infty} 8 + \lim_{n\to\infty} \frac{1}{n}}$$ *Apply limit theorems.*

$$= \frac{1}{8}$$ *$\lim_{n\to\infty} 1 = 1$, $\lim_{n\to\infty} 8 = 8$, and $\lim_{n\to\infty} \frac{1}{n} = 0$*

When n is even, $(-1)^n = 1$. When n is odd, $(-1)^n = -1$. Thus, the odd-numbered terms of the sequence described by $\frac{(-1)^n n}{8n + 1}$ approach $-\frac{1}{8}$, and the even-numbered terms approach $+\frac{1}{8}$. Therefore, the sequence has no limit.

An **infinite series** is the indicated sum of the terms of an infinite sequence. Consider the series $\frac{1}{5} + \frac{1}{25} + \frac{1}{125} + \cdots$. Since this is a geometric series, you can find the sum of the first 100 terms by using the formula $S_n = \frac{a_1 - a_1r^n}{1 - r}$, where $r = \frac{1}{5}$.

$$S_{100} = \frac{\frac{1}{5} - \frac{1}{5}\left(\frac{1}{5}\right)^{100}}{1 - \frac{1}{5}}$$

$$= \frac{\frac{1}{5} - \frac{1}{5}\left(\frac{1}{5}\right)^{100}}{\frac{4}{5}}$$

$$= \frac{5}{4}\left[\frac{1}{5} - \frac{1}{5}\left(\frac{1}{5}\right)^{100}\right] \text{ or } \frac{1}{4} - \frac{1}{4}\left(\frac{1}{5}\right)^{100}$$

Since $\left(\frac{1}{5}\right)^{100}$ is very close to 0, S_{100} is nearly equal to $\frac{1}{4}$. No matter how many terms are added, the sum of the infinite series will never exceed $\frac{1}{4}$, and the difference from $\frac{1}{4}$ gets smaller as $n \to \infty$. Thus, $\frac{1}{4}$ is the sum of the infinite series.

Sum of an Infinite Series

If S_n is the sum of the first n terms of a series, and S is a number such that $S - S_n$ approaches zero as n increases without bound, then the sum of the infinite series is S.

$$\lim_{n \to \infty} S_n = S$$

If the sequence of partial sums S_n has a limit, then the corresponding infinite series has a sum, and the nth term a_n of the series approaches 0 as $n \to \infty$. If $\lim_{n \to \infty} a_n \neq 0$, the series has no sum. If $\lim_{n \to \infty} a_n = 0$, the series may or may not have a sum.

The formula for the sum of the first n terms of a geometric series can be written as follows.

$$S_n = a_1 \frac{(1 - r^n)}{1 - r}, r \neq 1$$

Recall that $|r| < 1$ means $-1 < r < 1$.

Suppose $n \to \infty$; that is, the number of terms increases without limit. If $|r| > 1$, r^n increases without limit as $n \to \infty$. However, when $|r| < 1$, r^n approaches 0 as $n \to \infty$. Under this condition, the above formula for S_n approaches a value of $\frac{a_1}{1 - r}$.

Sum of an Infinite Geometric Series

The sum S of an infinite geometric series for which $|r| < 1$ is given by

$$S = \frac{a_1}{1 - r}.$$

Example 4 **Find the sum of the series $21 - 3 + \frac{3}{7} - \cdots$.**

In the series, $a_1 = 21$ and $r = -\frac{1}{7}$. Since $|r| < 1$, $S = \frac{a_1}{1 - r}$.

$$S = \frac{a_1}{1 - r}$$

$$= \frac{21}{1 - \left(-\frac{1}{7}\right)} \qquad a_1 = 21,\ r = -\frac{1}{7}$$

$$= \frac{147}{8} \text{ or } 18\frac{3}{8}$$

The sum of the series is $18\frac{3}{8}$.

In economics, finding the sum of an infinite series is useful in determining the overall effect of economic trends.

Example 5 Real World Application

ECONOMICS **Refer to the application at the beginning of the lesson. What would be the total amount of money spent in the economy as a result of just one family's tax savings?**

For the geometric series modeling this situation, $a_1 = 72$ and $r = 0.75$.

Since $|r| < 1$, the sum of the series is equal to $\frac{a_1}{1 - r}$.

$$S = \frac{a_1}{1 - r}$$

$$= \frac{72}{1 - 0.75} \text{ or } 288$$

Therefore, the total amount of money spent is \$288.

You can use what you know about infinite series to write repeating decimals as fractions. The first step is to write the repeating decimal as an infinite geometric series.

Example 6 **Write $0.\overline{762}$ as a fraction.**

$$0.\overline{762} = \frac{762}{1000} + \frac{762}{1{,}000{,}000} + \frac{762}{1{,}000{,}000{,}000} + \cdots$$

In this series, $a_1 = \frac{762}{1000}$ and $r = \frac{1}{1000}$.

(continued on the next page)

$$S = \frac{a_1}{1 - r}$$

$$= \frac{\frac{762}{1000}}{1 - \frac{1}{1000}}$$

$$= \frac{762}{999} \text{ or } \frac{254}{333}$$

Thus, $0.762762 \cdots = \frac{254}{333}$. *Check this with a calculator.*

CHECK FOR UNDERSTANDING

Communicating Mathematics

Read and study the lesson to answer each question.

1. Consider the sequence given by the general expression $a_n = \frac{n - 1}{n}$.
   a. **Graph** the first ten terms of the sequence with the term number on the x-axis and the value of the term on the y-axis.
   b. **Describe** what happens to the value of a_n as n increases.
   c. **Make a conjecture** based on your observation in part **a** as to the limit of the sequence as n approaches infinity.
   d. **Apply** the techniques presented in the lesson to evaluate $\lim_{n \to \infty} \frac{n - 1}{n}$. How does your answer compare to your conjecture made in part **c**?

2. Consider the infinite geometric sequence given by the general expression r^n.
   a. **Determine** the limit of the sequence for $r = \frac{1}{2}$, $r = \frac{1}{4}$, $r = 1$, $r = 2$, and $r = 5$.
   b. **Write** a general rule for the limit of the sequence, placing restrictions on the value of r.

3. **Give an example** of an infinite geometric series having no sum.

4. **You Decide** Tyree and Zonta disagree on whether the infinite sequence described by the general expression $2n - 3$ has a limit. Tyree says that after dividing by the highest-powered term, the expression simplifies to $2 - \frac{3}{n}$, which has a limit of 2 as n approaches infinity. Zonta says that the sequence has no limit. Who is correct? Explain.

Guided Practice

Find each limit, or state that the limit does not exist and explain your reasoning.

5. $\lim_{n \to \infty} \frac{1}{5^n}$

6. $\lim_{n \to \infty} \frac{5 - n^2}{2n}$

7. $\lim_{n \to \infty} \frac{3n - 6}{7n}$

Write each repeating decimal as a fraction.

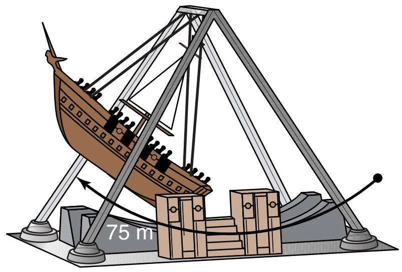
8. $0.\overline{7}$

9. $5.\overline{126}$

Find the sum of each infinite series, or state that the sum does not exist and explain your reasoning.

10. $-6 + 3 - \frac{3}{2} + \cdots$ **11.** $\frac{3}{4} + \frac{1}{4} + \frac{1}{12} + \cdots$ **12.** $\sqrt{3} + 3 + \sqrt{27} + \cdots$

13. Entertainment Pete's Pirate Ride operates like the bob of a pendulum. On its longest swing, the ship travels through an arc 75 meters long. Each successive swing is two-fifths the length of the preceding swing. If the ride is allowed to continue without intervention, what is the total distance the ship will travel before coming to rest?



EXERCISES

Practice

Find each limit, or state that the limit does not exist and explain your reasoning.

14. $\lim_{n\to\infty} \frac{7 - 2n}{5n}$ **15.** $\lim_{n\to\infty} \frac{n^3 - 2}{n^2}$ **16.** $\lim_{n\to\infty} \frac{6n^2 + 5}{3n^2}$

17. $\lim_{n\to\infty} \frac{9n^3 + 5n - 2}{2n^3}$ **18.** $\lim_{n\to\infty} \frac{(3n + 4)(1 - n)}{n^2}$ **19.** $\lim_{n\to\infty} \frac{8n^2 + 5n + 2}{3 + 2n}$

20. $\lim_{n\to\infty} \frac{4 - 3n + n^2}{2n^3 - 3n^2 + 5}$ **21.** $\lim_{n\to\infty} \frac{n}{3^n}$ **22.** $\lim_{n\to\infty} \frac{(-2)^n n}{4 + n}$

23. Find the limit of the sequence described by the general expression $\frac{5n + (-1)^n}{n^2}$, or state that the limit does not exist. Explain your reasoning.

Write each repeating decimal as a fraction.

24. $0.\overline{4}$ **25.** $0.\overline{51}$ **26.** $0.\overline{370}$

27. $6.\overline{259}$ **28.** $0.\overline{15}$ **29.** $0.\overline{263}$

30. Explain why the sum of the series $0.2 + 0.02 + 0.002 + \cdots$ exists. Then find the sum.

Find the sum of each series, or state that the sum does not exist and explain your reasoning.

31. $16 + 12 + 9 + \cdots$ **32.** $5 + 7.5 + 11.25 + \cdots$ **33.** $10 + 5 + 2.5 + \cdots$

34. $6 + 5 + 4 + \cdots$ **35.** $\frac{1}{8} + \frac{1}{4} + \frac{1}{2} + \cdots$ **36.** $-\frac{2}{3} + \frac{1}{9} - \frac{1}{54} + \cdots$

37. $\frac{6}{5} + \frac{4}{5} + \frac{8}{15} + \cdots$ **38.** $\sqrt{5} + 1 + \frac{\sqrt{5}}{5} + \cdots$ **39.** $8 - 4\sqrt{3} + 6 - \cdots$

Applications and Problem Solving

40. Physics A basketball is dropped from a height of 35 meters and bounces $\frac{2}{5}$ of the distance after each fall.

a. Find the first five terms of the infinite series representing the vertical distance traveled by the ball.

b. What is the total vertical distance the ball travels before coming to rest? (*Hint:* Rewrite the series found in part **a** as the sum of two infinite geometric series.)

41. Critical Thinking Consider the sequence whose nth term is described by $\frac{n^2}{2n+1} - \frac{n^2}{2n-1}$.

a. Explain why $\lim\limits_{n \to \infty} \left(\frac{n^2}{2n+1} - \frac{n^2}{2n-1} \right) \neq \lim\limits_{n \to \infty} \frac{n^2}{2n+1} - \lim\limits_{n \to \infty} \frac{n^2}{2n-1}$.

b. Find $\lim\limits_{n \to \infty} \left(\frac{n^2}{2n+1} - \frac{n^2}{2n-1} \right)$.

42. Engineering Francisco designs a toy with a rotary flywheel that rotates at a maximum speed of 170 revolutions per minute. Suppose the flywheel is operating at its maximum speed for one minute and then the power supply to the toy is turned off. Each subsequent minute thereafter, the flywheel rotates two-fifths as many times as in the preceding minute. How many complete revolutions will the flywheel make before coming to a stop?

43. Critical Thinking Does $\lim\limits_{n \to \infty} \cos \frac{n\pi}{2}$ exist? Explain.

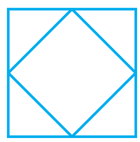
44. Medicine A certain drug developed to fight cancer has a half-life of about 2 hours in the bloodstream. The drug is formulated to be administered in doses of D milligrams every 6 hours. The amount of each dose has yet to be determined.

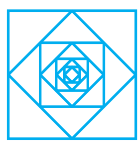
a. What fraction of the first dose will be left in the bloodstream before the second dose is administered?

b. Write a general expression for the geometric series that models the number of milligrams of drug left in the bloodstream after the nth dose.

c. About what amount of medicine is present in the bloodstream for large values of n?

d. A level of more than 350 milligrams of this drug in the bloodstream is considered toxic. Find the largest possible dose that can be given repeatedly over a long period of time without harming the patient.





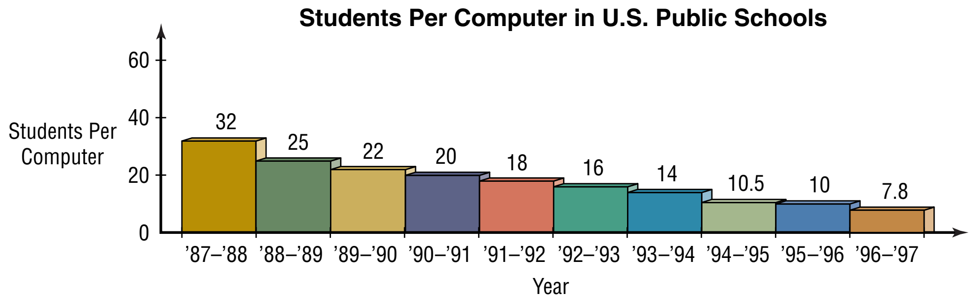
45. Geometry If the midpoints of a square are joined by straight lines, the new figure will also be a square.

a. If the original square has a perimeter of 20 feet, find the perimeter of the new square. (*Hint:* Use the Pythagorean Theorem.)

b. If this process is continued to form a sequence of "nested" squares, what will be the sum of the perimeters of all the squares?

CONTENTS

46. **Technology** Since the mid-1980s, the number of computers in schools has steadily increased. The graph below shows the corresponding decline in the student-computer ratio.



Source: *QED's Technology in Public Schools*, 16th Edition

Another publication states that the average number of students per computer in U.S. public schools can be estimated by the sequence model $a_n = 35.812791(0.864605)^n$, *for* $n = 1, 2, 3, \ldots$, with the 1987-1988 school year corresponding to $n = 1$.

a. Find the first ten terms of the model. Round your answers to the nearest tenth.

b. Use the model to estimate the average number of students having to share a computer during the 1995-1996 school year. How does this estimate compare to the actual data given in the graph?

c. Make a prediction as to the average number of students per computer for the 2000-2001 school year.

d. Does this sequence approach a limit? If so, what is the limit?

e. Realistically, will the student computer ratio ever reach this limit? Explain.

Mixed Review

47. The first term of a geometric sequence is -3, and the common ratio is $\frac{2}{3}$. Find the next four terms of the sequence. *(Lesson 12-2)*

48. Find the 16th term of the arithmetic sequence for which $a_1 = 1.5$ and $d = 0.5$. *(Lesson 12-1)*

49. Name the coordinates of the center, foci, and vertices, and the equation of the asymptotes of the hyperbola that has the equation $x^2 - 4y^2 - 12x - 16y = -16$. *(Lesson 10-4)*

50. Graph $r = 6 \cos 3\theta$. *(Lesson 9-2)*

51. **Navigation** A ship leaving port sails for 125 miles in a direction 20° north of due east. Find the magnitude of the vertical and horizontal components. *(Lesson 8-1)*

52. Use a half-angle identity to find the exact value of cos 112.5°. *(Lesson 7-4)*

53. Graph $y = \cos x$ on the interval $-180° \leq x \leq 360°$. *(Lesson 6-3)*

54. List all possible rational zeros of the function $f(x) = 8x^3 + 3x - 2$. *(Lesson 4-4)*

55. **SAT/ACT Practice** If $a = 4b + 26$, and b is a positive integer, then a could be divisible by all of the following EXCEPT

A 2 **B** 4 **C** 5 **D** 6 **E** 7

Extra Practice See p. A49.

GRAPHING CALCULATOR EXPLORATION

12-3B Continued Fractions

An Extension of Lesson 12-3

OBJECTIVE

- Explore sequences generated by continued fractions.

An expression of the following form is called a *continued fraction.*

$$a_1 + \cfrac{b_1}{a_2 + \cfrac{b_2}{a_3 + \cfrac{b_3}{a_4 + \cdots}}}$$

By using only a finite number of "decks" and values of a_n and b_n that follow regular patterns, you can often obtain a sequence of terms that approaches a limit, which can be represented by a simple expression. For example, if all of the numbers a_n and b_n are equal to 1, then the continued fraction gives rise to the following sequence.

$$1,\ 1 + \frac{1}{1},\ 1 + \cfrac{1}{1 + \cfrac{1}{1}},\ 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1}}},\ \ldots$$

It can be shown that the terms of this sequence approach the limit $\frac{1 + \sqrt{5}}{2}$. This number is often called the *golden ratio.*

The golden ratio is closely related to the Fibonacci sequence, which you will learn about in Lesson 12-7.

Now consider the following more general sequence.

$$A,\ A + \frac{1}{A},\ A + \cfrac{1}{A + \cfrac{1}{A}},\ A + \cfrac{1}{A + \cfrac{1}{A + \cfrac{1}{A}}},\ \ldots$$

To help you visualize what this sequence represents, suppose $A = 5$. The sequence becomes $5,\ 5 + \frac{1}{5},\ 5 + \cfrac{1}{5 + \cfrac{1}{5}},\ 5 + \cfrac{1}{5 + \cfrac{1}{5 + \cfrac{1}{5}}},\ \ldots$ or $5, \frac{26}{5}, \frac{135}{26}, \frac{701}{135}, \ldots$.

A calculator approximation of this sequence is 5, 5.2, 5.192307692, 5,192592593, … .

Each term of the sequence is the sum of A and the reciprocal of the previous term. The program at the right calculates the value of the nth term of the above sequence for $n \geq 3$ and a specified value of A.

When you run the program it will ask you to input values for A and N.

```
PROGRAM: CFRAC
: Prompt A
: Disp "INPUT TERM"
: Disp "NUMBER N, N ≥ 3"
: Prompt N
: 1 → K
: A + 1/A → C
: Lbl 1
: A + 1/C → C
: K + 1 → K
: If K < N − 1
: Then: Goto 1
: Else: Disp C
```

interNET CONNECTION

Graphing Calculator Programs
To download this graphing calculator program, visit our website at **www.amc.glencoe.com**

TRY THESE **Enter the program into your calculator and use it for the exercises that follow.**

1. What output is given when the program is executed for $A = 1$ and $N = 10$?

2. With $A = 1$, determine the least value of N necessary to obtain an output that agrees with the calculator's nine decimal approximation of $\dfrac{1+\sqrt{5}}{2}$.

3. Use algebra to show that the continued fraction $1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}$ has a value of $\dfrac{1+\sqrt{5}}{2}$. (*Hint:* If $x = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}$, then $x = 1 + \dfrac{1}{x}$. Solve this last equation for x.)

4. Find the exact value of $3 + \cfrac{1}{3 + \cfrac{1}{3 + \cdots}}$.

5. Execute the program with $A = 3$ and $N = 40$. How does this output compare to the decimal approximation of the expression found in Exercise 4?

6. Find a radical expression for $A + \cfrac{1}{A + \cfrac{1}{A + \cdots}}$.

7. Write a modified version of the program that calculates the nth term of the following sequence for $n \geq 3$.

$$A,\ A + \frac{B}{2A},\ A + \cfrac{B}{2A + \cfrac{B}{2A}},\ A + \cfrac{B}{2A + \cfrac{B}{2A + \cfrac{B}{2A}}},\ \ldots$$

8. Choose several positive integer values for A and B and compare the program output with the decimal approximation of $\sqrt{A^2 + B}$ for several values of n, for $n \geq 3$. Describe your observations.

9. Use algebra to show that for $A > 0$ and $B > 0$, $A + \cfrac{B}{2A + \cfrac{B}{2A + \cdots}}$ has a value of $\sqrt{A^2 + B}$.

$$\left(\textit{Hint: } \text{If } x = A + \cfrac{B}{2A + \cfrac{B}{2A + \cdots}},\ \text{then } x + A = 2A + \cfrac{B}{2A + \cfrac{B}{2A + \cdots}}.\right)$$

WHAT DO YOU THINK?

10. If you execute the original program for $A = 1$ and $N = 20$ and then execute it for $A = -1$ and $N = 20$, how will the two outputs compare?

11. What values can you use for A and B in the program for Exercise 7 in order to approximate $\sqrt{15}$?