12-4

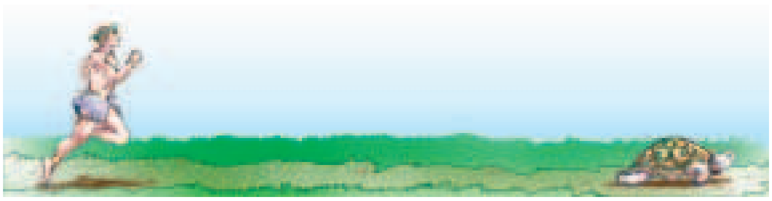
Convergent and Divergent Series

OBJECTIVE

- Determine whether a series is convergent or divergent.

Real World Application

HISTORY The Greek philosopher Zeno of Elea (c. 490–430 B.C.) proposed several perplexing riddles, or paradoxes. One of Zeno's paradoxes involves a race on a 100-meter track between the mythological Achilles and a tortoise. Zeno claims that even though Achilles can run twice as fast as the tortoise, if the tortoise is given a 10-meter head start, Achilles will never catch him. Suppose Achilles runs 10 meters per second and the tortoise a remarkable 5 meters per second. By the time Achilles has reached the 10-meter mark, the tortoise will be at 15 meters. By the time Achilles reaches the 15-meter mark, the tortoise will be at 17.5 meters, and so on. Thus, Achilles is always behind the tortoise and never catches up.



Is Zeno correct? Let us look at the distance between Achilles and the tortoise after specified amounts of time have passed. Notice that the distance between the two contestants will be zero as n approaches infinity since $\lim_{n\to\infty} \frac{10}{2^n} = 0$.

Time (seconds)	Distance Apart (meters)
0	10
1	$\frac{10}{2} = 5$
$1 + \frac{1}{2} = \frac{3}{2}$	$\frac{10}{4} = 2.5$
$1 + \frac{1}{2} + \frac{1}{4} = \frac{7}{4}$	$\frac{10}{8} = 1.25$
$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{15}{8}$ $\vdots$	$\frac{10}{16} = 0.625$ $\vdots$
$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$	$\frac{10}{2^n}$

To disprove Zeno's conclusion that Achilles will never catch up to the tortoise, we must show that there is a time value for which this 0 difference can be achieved. In other words, we need to show that the infinite series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$ has a sum, or limit. *This problem will be solved in Example 5.*

Starting with a time of 1 second, the partial sums of the time series form the sequence $1, \frac{3}{2}, \frac{7}{4}, \frac{15}{18}, \ldots$. As the number of terms used for the partial sums increases, the value of the partial sums also increases. If this sequence of partial sums approaches a limit, the related infinite series is said to **converge.** If this sequence of partial sums does not have a limit, then the related infinite series is said to **diverge.**

Convergent and Divergent Series	If an infinite series has a sum, or limit, the series is convergent. If a series is not convergent, it is divergent.

Example 1 **Determine whether each arithmetic or geometric series is convergent or divergent.**

There are many series that begin with the first few terms shown in this example. In this chapter, always assume that the expression for the general term is the simplest one possible.

a. $-\frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \cdots$

This is a geometric series with $r = -\frac{1}{2}$. Since $|r| < 1$, the series has a limit. Therefore, the series is convergent.

b. $2 + 4 + 8 + 16 + \cdots$

This is a geometric series with $r = 2$. Since $|r| > 1$, the series has no limit. Therefore, the series is divergent.

c. $10 + 8.5 + 7 + 5.5 + \cdots$

This is an arithmetic series with $d = -1.5$. Arithmetic series do not have limits. Therefore, the series is divergent.

When a series is neither arithmetic nor geometric, it is more difficult to determine whether the series is convergent or divergent. Several different techniques can be used. One test for convergence is the **ratio test.** This test can only be used when all terms of a series are positive. The test depends upon the ratio of consecutive terms of a series, which must be expressed in general form.

Ratio Test	Let a_n and a_{n+1} represent two consecutive terms of a series of positive terms. Suppose $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n}$ exists and that $r = \lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n}$. The series is convergent if $r < 1$ and divergent if $r > 1$. If $r = 1$, the test provides no information.

The ratio test is especially useful when the general form for the terms of a series contains powers.

Example 2 **Use the ratio test to determine whether each series is convergent or divergent.**

a. $\frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{4}{16} + \cdots$

First, find a_n and a_{n+1}. $\quad a_n = \frac{n}{2^n} \quad$ and $\quad a_{n+1} = \frac{n+1}{2^{n+1}}$

Then use the ratio test. $\quad r = \lim\limits_{n\to\infty} \dfrac{\frac{n+1}{2^{n+1}}}{\frac{n}{2^n}}$

(continued on the next page)

$$r = \lim_{n\to\infty} \frac{n+1}{2^{n+1}} \cdot \frac{2^n}{n}$$ *Multiply by the reciprocal of the divisor.*

$$r = \lim_{n\to\infty} \frac{1}{2} \cdot \frac{n+1}{n}$$ $\frac{2^n}{2^{n+1}} = \frac{1}{2}$

$$r = \lim_{n\to\infty} \frac{1}{2} \cdot \lim_{n\to\infty} \frac{n+1}{n}$$ *Limit of a Product*

$$r = \frac{1}{2} \lim_{n\to\infty} \frac{1 + \frac{1}{n}}{1}$$ *Divide by the highest power of n and then apply limit theorems.*

$$r = \frac{1}{2} \cdot \frac{1+0}{1} \text{ or } \frac{1}{2}$$ Since $r < 1$, the series is convergent.

b. $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \cdots$

$a_n = \frac{n}{n+1}$ and $a_{n+1} = \frac{n+1}{(n+1)+1}$ or $\frac{n+1}{n+2}$

$$r = \lim_{n\to\infty} \frac{\frac{n+1}{n+2}}{\frac{n}{n+1}}$$

$$r = \lim_{n\to\infty} \frac{n^2+2n+1}{n^2+2n}$$ $\frac{n+1}{n+2} \cdot \frac{n+1}{n} = \frac{n^2+2n+1}{n^2+2n}$

$$r = \lim_{n\to\infty} \frac{1 + \frac{2}{n} + \frac{1}{n^2}}{1 + \frac{2}{n}}$$ *Divide by the highest power of n and apply limit theorems.*

$$r = \frac{1+0+0}{1+0} \text{ or } 1$$ Since $r = 1$, the test provides no information.

The ratio test is also useful when the general form of the terms of a series contains products of consecutive integers.

Example 3 **Use the ratio test to determine whether the series**

$$1 + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{1 \cdot 2 \cdot 3 \cdot 4} + \cdots$$ **is convergent or divergent.**

First find the nth term and $(n+1)$th term. Then, use the ratio test.

$a_n = \frac{1}{1 \cdot 2 \cdot \cdots \cdot n}$ and $a_{n+1} = \frac{1}{1 \cdot 2 \cdot \cdots \cdot (n+1)}$

$$r = \lim_{n\to\infty} \frac{\frac{1}{1 \cdot 2 \cdot \cdots \cdot (n+1)}}{\frac{1}{1 \cdot 2 \cdot \cdots \cdot n}}$$

$$r = \lim_{n\to\infty} \frac{1 \cdot 2 \cdot \cdots \cdot n}{1 \cdot 2 \cdot \cdots \cdot (n+1)}$$ *Note that $1 \cdot 2 \cdot \cdots \cdot (n+1) = 1 \cdot 2 \cdot \cdots \cdot n \cdot (n+1)$.*

$$r = \lim_{n\to\infty} \frac{1}{n+1} \text{ or } 0$$ *Simplify and apply limit theorems.*

Since $r < 1$, the series is convergent.

When the ratio test does not determine if a series is convergent or divergent, other methods must be used.

Example 4 **Determine whether the series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$ is convergent or divergent.**

Suppose the terms are grouped as follows. Beginning after the second term, the number of terms in each successive group is doubled.

$$(1) + \left(\frac{1}{2}\right) + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \left(\frac{1}{9} + \cdots + \frac{1}{16}\right) + \cdots$$

Notice that the first enclosed expression is greater than $\frac{1}{2}$, and the second is equal to $\frac{1}{2}$. Beginning with the third expression, each sum of enclosed terms is greater than $\frac{1}{2}$. Since there are an unlimited number of such expressions, the sum of the series is unlimited. Thus, the series is divergent.

A series can be compared to other series that are known to be convergent or divergent. The following list of series can be used for reference.

Summary of Series for Reference

1. **Convergent:** $a_1 + a_1 r + a_1 r^2 + \cdots + a_1 r^{n-1} + \cdots, \ |r| < 1$
2. **Divergent:** $a_1 + a_1 r + a_1 r^2 + \cdots + a_1 r^{n-1} + \cdots, \ |r| > 1$
3. **Divergent:** $a_1 + (a_1 + d) + (a_1 + 2d) + (a_1 + 3d) + \cdots$
4. **Divergent:** $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots + \frac{1}{n} + \cdots$ *This series is known as the harmonic series.*
5. **Convergent:** $1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots, \ p > 1$

If a series has all positive terms, the **comparison test** can be used to determine whether the series is convergent or divergent.

Comparison Test

- A series of positive terms is convergent if, for $n > 1$, each term of the series is equal to or less than the value of the corresponding term of some convergent series of positive terms.
- A series of positive terms is divergent if, for $n > 1$, each term of the series is equal to or greater than the value of the corresponding term of some divergent series of positive terms.

Example 5 **Use the comparison test to determine whether the following series are convergent or divergent.**

a. $\frac{4}{5} + \frac{4}{7} + \frac{4}{9} + \frac{4}{11} + \cdots$

The general term of this series is $\frac{4}{2n + 3}$. The general term of the divergent series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$ is $\frac{1}{n}$. Since for all $n > 1$, $\frac{4}{2n + 3} > \frac{1}{n}$, the series $\frac{4}{5} + \frac{4}{7} + \frac{4}{9} + \frac{4}{11} + \cdots$ is also divergent.

b. $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots$

The general term of the series is $\frac{1}{(2n-1)^2}$. The general term of the convergent series $1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$ is $\frac{1}{n^2}$. Since $\frac{1}{(2n-1)^2} \leq \frac{1}{n^2}$ for all n, the series $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots$ is also convergent.

With a better understanding of convergent and divergent infinite series, we are now ready to tackle Zeno's paradox.

Example 6 Real World Application

HISTORY **Refer to the application at the beginning of the lesson. To disprove Zeno's conclusion that Achilles will never catch up to the tortoise, we must show that the infinite time series $1 + 0.5 + 0.25 + \cdots$ has a limit.**

To show that the series $1 + 0.5 + 0.25 + \cdots$ has a limit, we need to show that the series is convergent.

The general term of this series is $\frac{1}{2^n}$. Try using the ratio test for convergence of a series.

$$a_n = \frac{1}{2^n} \quad \text{and} \quad a_{n+1} = \frac{1}{2^{n+1}}$$

$$r = \frac{\frac{1}{2^{n+1}}}{\frac{1}{2^n}}$$

$$= \frac{1}{2} \qquad \frac{1}{2^{n+1}} \cdot \frac{2^n}{1} = \frac{1}{2}$$

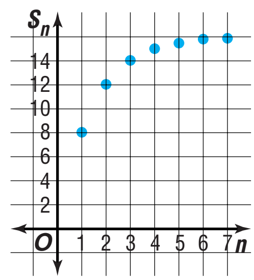
Since $r < 1$, the series converges and therefore has a sum. Thus, there is a time value for which the distance between Achilles and the tortoise will be zero. *You will determine how long it takes him to do so in Exercise 34.*

CHECK FOR UNDERSTANDING

Communicating Mathematics

Read and study the lesson to answer each question.

1. a. **Write** an example, of an infinite geometric series in which $|r| > 1$.
   b. **Determine** the 25th, 50th, and 100th terms of your series.
   c. **Identify** the sum of the first 25, 50, and 100 terms of your series.
   d. **Explain** why this type of infinite geometric series does not converge.
2. **Estimate** the sum S_n of the series whose partial sums are graphed at the right.



3. Consider the infinite series $\frac{1}{3} + \frac{2^2}{3^2} + \frac{3^2}{3^3} + \frac{4^2}{3^4} + \cdots$.

a. **Sketch** a graph of the first eight partial sums of this series.

b. **Make** a conjecture based on the graph found in part **a** as to whether the series is convergent or divergent.

c. **Determine** a general term for this series.

d. **Write a convincing argument** using the general term found in part **c** to support the conjecture you made in part **b**.

4. *Math Journal* **Make a list** of the methods presented in this lesson and in the previous lesson for determining convergence or divergence of an infinite series. Be sure to indicate any restrictions on a method's use. Then number your list as to the order in which these methods should be applied.

Guided Practice

Use the ratio test to determine whether each series is *convergent* or *divergent*.

5. $\frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \cdots$

6. $\frac{3}{4} + \frac{7}{8} + \frac{11}{12} + \frac{15}{16} + \cdots$

7. Use the comparison test to determine whether the series $\frac{2}{1} + \frac{3}{2} + \frac{4}{3} + \cdots$ is *convergent* or *divergent*.

Determine whether each series is *convergent* or *divergent*.

8. $\frac{1}{4} + \frac{5}{16} + \frac{3}{8} + \frac{7}{16} + \cdots$

9. $\frac{1}{2 + 1^2} + \frac{1}{2 + 2^2} + \frac{1}{2 + 3^2} + \cdots$

10. $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 2^2} + \frac{1}{3 \cdot 2^3} + \cdots$

11. $4 + 3 + \frac{9}{4} + \cdots$

12. **Ecology** An underground storage container is leaking a toxic chemical. One year after the leak began, the chemical had spread 1500 meters from its source. After two years, the chemical had spread 900 meters more, and by the end of the third year, it had reached an additional 540 meters.

a. If this pattern continues, how far will the spill have spread from its source after 10 years?

b. Will the spill ever reach the grounds of a school located 4000 meters away from the source? Explain.

EXERCISES

Practice

Use the ratio test to determine whether each series is *convergent* or *divergent*.

13. $\frac{4}{3} + \frac{4}{9} + \frac{4}{27} + \frac{4}{81} + \cdots$

14. $\frac{2}{5} + \frac{4}{10} + \frac{8}{15} + \cdots$

15. $2 + \frac{4}{2^2} + \frac{8}{3^2} + \frac{16}{4^2} + \cdots$

16. $\frac{2}{2 \cdot 3} + \frac{2}{3 \cdot 4} + \frac{2}{4 \cdot 5} + \cdots$

17. $1 + \frac{3}{1 \cdot 2 \cdot 3} + \frac{5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} + \cdots$

18. $5 + \frac{5^2}{1 \cdot 2} + \frac{5^3}{1 \cdot 2 \cdot 3} + \cdots$

19. Use the ratio test to determine whether the series $\frac{2 \cdot 4}{2} + \frac{4 \cdot 6}{4} + \frac{6 \cdot 8}{8} + \frac{8 \cdot 10}{16} + \cdots$ is convergent or divergent.

Use the comparison test to determine whether each series is *convergent* or *divergent*.

20. $\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \cdots$

21. $\frac{1}{2} + \frac{1}{9} + \frac{1}{28} + \frac{1}{65} + \cdots$

22. $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \cdots$

23. $\frac{5}{3} + \frac{5}{4} + 1 + \frac{5}{6} + \cdots$

24. Use the comparison test to determine whether the series $\frac{1}{3} + \frac{1}{5} + \frac{1}{9} + \frac{1}{17} \cdots$ is *convergent* or *divergent*.

Determine whether each series is *convergent* or *divergent*.

25. $\frac{1}{2} - \frac{3}{8} + \frac{9}{32} - \cdots$

26. $3 + \frac{5}{3} + \frac{7}{5} + \cdots$

27. $\frac{1}{5 + 1^2} + \frac{1}{5 + 2^2} + \frac{1}{5 + 3^2} + \cdots$

28. $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \cdots$

29. $\frac{4\pi}{3} + \frac{5\pi}{6} + \frac{\pi}{3} + \cdots$

30. $\frac{1}{4} + \frac{3}{8} + \frac{5}{16} + \frac{7}{32} + \cdots$

Applications and Problem Solving

31. Economics The MagicSoft software company has a proposal to the city council of Alva, Florida, to relocate there. The proposal claims that the company will generate \$3.3 million for the local economy by the \$1 million in salaries that will be paid. The city council estimates that 70% of the salaries will be spent in the local community, and 70% of that money will again be spent in the community, and so on.

a. According to the city council's estimates, is the claim made by MagicSoft accurate? Explain.

b. What is the correct estimate of the amount generated to the local economy?

32. Critical Thinking Give an example of a series $a_1 + a_2 + a_3 + \cdots + a_n + \cdots$ that diverges, but when its terms are squared, the resulting series ${a_1}^2 + {a_2}^2 + {a_3}^2 + \cdots + {a_n}^2 + \cdots$ converges.

33. Cellular Growth Leticia Cox is a biochemist. She is testing two different types of drugs that induce cell growth. She has selected two cultures of 1000 cells each. To culture A, she administers a drug that raises the number of cells by 200 each day and every day thereafter. Culture B gets a drug that increases cell growth by 8% each day and everyday thereafter.

a. Assuming no cells die, how many cells will have grown in each culture by the end of the seventh day?

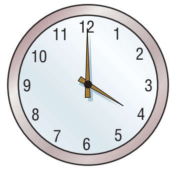
b. At the end of one month's time, which drug will prove to be more effective in promoting cell growth? Explain.

34. Critical Thinking Refer to Example 6 of this lesson. The sequence of partial sums, $S_1, S_2, S_3, \ldots, S_n, \ldots$, for the time series is $1, \frac{3}{2}, \frac{7}{4}, \frac{15}{8}, \ldots$.

a. Find a general expression for the nth term of this sequence.

b. To determine how long it takes for Achilles to catch-up to the tortoise, find the sum of the infinite time series. (*Hint:* Recall from the definition of the sum S of an infinite series that $\lim\limits_{n \to \infty} S_n = S$.)

35. **Clocks** The hour and minute hands of a clock travel around its face at different speeds, but at certain times of the day, the two hands coincide. In addition to noon and midnight, the hands also coincide at times occurring between the hours. According the figure at the right, it is 4:00.



a. When the minute hand points to 4, what fraction of the distance between 4 and 5 will the hour hand have traveled?
b. When the minute hand reaches the hour hand's new position, what additional fraction will the hour hand have traveled?
c. List the next two terms of this series representing the distance traveled by the hour hand as the minute hand "chases" its position.
d. At what time between the hours of 4 and 5 o'clock will the two hands coincide?

Mixed Review

36. Evaluate $\lim\limits_{n \to \infty} \dfrac{4n^2 + 5}{3n^2 - 2n}$. *(Lesson 12-3)*

37. Find the ninth term of the geometric sequence $\sqrt{2}, 2, 2\sqrt{2}, \ldots$. *(Lesson 12-2)*

38. Form an arithmetic sequence that has five arithmetic means between -11 and 19. *(Lesson 12-1)*

39. Solve $45.9 = e^{0.075t}$ *(Lesson 11-6)*

40. **Navigation** A submarine sonar is tracking a ship. The path of the ship is recorded as $6 = 12r \cos(\theta - 30°)$. Find the linear equation of the path of the ship. *(Lesson 9-4)*

41. Find an ordered pair that represents $\overrightarrow{AB}$ for $A(8, -3)$ and $B(5, -1)$. *(Lesson 8-2)*

42. **SAT/ACT Practice** How many numbers from 1 to 200 inclusive are equal to the cube of an integer?

A one **B** two **C** three **D** four **E** five

MID-CHAPTER QUIZ

1. Find the 19th term in the sequence for which $a_1 = 11$ and $d = -2$. (Lesson 12-1)

2. Find S_{20} for the arithmetic series for which $a_1 = -14$ and $d = 6$. (Lesson 12-1)

3. Form a sequence that has two geometric means between 56 and 189. (Lesson 12-2)

4. Find the sum of the first eight terms of the series $3 - 6 + 12 - \cdots$. (Lesson 12-2)

5. Find $\lim\limits_{n \to \infty} \dfrac{n^2 + 2n - 5}{n^2 - 1}$ or explain why the limit does not exist. (Lesson 12-3)

6. **Recreation** A bungee jumper rebounds 55% of the height jumped. If a bungee jump is made using a cord that stretches 250 feet, find the total distance traveled by the jumper before coming to rest. (Lesson 12-3)

7. Find the sum of the following series. $\dfrac{1}{25} + \dfrac{1}{250} + \dfrac{1}{2500} + \cdots$. (Lesson 12-3)

Determine whether each series is *convergent* or *divergent*. (Lesson 12-4)

8. $\dfrac{1}{10} + \dfrac{2}{100} + \dfrac{6}{1000} + \dfrac{24}{10{,}000} + \cdots$

9. $\dfrac{6}{5} + \dfrac{2}{5} + \dfrac{2}{15} + \cdots$

10. **Finance** Ms. Fuentes invests \$500 quarterly (January 1, April 1, July 1, and October 1) in a retirement account that pays an APR of 12% compounded quarterly. Interest for each quarter is posted on the last day of the quarter. Determine the value of her investment at the end of the year. (Lesson 12-2)

Extra Practice See p. A49.