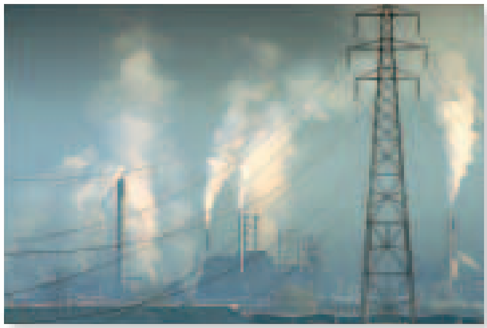
12-5

Sigma Notation and the *n*th Term

OBJECTIVE
- Use sigma notation.

Real World Application

MANUFACTURING Manufacturers are required by the Environmental Protection Agency to meet certain emission standards. If these standards are not met by a preassigned date, the manufacturer is fined. To encourage swift compliance, the fine increases a specified amount each day until the manufacturer is able to pass inspection. Suppose a manufacturing plant is charged \$2000 for not meeting its January 1st deadline. The next day it is charged \$2500, the next day \$3000, and so on, until it passes inspection on January 21st. What is the total amount of fines owed by the manufacturing plant? *This problem will be solved in Example 2.*



In mathematics, the uppercase Greek letter sigma, Σ, is often used to indicate a sum or series. A series like the one indicated above may be written using **sigma notation.**

$$\sum_{n=1}^{k} a_n$$

maximum value of n → k; *starting value of n* → $n=1$; *expression for general term* → a_n

Other variables besides n may be used for the index of summation.

The variable n used with the sigma notation is called the **index of summation.**

Sigma Notation of a Series

For any sequence $a_1, a_2, a_3, \ldots$, the sum of the first k terms may be written $\sum_{n=1}^{k} a_n$, which is read "the summation from $n = 1$ to k of a_n." Thus,

$$\sum_{n=1}^{k} a_n = a_1 + a_2 + a_3 + \cdots + a_k, \text{ where } k \text{ is an integer value.}$$

Example 1 (Real World Application) **Write each expression in expanded form and then find the sum.**

a. $\sum_{n=1}^{4} (n^2 - 3)$

First, write the expression in expanded form.

$$\sum_{n=1}^{4} (n^2 - 3) = \underbrace{(1^2 - 3)}_{n=1} + \underbrace{(2^2 - 3)}_{n=2} + \underbrace{(3^2 - 3)}_{n=3} + \underbrace{(4^2 - 3)}_{n=4}$$

Now find the sum.

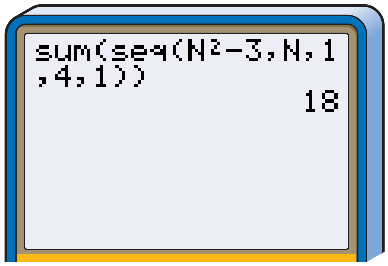
Method 1 Simplify the expanded form.
$(1^2 - 3) + (2^2 - 3) + (3^2 - 3) + (4^2 - 3) = -2 + 1 + 6 + 13$ or 18

Method 2 Use a graphing calculator.

You can combine **sum(** with **seq(** to obtain the sum of a finite sequence. In the **LIST** menu, select the **MATH** option to locate the **sum(** command. The **seq(** command can be found in the **OPS** option of the **LIST** menu.

sum(seq(N² − 3 , N , 1 , 4 , 1))

expression (N² − 3); *index of summation* (N); *starting value* (1); *maximum value* (4); *step size* (1)



b. $\sum_{n=1}^{\infty} 5\left(-\frac{2}{7}\right)^{n-1}$

$$\sum_{n=1}^{\infty} 5\left(-\frac{2}{7}\right)^{n-1} = \underbrace{5\left(-\frac{2}{7}\right)^{1-1}}_{n=1} + \underbrace{5\left(-\frac{2}{7}\right)^{2-1}}_{n=2} + \underbrace{5\left(-\frac{2}{7}\right)^{3-1}}_{n=3} + \underbrace{5\left(-\frac{2}{7}\right)^{4-1}}_{n=4} + \cdots$$

$$= 5 + \frac{10}{7} + \frac{20}{49} + \frac{40}{343} + \cdots$$

This is an infinite geometric series. Use the formula $S = \frac{a_1}{1 - r}$.

$$S = \frac{5}{1 - \left(-\frac{2}{7}\right)} \qquad a_1 = 5,\ r = -\frac{2}{7}$$

$$S = \frac{35}{9}$$

Therefore, $\sum_{n=1}^{\infty} 5\left(-\frac{2}{7}\right)^{n-1} = \frac{35}{9}$.

A series in expanded form can be written using sigma notation if a general formula can be written for the nth term of the series.

Example 2 Real World Application **MANUFACTURING** **Refer to the application at the beginning of this lesson.**

a. How much is the company fined on the 20th day?

b. What is the total amount in fines owed by the manufacturing plant?

c. Represent this sum using sigma notation.

a. Since this sequence is arithmetic, we can use the formula for the nth term of an arithmetic sequence to find the amount of the fine charged on the 20th day.

$$a_n = a_1 + (n - 1)d$$
$$a_{20} = 2000 + (20 - 1)500 \qquad a_1 = 2000,\ n = 20, \text{ and } d = 500$$
$$= 11{,}500$$

The fine on the 20th day will be $11,500.

b. To determine the total amount owed in fines, we can use the formula for the sum of an arithmetic series. The plant will not be charged for the day it passes inspection, so it is assessed 20 days in fines.

$$S_n = \frac{n}{2}(a_1 + a_n)$$
$$= \frac{20}{2}(2000 + 11{,}500) \quad n = 20,\ a_1 = 2000,\text{ and } a_{20} = 11{,}500$$
$$= 135{,}000$$

The plant must pay a total of $135,000 in fines.

c. To determine the fine for the nth day, we can again use the formula for the nth term of an arithmetic sequence.

$$a_n = a_1 + (n - 1)d$$
$$= 2000 + (n - 1)500 \quad a_1 = 2000 \text{ and } d = 500$$
$$= 2000 + 500n - 500$$
$$= 500n + 1500$$

The fine on the nth day is $\$500n + \$1500$.

Since $2000 is the fine on the first day and $12,000 is the fine on the 20th day, the index of summation goes from $n = 1$ to $n = 20$.

Therefore, $2000 + 2500 + 3000 + \cdots + 11{,}500 = \sum_{n=1}^{20}(500n + 1500)$ or $\sum_{n=1}^{20} 500(n + 3)$.

When using sigma notation, it is not always necessary that the sum start with the index equal to 1.

Example 3 **Express the series 15 + 24 + 35 + 48 + ⋯ + 143 using sigma notation.**

Notice that each term is 1 less than a perfect square. Thus, the nth term of the series is $n^2 - 1$. Since $4^2 - 1 = 15$ and $12^2 - 1 = 143$, the index of summation goes from $n = 4$ to $n = 12$.

Therefore, $15 + 24 + 35 + 48 + \cdots + 143 = \sum_{n=4}^{12} (n^2 - 1)$.

As you have seen, not all sequences are arithmetic or geometric. Some important sequences are generated by products of consecutive integers. The product $n(n - 1)(n - 2) \cdots 3 \cdot 2 \cdot 1$ is called ***n* factorial** and symbolized $n!$.

n Factorial

The expression $n!$ (n factorial) is defined as follows for n, an integer greater than zero.

$$n! = n(n - 1)(n - 2) \cdots 1$$

By definition $0! = 1$.

Factorial notation can be used to express the general form of a series.

Example 4 **Express the series $\frac{2}{2} - \frac{4}{6} + \frac{6}{24} - \frac{8}{120} + \frac{10}{720}$ using sigma notation.**

The sequence representing the numerators is 2, 4, 6, 8, 10. This is an arithmetic sequence with a common difference of 2. Thus the nth term can be represented by $2n$.

Because the series has alternating signs, one factor for the general term of the series is $(-1)^{n+1}$. Thus, when n is odd, the terms are positive, and when n is even, the terms are negative.

The sequence representing the denominators is 2, 6, 24, 120, 720. This sequence is generated by factorials.

$2! = 2$
$3! = 6$
$4! = 24$
$5! = 120$
$6! = 720$

Therefore, $\frac{2}{2} - \frac{4}{6} + \frac{6}{24} - \frac{8}{120} + \frac{10}{720} = \sum_{n=1}^{5} \frac{(-1)^{n+1}\,2n}{(n+1)!}$.

You can check this answer by substituting values of n into the general term.

Check for Understanding

Communicating Mathematics

Read and study the lesson to answer each question.

1. **Find a counterexample** to the following statement: "The summation notion used to represent a series is unique."
2. In Example 4 of this lesson, the alternating signs of the series were represented by a factor of $(-1)^{n+1}$.
   a. **Write** a different factor that could have been used in the general form of the nth term of the series.
   b. **Determine** a factor that could be used if the alternating signs of the series began with a negative first term.
3. Consider the series $\sum_{j=2}^{10} (-2j + 1)$.
   a. **Identify** the number of terms in this series.
   b. **Write** a formula that determines the number of terms t in a finite series if the index of summation has a minimum value of a and a maximum value of b.
   c. **Use** the formula in part **b** to identify the number of terms in the series $\sum_{k=-2}^{3} \frac{1}{k+3}$.
   d. **Verify** your answer in part **c** by writing the series $\sum_{k=-2}^{3} \frac{1}{k+3}$ in expanded form.

Guided Practice

Write each expression in expanded form and then find the sum.

4. $\sum_{n=1}^{6} (n - 3)$
5. $\sum_{k=2}^{5} 4k$
6. $\sum_{a=0}^{4} \frac{1}{2^a}$
7. $\sum_{p=0}^{\infty} 5\left(\frac{3}{4}\right)^p$

Express each series using sigma notation.

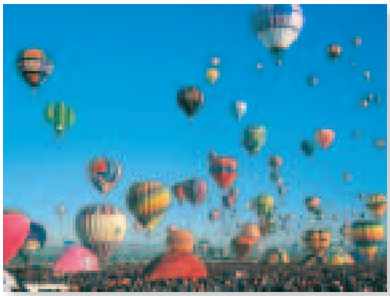
8. $5 + 10 + 15 + 20 + 25$

9. $2 + 4 + 10 + 28$

10. $2 - 4 - 10 - 16$

11. $\frac{3}{4} + \frac{3}{8} + \frac{3}{16} + \frac{3}{32} + \cdots$

12. $-3 + 9 - 27 + \cdots$



13. Aviation Each October Albuquerque, New Mexico, hosts the Balloon Fiesta. In 1998, 873 hot air balloons participated in the opening day festivities. One of these balloons rose 389 feet after 1 minute. Because the air in the balloon was not reheated, each succeeding minute the balloon rose 63% as far as it did the previous minute.

a. Use sigma notation to represent the height of the balloon above the ground after one hour. Then calculate the total height of the balloon after one hour to the nearest foot.

b. What was the maximum height achieved by this balloon?

EXERCISES

Practice

Write each expression in expanded form and then find the sum.

14. $\sum_{n=1}^{4} (2n - 7)$

15. $\sum_{a=2}^{5} 5a$

16. $\sum_{b=3}^{8} (6 - 4b)$

17. $\sum_{k=2}^{6} (k + k^2)$

18. $\sum_{n=5}^{8} \frac{n}{n - 4}$

19. $\sum_{j=4}^{8} 2^j$

20. $\sum_{m=0}^{3} 3^{m-1}$

21. $\sum_{r=1}^{3} \left(\frac{1}{2} + 4^r\right)$

22. $\sum_{i=3}^{5} (0.5)^{-i}$

23. $\sum_{k=3}^{7} k!$

24. $\sum_{p=0}^{\infty} 4(0.75)^p$

25. $\sum_{n=1}^{\infty} 4\left(\frac{2}{5}\right)^n$

26. Write $\sum_{n=2}^{5} n + \boldsymbol{i}^n$ in expanded form. Then find the sum.

Express each series using sigma notation.

27. $6 + 9 + 12 + 15$

28. $1 + 4 + 16 + \cdots + 256$

29. $8 + 10 + 12 + \cdots + 24$

30. $-8 + 4 - 2 + 1$

31. $10 + 50 + 250 + 1250$

32. $13 + 9 + 5 + 1$

33. $\frac{1}{9} + \frac{1}{14} + \frac{1}{19} + \cdots + \frac{1}{49}$

34. $\frac{2}{3} + \frac{4}{5} + \frac{8}{7} + \frac{16}{9} + \cdots$

35. $4 - 9 + 16 - 25 + \cdots$

36. $5 + 5 + \frac{5}{2} + \frac{5}{6} + \frac{5}{24} + \cdots$

37. $-32 + 16 - 8 + 4 - \cdots$

38. $2 + \frac{6}{2} + \frac{24}{3} + \frac{120}{4} + \cdots$

39. $\frac{1}{5} + \frac{2}{7} + \frac{3}{11} + \frac{4}{19} + \frac{5}{35} + \cdots$

40. $\frac{3}{9 \cdot 2} + \frac{8}{27 \cdot 6} + \frac{15}{81 \cdot 24} + \cdots$

41. Express the series $\frac{\sqrt{2}}{3} + \frac{2}{6} + \frac{\sqrt{8}}{18} + \frac{4}{72} + \frac{\sqrt{32}}{360} + \cdots$ using sigma notation.

Simplify. Assume that *n* and *m* are positive integers, *a* > *b*, and *a* > 2.

42. $\frac{(a - 2)!}{a!}$

43. $\frac{(a + 1)!}{(a - 2)!}$

44. $\frac{(a + b)!}{(a + b - 1)!}$

Graphing Calculator

45. Use a graphing calculator to find the sum of $\sum_{n=1}^{100} \frac{8n^3 - 2n^2 + 5}{n^4}$. Round to the nearest hundredth.

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CONTENTS

Applications and Problem Solving

46. Advertising A popular shoe manufacturer is planning to market a new style of tennis shoe in a city of 500,000 people. Using a prominent professional athlete as their spokesperson, the company's ad agency hopes to induce 35% of the people to buy the product. The ad agency estimates that these satisfied customers will convince 35% of 35% of 500,000 to buy a pair of shoes, and those will persuade 35% of 35% of 35% of 500,000, and so on.

a. Model this situation using sigma notation.

b. Find the total number of people that will buy the product as a result of the advertising campaign.

c. What percentage of the population is this?

d. What important assumption does the advertising agency make in proposing the figure found in part **b** to the shoe manufacturer?

47. Critical Thinking Solve each equation for x.

a. $\sum_{n=1}^{6} (x - 3n) = -3$

b. $\sum_{n=0}^{5} n(n - x) = 25$

48. Critical Thinking Determine whether each equation is *true* or *false*. Explain your answer.

a. $\sum_{k=3}^{7} 3^k + \sum_{b=7}^{9} 3^b = \sum_{a=3}^{9} 3^a$

b. $\sum_{n=2}^{8} (2n - 3) = \sum_{m=3}^{9} (2m - 5)$

c. $2 \sum_{n=3}^{7} n^2 = \sum_{n=3}^{7} 2n^2$

d. $\sum_{k=1}^{10} (5 + k) = \sum_{p=0}^{9} (4 + p)$

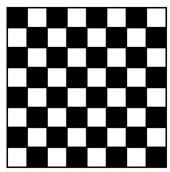
49. Word Play An *anagram* is a word or phrase that is made by rearranging the letters of another word or phrase. Consider the word "SILENT."

a. How many different arrangements of the letters in this word are possible? Write this number as a factorial. (*Hint:* First solve a simpler problem to see a pattern, such as how many different arrangements are there of just 2 letters? 3 letters?)

b. If a friend gives you a hint and tells you that an anagram of this word starts with "L," how many different arrangements still remain?

c. Your friend gives you one more hint. The last letter in the anagram is "N." Determine how many possible arrangements remain and then determine the anagram your friend is suggesting.

50. Chess A standard chess board contains 64 small black or white squares. These squares make up many other larger squares of various sizes.



a. How many 8×8 squares are there on a standard 8×8 chessboard? How many 7×7 squares?

b. Continue this list until you have accounted for all 8 sizes of squares.

c. Use sigma notation to represent the total number of squares found on an 8×8 chessboard. Then calculate this sum.

Mixed Review

51. Use the comparison test to determine whether the series $\frac{3}{3} + \frac{3}{4} + \frac{3}{5} + \frac{3}{6} + \cdots$ is convergent or divergent. *(Lesson 12-4)*

52. Chemistry A vacuum pump removes 20% of the air in a sealed jar on each stroke of its piston. The jar contains 21 liters of air before the pump starts. After how many strokes will only 42% of the air remain? *(Lesson 12-3)*

Extra Practice See p. A49.

CONTENTS

53. Find the first four terms of the geometric sequence for which $a_5 = 32\sqrt{2}$ and $r = -\sqrt{2}$. *(Lesson 12-2)*

54. Evaluate $\log_{10} 0.001$. *(Lesson 11-4)*

55. Write the standard form of the equation of the circle that passes through points at (0, 9), (−7, 2), and (0, −5). *(Lesson 10-2)*

56. Simplify $\left(\sqrt{2} + \boldsymbol{i}\right)\left(4\sqrt{2} + \boldsymbol{i}\right)$. *(Lesson 9-5)*

57. **Sports** Find the initial vertical and horizontal velocities of a javelin thrown with an initial velocity of 59 feet per second at an angle of 63° with the horizontal. *(Lesson 8-7)*

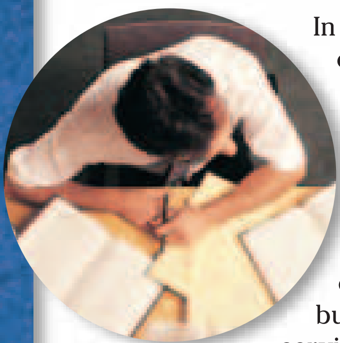
58. Find the equation of the line that bisects the obtuse angle formed by the graphs of $2x - 3y + 9 = 0$ and $x + 4y + 4 = 0$. *(Lesson 7-7)*

59. **SAT/ACT Practice** If $\frac{5 + m}{9 + m} = \frac{2}{3}$, then m = ?

A 8 **B** 6 **C** 5 **D** 3 **E** 2

CAREER CHOICES

Operations Research Analyst



In the changing economy of today, it is difficult to start and maintain a successful business. A business operator needs to be sure that the income from the business exceeds the expenses. Sometimes, businesses need the services of an operations research analyst. If you enjoy mathematics and solving tough problems, then you may want to consider a career as an operations research analyst.

In this occupation, you would gather many types of data about a business and analyze that data using mathematics and statistics. Examples of ways you might assist a business are: help a retail store determine the best store layout, help a bank in processing deposits more efficiently, or help a business set prices. Most operations research analysts work for private industry, private consulting firms, or the government.

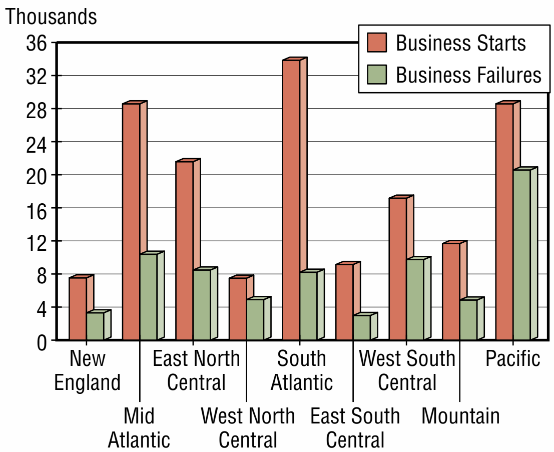
Career Overview

Degree Preferred:
bachelor's degree in applied mathematics

Related Courses:
mathematics, statistics, computer science, English

Outlook:
faster than average through the year 2006

1997 Business Starts and Failures



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