2-2 Solving Systems of Equations in Three Variables

OBJECTIVE

- Solve systems of equations involving three variables algebraically.

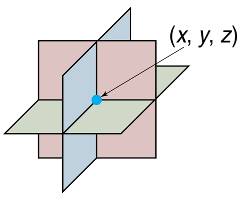
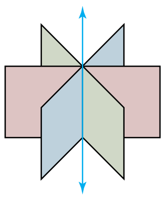
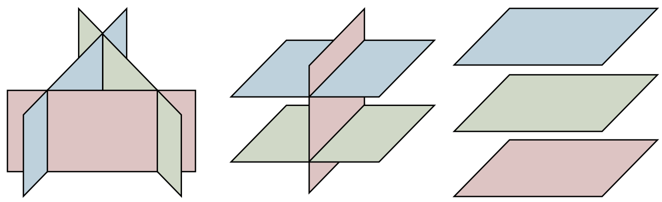
Real World Application

SPORTS In 1998, Cynthia Cooper of the WNBA Houston Comets basketball team was named Team Sportswoman of the Year by the Women's Sports Foundation. Cooper scored 680 points in the 1998 season by hitting 413 of her 1-point, 2-point, and 3-point attempts. She made 40% of her 160 3-point field goal attempts. How many 1-, 2-, and 3-point baskets did Ms. Cooper complete? *This problem will be solved in Example 3.*

You will learn more about graphing in three-dimensional space in Chapter 8.

This situation can be described by a system of three equations in three variables. You used graphing to solve a system of equations in two variables. For a system of equations in three variables, the graph of each equation is a plane in space rather than a line. The three planes can appear in various configurations. This makes solving a system of equations in three variables by graphing rather difficult. However, the pictorial representations provide information about the types of solutions that are possible. Some of them are shown below.

Systems of Equations in Three Variables

Unique Solution	Infinite Solutions	No Solution
		
The three planes intersect at one point.	The three planes intersect in a line.	The three planes have no points in common.

You can solve systems of three equations more efficiently than graphing by using the same algebraic techniques that you used to solve systems of two equations.

Example 1 **Solve the system of equations by elimination.**

$$x - 2y + z = 15$$
$$2x + 3y - 3z = 1$$
$$4x + 10y - 5z = -3$$

One way to solve a system of three equations is to choose pairs of equations and then eliminate one of the variables. Because the coefficient of x is 1 in the first equation, it is a good choice for eliminating x from the second and third equations.

(continued on the next page)

To eliminate x using the first and second equations, multiply each side of the first equation by -2.

$$-2(x - 2y + z) = -2(15)$$
$$-2x + 4y - 2z = -30$$

Then add that result to the second equation.

$$\begin{aligned} -2x + 4y - 2z &= -30 \\ 2x + 3y - 3z &= 1 \\ \hline 7y - 5z &= -29 \end{aligned}$$

To eliminate x using the first and third equations, multiply each side of the first equation by -4.

$$-4(x - 2y + z) = -4(15)$$
$$-4x + 8y - 4z = -60$$

Then add that result to the third equation .

$$\begin{aligned} -4x + 8y - 4z &= -60 \\ 4x + 10y - 5z &= -3 \\ \hline 18y - 9z &= -63 \end{aligned}$$

Now you have two linear equations in two variables. Solve this system. Eliminate z by multiplying each side of the first equation by -9 and each side of the second equation by 5. Then add the two equations.

$$\begin{aligned} -9(7y - 5z) &= -9(-29) \\ 5(18y - 9z) &= 5(-63) \end{aligned} \quad \Rightarrow \quad \begin{aligned} -63y + 45z &= 261 \\ 90y - 45z &= -315 \\ \hline 27y &= -54 \\ y &= -2 \end{aligned}$$

The value of y is -2.

By substituting the value of y into one of the equations in two variables, we can solve for the value of z.

$$7y - 5z = -29$$
$$7(-2) - 5z = -29 \quad y = -2$$
$$z = 3 \quad \text{The value of } z \text{ is 3.}$$

Finally, use one of the original equations to find the value of x.

$$x - 2y + z = 15$$
$$x - 2(-2) + 3 = 15 \quad y = -2, z = 3$$
$$x = 8 \quad \text{The value of } x \text{ is 8.}$$

The solution is $x = 8$, $y = -2$, and $z = 3$. This can be written as the **ordered triple** $(8, -2, 3)$. *Check by substituting the values into each of the original equations.*

The substitution method of solving systems of equations also works with systems of three equations.

Example 2 **Solve the system of equations by substitution.**

$$\mathbf{4x = -8z}$$
$$\mathbf{3x - 2y + z = 0}$$
$$\mathbf{-2x + y - z = -1}$$

You can easily solve the first equation for x.

$$4x = -8z$$
$$x = -2z \quad \textit{Divide each side by 4.}$$

Then substitute $-2z$ for x in each of the other two equations. Simplify each equation.

$3x - 2y + z = 0$
$3(-2z) - 2y + z = 0$ *$x = -2z$*
$-2y - 5z = 0$

$-2x + y - z = -1$
$-2(-2z) + y - z = -1$ *$x = -2z$*
$y + 3z = -1$

Solve $y + 3z = -1$ for y. $y + 3z = -1$
$y = -1 - 3z$ *Subtract $3z$ from each side.*

Substitute $-1 - 3z$ for y in $-2y - 5z = 0$. Simplify.

$-2y - 5z = 0$
$-2(-1 - 3z) - 5z = 0$ *$y = -1 - 3z$*
$z = -2$

Now, find the values of y and x. Use $y = -1 - 3z$ and $x = -2z$. Replace z with -2.

$y = -1 - 3z$
$y = -1 - 3(-2)$ *$z = -2$*
$y = 5$

$x = -2z$
$x = -2(-2)$ *$z = -2$*
$x = 4$

The solution is $x = 4$, $y = 5$, *and* $z = -2$. *Check each value in the original system.*

Many real-world situations can be represented by systems of three equations.

Example 3 Real World Application

SPORTS Refer to the application at the beginning of the lesson. Find the number of 1point free throws, 2-point field goals, and 3-point field goals Cynthia Cooper scored in the 1998 season.

Write a system of equations. Define the variables as follows.

x = the number of 1-point free throws

y = the number of 2-point field goals

z = the number of 3-point field goals

The system is:

$x + 2y + 3z = 680$ *total number of points*

$x + y + z = 413$ *total number of baskets*

$\frac{z}{160} = 0.40$ *percent completion*

The third equation is a simple linear equation. Solve for z.

$\frac{z}{160} = 0.40$, so $z = 160(0.40)$ or 64.

Now substitute 64 for z to make a system of two equations.

$x + 2y + 3z = 680$
$x + 2y + 3(64) = 680$ *$z = 64$*
$x + 2y = 488$

$x + y + z = 413$
$x + y + 64 = 413$ *$z = 64$*
$x + y = 349$

Solve $x + y = 349$ for x. Then substitute that value for x in $x + 2y = 488$ and solve for y.

$x + y = 349$
$x = 349 - y$

$x + 2y = 488$
$(349 - y) + 2y = 488$ *$x = 349 - y$*
$y = 139$

(continued on the next page)

Solve for x.
$x = 349 - y$
$x = 349 - 139 \quad y = 139$
$x = 210$

In 1998, Ms. Cooper made 210 1-point free throws, 139 2-point field goals, and 64 3-point field goals. *Check your answer in the original problem.*

CHECK FOR UNDERSTANDING

Communicating Mathematics

Read and study the lesson to answer each question.

1. **Compare and contrast** solving a system of three equations to solving a system of two equations.
2. **Describe** what you think would happen if two of the three equations in a system were consistent and dependent. Give an example.
3. **Write** an explanation of how to solve a system of three equations using the elimination method.

Guided Practice

Solve each system of equations.

4. $4x + 2y + z = 7$
   $2x + 2y - 4z = -4$
   $x + 3y - 2z = -8$

5. $x - y - z = 7$
   $-x + 2y - 3z = -12$
   $3x - 2y + 7z = 30$

6. $2x - 2y + 3z = 6$
   $2x - 3y + 7z = -1$
   $4x - 3y + 2z = 0$

7. **Physics** The height of an object that is thrown upward with a constant acceleration of a feet per second per second is given by the equation $s = \frac{1}{2}at^2 + v_0t + s_0$. The height is s feet, t represents the time in seconds, v_0 is the initial velocity in feet per second, and s_0 is the initial height in feet. Find the acceleration, the initial velocity, and the initial height if the height at 1 second is 75 feet, the height at 2.5 seconds is 75 feet, and the height at 4 seconds is 3 feet.

EXERCISES

Practice

Solve each system of equations.

8. $x + 2y + 3z = 5$
   $3x + 2y - 2z = -13$
   $5x + 3y - z = -11$

9. $7x + 5y + z = 0$
   $-x + 3y + 2z = 16$
   $x - 6y - z = -18$

10. $x - 3z = 7$
    $2x + y - 2z = 11$
    $-x - 2y + 9z = 13$

11. $3x - 5y + z = 9$
    $x - 3y - 2z = -8$
    $5x - 6y + 3z = 15$

12. $8x - z = 4$
    $y + z = 5$
    $11x + y = 15$

13. $4x - 3y + 2z = 12$
    $x + y - z = 3$
    $-2x - 2y + 2z = 5$

14. $36x - 15y + 50z = -10$
    $2x + 25y = 40$
    $54x - 5y + 30z = -160$

15. $-x - 3y + z = 54$
    $4x + 2y - 3z = -32$
    $2y + 8z = 78$

16. $1.8x - z = 0.7$
    $1.2y + z = -0.7$
    $1.5x - 3y = 3$

17. If possible, find the solution of $y = x + 2z$, $z = -1 - 2x$, and $x = y - 14$.

18. What is the solution of $\frac{1}{8}x - \frac{2}{3}y + \frac{5}{6}z = -8$, $\frac{3}{4}x + \frac{1}{6}y - \frac{1}{3}z = -12$, and $\frac{3}{16}x - \frac{5}{8}y - \frac{7}{12}z = -25$? If there is no solution, write *impossible*.

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Applications and Problem Solving

19. Finance Ana Colón asks her broker to divide her 401K investment of \$2000 among the International Fund, the Fixed Assets Fund, and company stock. She decides that her investment in the International Fund should be twice her investment in company stock. During the first quarter, the International Fund earns 4.5%, the Fixed Assets Fund earns 2.6%, and the company stock falls 0.2%. At the end of the first quarter, Ms. Colón receives a statement indicating a return of \$58 on her investment. How did the broker divide Ms. Colón's initial investment?

20. Critical Thinking Write a system of equations that fits each description.

a. The system has a solution of $x = -5$, $y = 9$, $z = 11$.

b. There is no solution to the system.

c. The system has an infinite number of solutions.

21. Physics Each year the Punkin' Chunkin' contest is held in Lewes, Delaware. The object of the contest is to propel an 8- to 10-pound pumpkin as far as possible. Steve Young of Hopewell, Illinois, set the 1998 record of 4026.32 feet. Suppose you build a machine that fires the pumpkin so that it is at a height of 124 feet after 1 second, the height at 3 seconds is 272 feet, and the height at 8 seconds is 82 feet. Refer to the formula in Exercise 7 to find the acceleration, the initial velocity, and the initial height of the pumpkin.

22. Critical Thinking Suppose you are using elimination to solve a system of equations.

a. How do you know that a system has no solution?

b. How do you know when it has an infinite number of solutions?

23. Number Theory Find all of the ordered triples (x, y, z) such that when any one of the numbers is added to the product of the other two, the result is 2.

Mixed Review

24. Solve the system of equations, $3x + 4y = 375$ and $5x + 2y = 345$. *(Lesson 2-1)*

25. Graph $y \leq -\frac{1}{3}x + 2$. *(Lesson 1-8)*

26. Show that points with coordinates $(-1, 3)$, $(3, 6)$, $(6, 2)$, and $(2, -1)$ are the vertices of a square. *(Lesson 1-5)*

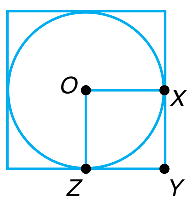
27. Manufacturing It costs ABC Corporation \$3000 to produce 20 of a particular model of color television and \$5000 to produce 60 of that model. *(Lesson 1-4)*

a. Write an equation to represent the cost function.

b. Determine the fixed cost and variable cost per unit.

c. Sketch the graph of the cost function.

28. SAT/ACT Practice In the figure, the area of square $OXYZ$ is 2. What is the area of the circle?



A $\frac{\pi}{4}$ **B** $\pi\sqrt{2}$ **C** 2π

D 4π **E** 8π

Extra Practice See p. A28.

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