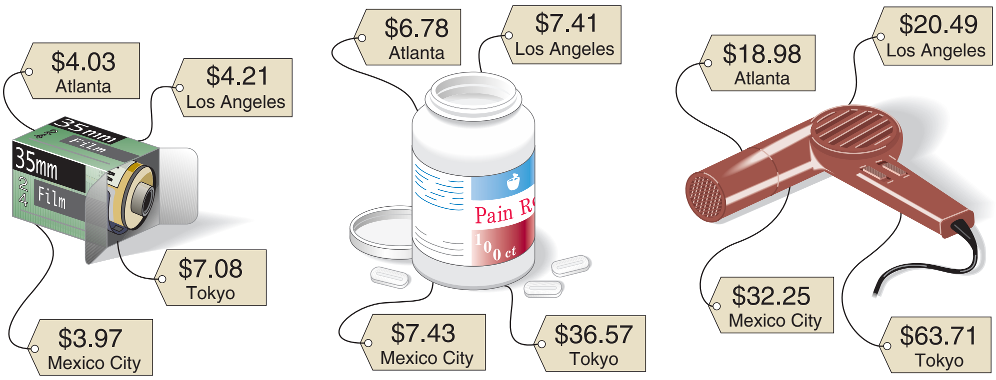
2-3 Modeling Real-World Data with Matrices

OBJECTIVES

- Model data using matrices.
- Add, subtract, and multiply matrices.

TRAVEL Did you ever go on a vacation and realize that you forgot to pack something that you needed? Sometimes purchasing those items while traveling can be expensive. The average cost of some items bought in various cities is given below.



Source: Runzheimer International

Data like these can be displayed or modeled using a matrix. *A problem related to this will be solved in Example 1.*

The plural of matrix is matrices.

A **matrix** is a rectangular array of terms called **elements.** The elements of a matrix are arranged in rows and columns and are usually enclosed by brackets. A matrix with m rows and n columns is an $\boldsymbol{m \times n}$ **matrix** (read "m by n"). The **dimensions** of the matrix are m and n. Matrices can have various dimensions and can contain any type of numbers or other information.

2 × 2 matrix

$$\begin{bmatrix} -\frac{3}{5} & \frac{1}{2} \\ 3 & -\frac{3}{4} \end{bmatrix}$$

2 × 5 matrix

$$\begin{bmatrix} 0.2 & 3.4 & -1.1 & 2.5 & 6.7 \\ 3.4 & -3.4 & -22 & 0.5 & 7.2 \end{bmatrix}$$

3 × 1 matrix

$$\begin{bmatrix} -2 \\ 3 \\ 11 \end{bmatrix}$$

The element 3 is in row 2, column 1.

Special names are given to certain matrices. A matrix that has only one row is called a **row matrix,** and a matrix that has only one column is called a **column matrix.** A **square matrix** has the same number of rows as columns. Sometimes square matrices are called matrices of ***n*th order,** where n is the number of rows and columns. The elements of an $m \times n$ matrix can be represented using double subscript notation; that is, a_{24} would name the element in the second row and fourth column.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

a_{ij} is the element in the ith row and the jth column.

Example 1 **TRAVEL** **Refer to the application at the beginning of the lesson.**

a. Use a matrix to represent the data.

b. Use a symbol to represent the price of pain reliever in Mexico City.

a. To represent data using a matrix, choose which category will be represented by the columns and which will be represented by the rows. Let's use the columns to represent the prices in each city and the rows to represent the prices of each item. Then write each data piece as you would if you were placing the data in a table.

	Atlanta	Los Angeles	Mexico City	Tokyo
film (24 exp.)	\$4.03	\$4.21	\$3.97	\$7.08
pain reliever (100 ct)	\$6.78	\$7.41	\$7.43	\$36.57
blow dryer	\$18.98	\$20.49	\$32.25	\$63.71

Notice that the category names appear outside of the matrix.

b. The price of pain reliever in Mexico City is found in the row 2, column 3 of the matrix. This element is represented by the symbol a_{23}.

Just as with numbers or algebraic expressions, matrices are equal under certain conditions.

Equal Matrices — Two matrices are equal if and only if they have the same dimensions and are identical, element by element.

Example 2 **Find the values of x and y for which the matrix equation $\begin{bmatrix} y \\ x \end{bmatrix} = \begin{bmatrix} 2x - 6 \\ 2y \end{bmatrix}$ is true.**

Since the corresponding elements are equal, we can express the equality of the matrices as two equations.

$y = 2x - 6$
$x = 2y$

Solve the system of equations by using substitution.

$y = 2x - 6$
$y = 2(2y) - 6$ *Substitute 2y for x.*
$y = 2$ *Solve for y.*

$x = 2(2)$ *Substitute 2 for y in the second equation to find x.*
$x = 4$

The matrices are equal if $x = 4$ and $y = 2$. *Check by substituting into the matrices.*

Matrices are usually named using capital letters. The sum of two matrices, $A + B$, exists only if the two matrices have the same dimensions. The ijth element of $A + B$ is $a_{ij} + b_{ij}$.

Addition of Matrices — The sum of two $m \times n$ matrices is an $m \times n$ matrix in which the elements are the sum of the corresponding elements of the given matrices.

Example 3 Find $A + B$ if $A = \begin{bmatrix} -2 & 0 & 1 \\ 0 & 5 & -8 \end{bmatrix}$ and $B = \begin{bmatrix} -6 & 7 & -1 \\ 4 & -3 & 10 \end{bmatrix}$.

$$A + B = \begin{bmatrix} -2 + (-6) & 0 + 7 & 1 + (-1) \\ 0 + 4 & 5 + (-3) & -8 + 10 \end{bmatrix}$$

$$= \begin{bmatrix} -8 & 7 & 0 \\ 4 & 2 & 2 \end{bmatrix}$$

Graphing Calculator Appendix

For keystroke instruction on entering matrices and performing operations on them, see pages A16-A17.

You know that 0 is the additive identity for real numbers because $a + 0 = a$. Matrices also have additive identities. For every matrix A, another matrix can be found so that their sum is A. For example,

$$\text{if } A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \text{ then } \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}.$$

The matrix $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is called a **zero matrix.** The $m \times n$ zero matrix is the **additive identity matrix** for any $m \times n$ matrix.

You also know that for any number a, there is a number $-a$, called the additive inverse of a, such that $a + (-a) = 0$. Matrices also have additive inverses. If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, then the matrix that must be added to A in order to have a sum of a zero matrix is $\begin{bmatrix} -a_{11} & -a_{12} \\ -a_{21} & -a_{22} \end{bmatrix}$ or $-A$. Therefore, $-A$ is the additive inverse of A. The additive inverse is used when you subtract matrices.

Subtraction of Matrices	The difference $A - B$ of two $m \times n$ matrices is equal to the sum $A + (-B)$, where $-B$ represents the additive inverse of B.

Example 4 Find $C - D$ if $C = \begin{bmatrix} 9 & 4 \\ -1 & 3 \\ 0 & -4 \end{bmatrix}$ and $D = \begin{bmatrix} 8 & -2 \\ -6 & 1 \\ 5 & -5 \end{bmatrix}$.

$$C - D = C + (-D)$$

$$= \begin{bmatrix} 9 & 4 \\ -1 & 3 \\ 0 & -4 \end{bmatrix} + \begin{bmatrix} -8 & 2 \\ 6 & -1 \\ -5 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 9 + (-8) & 4 + 2 \\ -1 + 6 & 3 + (-1) \\ 0 + (-5) & -4 + 5 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 6 \\ 5 & 2 \\ -5 & 1 \end{bmatrix}$$

You can multiply a matrix by a number; when you do, the number is called a **scalar.** The product of a scalar and a matrix A is defined as follows.

Scalar Product	The product of a scalar k and an $m \times n$ matrix A is an $m \times n$ matrix denoted by kA. Each element of kA equals k times the corresponding element of A.

Example 5 If $A = \begin{bmatrix} -4 & 1 & -1 \\ 3 & 7 & 0 \\ -3 & -1 & 8 \end{bmatrix}$, find $3A$.

$$3\begin{bmatrix} -4 & 1 & -1 \\ 3 & 7 & 0 \\ -3 & -1 & 8 \end{bmatrix} = \begin{bmatrix} 3(-4) & 3(1) & 3(-1) \\ 3(3) & 3(7) & 3(0) \\ 3(-3) & 3(-1) & 3(8) \end{bmatrix} \quad \textit{Multiply each element by 3.}$$

$$= \begin{bmatrix} -12 & 3 & -3 \\ 9 & 21 & 0 \\ -9 & -3 & 24 \end{bmatrix}$$

You can also multiply a matrix by a matrix. For matrices A and B, you can find AB if the number of columns in A is the same as the number of rows in B.

$$\begin{bmatrix} 3 & -8 & 1 \\ 1 & 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} 0 & 2 & 0 & 1 \\ -4 & 0 & -2 & 1 \\ 1 & -3 & -1 & 6 \end{bmatrix} \qquad \begin{bmatrix} 5 & 3 & 1 & 0 \\ 6 & 0 & 2 & -3 \\ -5 & 3 & 2 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 12 & 9 \\ 0 & 0 & -4 & -8 \\ -2 & 3 & 4 & 3 \end{bmatrix}$$

2 × 3　*3 × 4*　　*3 × 4*　*3 × 4*

Since 3 = 3, multiplication is possible.　*Since 4 ≠ 3, multiplication is not possible.*

The product of two matrices is found by multiplying columns and rows. Suppose $A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$ and $X = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix}$. Each element of matrix AX is the product of one row of matrix A and one column of matrix X.

$$AsX = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \cdot \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix} = \begin{bmatrix} a_1x_1 + b_1x_2 & a_1y_1 + b_1y_2 \\ a_2x_1 + b_2x_2 & a_2y_2 + b_2y_2 \end{bmatrix}$$

In general, the product of two matrices is defined as follows.

Product of Two Matrices

The product of an $m \times n$ matrix A and an $n \times r$ matrix B is an $m \times r$ matrix AB. The ijth element of AB is the sum of the products of the corresponding elements in the ith row of A and the jth column of B.

Example 6 Use matrices $A = \begin{bmatrix} 7 & 0 \\ 5 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -3 & 6 \\ 5 & 4 & -2 \end{bmatrix}$, and $C = \begin{bmatrix} 6 & -1 & 4 \\ 2 & -2 & -1 \end{bmatrix}$ to find each product.

a. AB

$$AB = \begin{bmatrix} 7 & 0 \\ 5 & 3 \end{bmatrix} \cdot \begin{bmatrix} 3 & -3 & 6 \\ 5 & 4 & -2 \end{bmatrix}$$

$$AB = \begin{bmatrix} 7(3) + 0(5) & 7(-3) + 0(4) & 7(6) + 0(-2) \\ 5(3) + 3(5) & 5(-3) + 3(4) & 5(6) + 3(-2) \end{bmatrix} \text{ or } \begin{bmatrix} 21 & -21 & 42 \\ 30 & -3 & 24 \end{bmatrix}$$

b. BC

B is a 2×3 matrix and C is a 2×3 matrix. Since B does not have the same number of columns as C has rows, the product BC does not exist. BC is undefined.

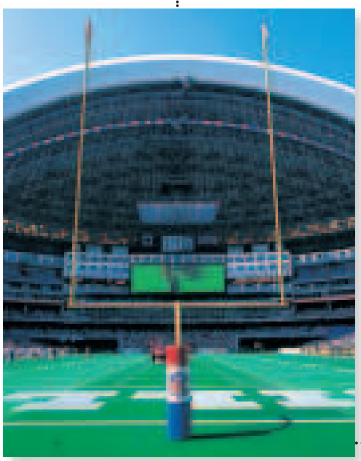
Example 7 Real World Application

SPORTS **In football, a player scores 6 points for a touchdown (TD), 3 points for kicking a field goal (FG), and 1 point for kicking the extra point after a touchdown (PAT). The chart lists the records of the top five all-time professional football scorers (as of the end of the 1997 season). Use matrix multiplication to find the number of points each player scored.**

Scorer	TD	FG	PAT
George Blanda	9	335	943
Nick Lowery	0	383	562
Jan Stenerud	0	373	580
Gary Anderson	0	385	526
Morten Andersen	0	378	507

Source: *The World Almanac and Book of Facts*, 1999

Write the scorer information as a 5 × 3 matrix and the points per play as a 3 × 1 matrix. Then multiply the matrices.



$$\begin{array}{l} \text{Blanda} \\ \text{Lowery} \\ \text{Stenerud} \\ \text{Anderson} \\ \text{Andersen} \end{array} \overset{\begin{array}{ccc}\text{TD} & \text{FG} & \text{PAT}\end{array}}{\begin{bmatrix} 9 & 335 & 943 \\ 0 & 383 & 562 \\ 0 & 373 & 580 \\ 0 & 385 & 526 \\ 0 & 378 & 507 \end{bmatrix}} \cdot \begin{array}{l} \text{TD} \\ \text{FG} \\ \text{PAT} \end{array} \overset{\text{pts}}{\begin{bmatrix} 6 \\ 3 \\ 1 \end{bmatrix}} =$$

$$\begin{array}{l} \text{Blanda} \\ \text{Lowery} \\ \text{Stenerud} \\ \text{Anderson} \\ \text{Andersen} \end{array} \overset{\text{pts}}{\begin{bmatrix} 9(6) + 335(3) + 943(1) \\ 0(6) + 383(3) + 562(1) \\ 0(6) + 373(3) + 580(1) \\ 0(6) + 385(3) + 526(1) \\ 0(6) + 378(3) + 507(1) \end{bmatrix}} = \begin{array}{l} \text{Blanda} \\ \text{Lowery} \\ \text{Stenerud} \\ \text{Anderson} \\ \text{Andersen} \end{array} \overset{\text{pts}}{\begin{bmatrix} 2002 \\ 1711 \\ 1699 \\ 1681 \\ 1641 \end{bmatrix}}$$

CHECK FOR UNDERSTANDING

Communicating Mathematics

Read and study the lesson to answer each question.

1. **Write** a matrix other than the one given in Example 1 to represent the data on travel prices.
2. **Tell** the dimensions of the matrix $\begin{bmatrix} 4 & 0 & -2 & 4 \\ -1 & 3 & -1 & 5 \end{bmatrix}$.
3. **Explain** how you determine whether the sum of two matrices exists.
4. **You Decide** Sarah says that $\begin{bmatrix} 3 & 2 & 3 \\ -4 & 2 & 0 \\ 0 & -1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$ is a third-order matrix. Anthony disagrees. Who is correct and why?

Guided Practice

Find the values of *x* and *y* for which each matrix equation is true.

5. $\begin{bmatrix} 2y \\ x \end{bmatrix} = \begin{bmatrix} x - 3 \\ y + 5 \end{bmatrix}$
6. $[18 \quad 24] = [4x - y \quad 12y]$
7. $[16 \quad 0 \quad 2x] = [4x \quad y \quad 8 - y]$

Use matrices X, Y, and Z to find each of the following. If the matrix does not exist, write *impossible*.

$X = \begin{bmatrix} 4 & 1 \\ -2 & 6 \end{bmatrix}$ $\quad Y = \begin{bmatrix} 0 & -3 \end{bmatrix}$ $\quad Z = \begin{bmatrix} -1 & 3 \\ 0 & -2 \end{bmatrix}$

8. $X + Z$ **9.** $Z - Y$ **10.** $Z - X$ **11.** $4X$ **12.** ZY **13.** YX

14. Advertising A newspaper surveyed companies on the annual amount of money spent on television commercials and the estimated number of people who remember seeing those commercials each week. A soft-drink manufacturer spends $40.1 million a year and estimates 78.6 million people remember the commercials. For a package-delivery service, the budget is $22.9 million for 21.9 million people. A telecommunications company reaches 88.9 million people by spending a whopping $154.9 million. Use a matrix to represent this data.

EXERCISES

Practice

Find the values of x and y for which each matrix equation is true.

15. $\begin{bmatrix} y \\ x \end{bmatrix} = \begin{bmatrix} 2x - 1 \\ y - 5 \end{bmatrix}$

16. $\begin{bmatrix} 9 & 13 \end{bmatrix} = \begin{bmatrix} x + 2y & 4x + 1 \end{bmatrix}$

17. $\begin{bmatrix} 4x \\ 5 \end{bmatrix} = \begin{bmatrix} 15 + x \\ 2y \end{bmatrix}$

18. $\begin{bmatrix} x & y \end{bmatrix} = \begin{bmatrix} 2y & 2x - 6 \end{bmatrix}$

19. $\begin{bmatrix} 27 \\ 8 \end{bmatrix} = \begin{bmatrix} 3y \\ 5x - 3y \end{bmatrix}$

20. $\begin{bmatrix} 4x - 3y \\ x + y \end{bmatrix} = \begin{bmatrix} 11 \\ 1 \end{bmatrix}$

21. $\begin{bmatrix} 2x & y & -y \end{bmatrix} = \begin{bmatrix} -10 & 3x & 15 \end{bmatrix}$

22. $\begin{bmatrix} -12 \\ 2 \\ 12y \end{bmatrix} = \begin{bmatrix} 6x \\ y + 1 \\ 10 - x \end{bmatrix}$

23. $\begin{bmatrix} x + y & 3 \\ y & 6 \end{bmatrix} = \begin{bmatrix} 0 & 2y - x \\ y^2 & 4 - 2x \end{bmatrix}$

24. $\begin{bmatrix} x^2 + 1 & 5 - y \\ x + y & y - 4 \end{bmatrix} = \begin{bmatrix} 2 & x \\ 5 & 2 \end{bmatrix}$

25. Find the values of x, y, and z for $3\begin{bmatrix} x & y - 1 \\ 4 & 3z \end{bmatrix} = \begin{bmatrix} 15 & 6 \\ 6z & 3x + y \end{bmatrix}$.

26. Solve $-2\begin{bmatrix} w + 5 & x - z \\ 3y & 8 \end{bmatrix} = \begin{bmatrix} -16 & -4 \\ 6 & 2x + 8z \end{bmatrix}$ for w, x, y, and z.

Use matrices A, B, C, D, E, and F to find each of the following. If the matrix does not exist, write *impossible*.

$A = \begin{bmatrix} 5 & 7 \\ -6 & 1 \end{bmatrix}$ $\quad B = \begin{bmatrix} 3 & 5 \\ -1 & 8 \end{bmatrix}$ $\quad C = \begin{bmatrix} 4 & -2 & 3 \\ 5 & 0 & -1 \\ 9 & 0 & 1 \end{bmatrix}$ $\quad D = \begin{bmatrix} 0 & 1 & 2 \\ -2 & 3 & 0 \\ 4 & 4 & -2 \end{bmatrix}$

$E = \begin{bmatrix} 8 & -4 & 2 \\ 3 & 1 & -5 \end{bmatrix}$ $\quad F = \begin{bmatrix} -6 & -1 & 0 \\ 1 & 4 & 0 \end{bmatrix}$

27. $A + B$ **28.** $A + C$ **29.** $D + B$ **30.** $D + C$ **31.** $B - A$

32. $C - D$ **33.** $4D$ **34.** $-2F$ **35.** $F - E$ **36.** $E - F$

37. $5A$ **38.** BA **39.** CF **40.** FC **41.** ED

42. AA **43.** $E + FD$ **44.** $-3AB$ **45.** $(BA)E$ **46.** $F - 2EC$

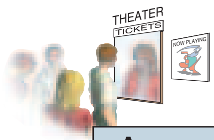
47. Find $3XY$ if $X = \begin{bmatrix} 2 & 4 \\ 8 & -4 \\ -2 & 6 \end{bmatrix}$ and $Y = \begin{bmatrix} 3 & -3 & 6 \\ 5 & 4 & -2 \end{bmatrix}$.

48. If $K = \begin{bmatrix} 1 & -7 \\ 3 & 2 \end{bmatrix}$ and $J = \begin{bmatrix} -4 & 5 \\ 1 & -1 \end{bmatrix}$, find $2K - 3J$.

Applications and Problem Solving

Real World Application

49. Entertainment How often do you go to the movies? The graph below shows the projected number of adults of different ages who attend movies at least once a month. Organize the information in a matrix.



Projected Movie Attendance

	Year		
Age group	**1996**	**2000**	**2006**
18 to 24	8485	8526	8695
25 to 34	10,102	9316	9078
35 to 44	8766	9039	8433
45 to 54	6045	6921	7900
55 to 64	2444	2741	3521
65 and older	2381	2440	2572

Source: American Demographics

interNET CONNECTION

Data Update For the latest National Endowment for the Arts survey, visit **www.amc.glencoe.com**

50. Music The National Endowment for the Arts exists to broaden public access to the arts. In 1992, it performed a study to find what types of arts were most popular and how they could attract more people. The matrices below represent their findings.

Percent of People Listening or Watching Performances

1982

	TV	Radio	Recording
Classical	25	20	22
Jazz	18	18	20
Opera	12	7	8
Musicals	21	4	8

1992

	TV	Radio	Recording
Classical	25	31	24
Jazz	21	28	21
Opera	12	9	7
Musicals	15	4	6

Source: National Endowment for the Arts

a. Find the difference in arts patronage from 1982 to 1992. Express your answer as a matrix.

b. Which areas saw the greatest increase and decrease in this time?

51. Critical Thinking Consider the matrix equation $\begin{bmatrix} -2 & 3 \\ 4 & -5 \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} -2 & 3 \\ 4 & -5 \end{bmatrix}$.

a. Find the values of a, b, c, and d to make the statement true.

b. If any matrix containing two columns were multiplied by the matrix containing a, b, c, and d, what would you expect the result to be? Explain.

52. **Finance** Investors choose different stocks to comprise a balanced portfolio. The matrix below shows the prices of one share of each of several stocks on the first business day of July, August, and September of 1998.

$$\begin{array}{r} \\ \text{Stock A} \\ \text{Stock B} \\ \text{Stock C} \\ \text{Stock D} \end{array} \begin{array}{c} \begin{array}{ccc} \textbf{July} & \textbf{August} & \textbf{September} \end{array} \\ \begin{bmatrix} \$33\frac{13}{16} & \$30\frac{15}{16} & \$27\frac{1}{4} \\ \$15\frac{1}{16} & \$13\frac{1}{4} & \$8\frac{3}{4} \\ \$54 & \$54 & \$46\frac{7}{16} \\ \$52\frac{1}{16} & \$44\frac{11}{16} & \$34\frac{3}{8} \end{bmatrix} \end{array}$$

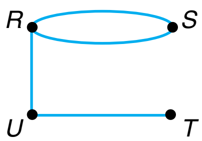
a. Mrs. Numkena owns 42 shares of stock A, 59 shares of stock B, 21 shares of stock C, and 18 shares of stock D. Write a row matrix to represent Mrs. Numkena's portfolio.

b. Use matrix multiplication to find the total value of Mrs. Numkena's portfolio for each month to the nearest cent.

53. **Critical Thinking** Study the matrix at the right. In which row and column will 2001 occur? Explain your reasoning.

$$\begin{bmatrix} 1 & 3 & 6 & 10 & 15 & \dots \\ 2 & 5 & 9 & 14 & 20 & \dots \\ 4 & 8 & 13 & 19 & 26 & \dots \\ 7 & 12 & 18 & 25 & 33 & \dots \\ 11 & 17 & 24 & 32 & 41 & \dots \\ 16 & 23 & 31 & 40 & 50 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

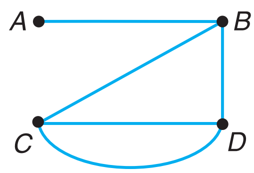
54. **Discrete Math** Airlines and other businesses often use *finite graphs* to represent their routes. A finite graph contains points called *nodes* and segments called *edges*. In a graph for an airline, each node represents a city, and each edge represents a route between the cities. Graphs can be represented by square matrices. The elements of the matrix are the numbers of edges between each pair of nodes. Study the graph and its matrix at the right.



$$\begin{array}{c} \\ R \\ S \\ T \\ U \end{array} \begin{array}{c} \begin{array}{cccc} R & S & T & U \end{array} \\ \begin{bmatrix} 0 & 2 & 0 & 1 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

a. Represent the graph with nodes *A, B, C,* and *D* at the right using a matrix.

b. Equivalent graphs have the same number of nodes and edges between corresponding nodes. Can different graphs be represented by the same matrix? Explain your answer.



Mixed Review

55. Solve the system $2x + 6y + 8z = 5$, $-2x + 9y + 12z = 5$, and $4x + 6y - 4z = 3$. *(Lesson 2-2)*

56. State whether the system $4x - 2y = 7$ and $-12x + 6y = -21$ is *consistent and independent, consistent and dependent,* or *inconsistent.* *(Lesson 2-1)*

57. Graph $-6 \le 3x - y \le 12$. *(Lesson 1-8)*

58. Graph $f(x) = |3x| + 2$. *(Lesson 1-7)*

59. Education Many educators believe that taking practice tests will help students succeed on actual tests. The table below shows data gathered about students studying for an algebra test. Use the data to write a prediction equation. *(Lesson 1-6)*

Practice Test Time (minutes)	15	75	60	45	90	60	30	120	10	120
Test scores (percents)	68	87	92	73	95	83	77	98	65	94

60. Write the slope-intercept form of the equation of the line through points at (1, 4) and (5, 7). *(Lesson 1-4)*

61. Find the zero of $f(x) = 5x - 3$ *(Lesson 1-3)*

62. Find $[f \cdot g](x)$ if $f(x) = \frac{2}{5}x$ and $g(x) = 40x - 10$. *(Lesson 1-2)*

63. Given $f(x) = 4 + 6x - x^3$, find $f(14)$. *(Lesson 1-1)*

64. SAT/ACT Practice If $\frac{2x - 3}{x} = \frac{3 - x}{2}$, which of the following could be a value of x?

A -3 **B** -1 **C** 37 **D** 5 **E** 15

GRAPHING CALCULATOR EXPLORATION

Remember the properties of real numbers:

Properties of Addition

Commutative $a + b = b + a$

Associative $(a + b) + c = a + (b + c)$

Properties of Multiplication

Commutative $ab = ba$

Associative $(ab)c = a(bc)$

Distributive Property

$a(b + c) = ab + ac$

Do these properties hold for operations with matrices?

TRY THESE

Use matrices A, B, and C to investigate each of the properties shown above.

$$A = \begin{bmatrix} -1 & 2 \\ 3 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 4 & 0 \\ -2 & -3 \end{bmatrix} \quad C = \begin{bmatrix} -2 & -3 \\ 4 & 2 \end{bmatrix}$$

WHAT DO YOU THINK?

1. Do these three matrices satisfy the properties of real numbers listed at the left? Explain.

2. Would these properties hold for any 2×2 matrices? Prove or disprove each statement below using variables as elements of each 2×2 matrix.

a. Addition of matrices is commutative.

b. Addition of matrices is associative.

c. Multiplication of matrices is commutative.

d. Multiplication of matrices is associative.

3. Which properties do you think hold for $n \times n$ matrices? Explain.

Extra Practice See p. A28.

CONTENTS

2-4A Transformation Matrices

OBJECTIVE
- Determine the effect of matrix multiplication on a vertex matrix.

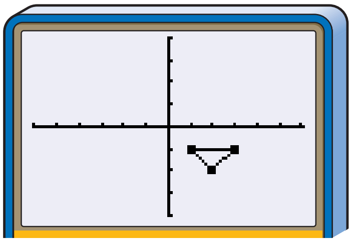
An Introduction to Lesson 2-4

The coordinates of a figure can be represented by a matrix with the x-coordinates in the first row and the y-coordinates in the second. When this matrix is multiplied by certain other matrices, the result is a new matrix that contains the coordinates of the vertices of a different figure. You can use **List** and **Matrix** operations on your calculator to visualize some of these multiplications.

TRY THESE

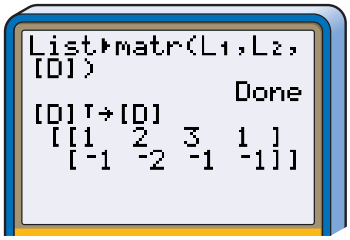
Step 1 Enter $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, and $C = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

Step 2 Graph the triangle LMN with $L(1, -1)$, $M(2, -2)$, and $N(3, -1)$ by using **STAT PLOT**. Enter the x-coordinates in **L1** and the y-coordinates in **L2**, repeating the first point to close the figure. In **STAT PLOT**, turn **Plot 1** on and select the connected graph. After graphing the figure, press ZOOM **5** to reflect a less distorted viewing window.



[−4.548 ..., 4.548 ...] scl: 1 by [−3, 3] scl: 1

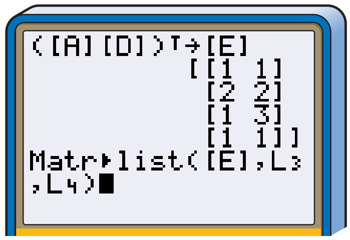
Step 3 To transfer the coordinates from the lists to a matrix, use the **9:List▶matr** command from the **MATH** submenu in the **MATRX** menu. Store this as matrix D. Matrix D has the x-coordinates in Column 1 and y-coordinates in Column 2. But a vertex matrix has the x-coordinates in *Row* 1 and the y-coordinates in *Row 2.* This switch can be easily done by using the **2:^{T}** (transpose) command found in the **MATH** submenu as shown in the screen.



The **List▶matr** *and* **Matr▶list** *commands transfer the data column for column. That is, the data in List 1 goes to Column 1 of the matrix and vice versa.*

Step 4 Multiply matrix D by matrix A. To graph the result we need to put the ordered pairs back into the **LIST** menu.

- This means we need to transpose AD first. Store as new matrix E.
- Use the **8:Matr▶list** command from the math menu to store values into **L3** and **L4**.



Step 5 Assign **Plot 2** as a connected graph of the **L3** and **L4** data and view the graph.

WHAT DO YOU THINK?

1. What is the relationship between the two plots?
2. Repeat Steps 4 and 5 replacing matrix A with matrix B. Compare the graphs.
3. Repeat Steps 4 and 5 replacing matrix A with matrix C. Compare the graphs.
4. Select a new figure and repeat this activity using each of the 2×2 matrices. Make a conjecture about these 2×2 matrices.