2-4 Modeling Motion with Matrices

OBJECTIVE

- Use matrices to determine the coordinates of polygons under a given transformation.

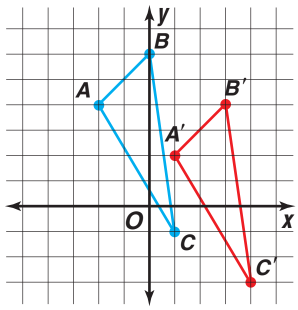
Real World Application

COMPUTER ANIMATION In 1995, animation took a giant step forward with the release of the first major motion picture to be created entirely on computers. Animators use computer software to create three-dimensional computer models of characters, props, and sets. These computer models describe the shape of the object as well as the motion controls that the animators use to create movement and expressions. The animation models are actually very large matrices.

Even though large matrices are used for computer animation, you can use a simple matrix to describe many of the motions called **transformations** that you learned about in geometry. Some of the transformations we will examine in this lesson are **translations** (slides), **reflections** (flips), **rotations** (turns), and **dilations** (enlargements or reductions).

An n-gon is a polygon with n sides.

A $2 \times n$ matrix can be used to express the vertices of an n-gon with the first row of elements representing the x-coordinates and the second row the y-coordinates of the vertices.



Triangle ABC can be represented by the following **vertex matrix.**

$$\begin{array}{r} \\ x\text{-coordinate} \\ y\text{-coordinate} \end{array} \begin{array}{c} \begin{matrix} A & B & C \end{matrix} \\ \begin{bmatrix} -2 & 0 & 1 \\ 4 & 6 & -1 \end{bmatrix} \end{array}$$

Triangle $A'B'C'$ is congruent to and has the same orientation as $\triangle ABC$, but is moved 3 units right and 2 units down from $\triangle ABC$'s location. The coordinates of $\triangle A'B'C'$ can be expressed as the following vertex matrix.

$$\begin{array}{r} \\ x\text{-coordinate} \\ y\text{-coordinate} \end{array} \begin{array}{c} \begin{matrix} A' & B' & C' \end{matrix} \\ \begin{bmatrix} 1 & 3 & 4 \\ 2 & 4 & -3 \end{bmatrix} \end{array}$$

Compare the two matrices. If you add $\begin{bmatrix} 3 & 3 & 3 \\ -2 & -2 & -2 \end{bmatrix}$ to the first matrix, you get the second matrix. Each 3 represents moving 3 units right for each x-coordinate. Likewise, each -2 represents moving 2 units down for each y-coordinate. This type of matrix is called a **translation matrix.** In this transformation, $\triangle ABC$ is the **pre-image,** and $\triangle A'B'C'$ is the **image** after the translation.

Example 1 **Suppose quadrilateral *ABCD* with vertices $A(-1, 1)$, $B(4, 0)$, $C(4, -5)$, and $D(-1, -3)$ is translated 2 units left and 4 units up.**

Note that the image under a translation is the same shape and size as the pre-image. The figures are congruent.

a. Represent the vertices of the quadrilateral as a matrix.

b. Write the translation matrix.

c. Use the translation matrix to find the vertices of *A′B′C′D′*, the translated image of the quadrilateral.

d. Graph quadrilateral *ABCD* and its image.

a. The matrix representing the coordinates of the vertices of quadrilateral *ABCD* will be a 2×4 matrix.

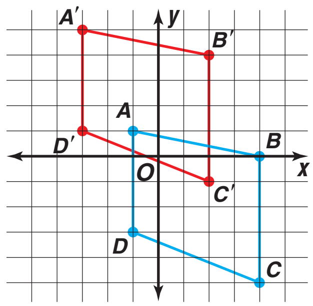
$$\begin{matrix} \text{x-coordinate} \\ \text{y-coordinate} \end{matrix} \overset{\begin{matrix} A & B & C & D \end{matrix}}{\begin{bmatrix} -1 & 4 & 4 & -1 \\ 1 & 0 & -5 & -3 \end{bmatrix}}$$

b. The translation matrix is $\begin{bmatrix} -2 & -2 & -2 & -2 \\ 4 & 4 & 4 & 4 \end{bmatrix}$.

c. Add the two matrices.

$$\begin{bmatrix} -1 & 4 & 4 & -1 \\ 1 & 0 & -5 & -3 \end{bmatrix} + \begin{bmatrix} -2 & -2 & -2 & -2 \\ 4 & 4 & 4 & 4 \end{bmatrix} = \overset{\begin{matrix} A' & B' & C' & D' \end{matrix}}{\begin{bmatrix} -3 & 2 & 2 & -3 \\ 5 & 4 & -1 & 1 \end{bmatrix}}$$

d. Graph the points represented by the resulting matrix.

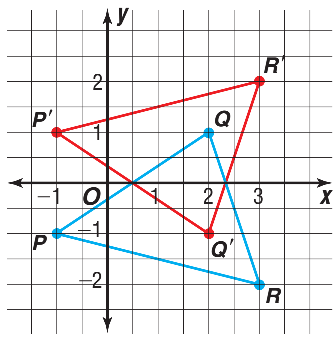


There are three lines over which figures are commonly reflected.

- the x-axis
- the y-axis, and
- the line $y = x$

The preimage and the image under a reflection are congruent.

In the figure at the right, $\triangle P'Q'R'$ is a reflection of $\triangle PQR$ over the x-axis. There is a 2×2 **reflection matrix** that, when multiplied by the vertex matrix of $\triangle PQR$, will yield the vertex matrix of $\triangle P'Q'R'$.



Let $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ represent the unknown square matrix.

Thus, $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} -1 & 2 & 3 \\ -1 & 1 & -2 \end{bmatrix} = \begin{bmatrix} -1 & 2 & 3 \\ 1 & -1 & 2 \end{bmatrix}$, or

$\begin{bmatrix} -a - b & 2a + b & 3a - 2b \\ -c - d & 2c + d & 3c - 2d \end{bmatrix} = \begin{bmatrix} -1 & 2 & 3 \\ 1 & -1 & 2 \end{bmatrix}$.

Since corresponding elements of equal matrices are equal, we can write equations to find the values of the variables. These equations form two systems.

$$\begin{aligned} -a - b &= -1 \\ 2a + b &= 2 \\ 3a - 2b &= 3 \end{aligned} \qquad \begin{aligned} -c - d &= 1 \\ 2c + d &= -1 \\ 3c - 2d &= 2 \end{aligned}$$

When you solve each system of equations, you will find that $a = 1$, $b = 0$, $c = 0$, and $d = -1$. Thus, the matrix that results in a reflection over the x-axis is $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. This matrix will work for any reflection over the x-axis.

The matrices for a reflection over the y-axis or the line $y = x$ can be found in a similar manner. These are summarized below.

Reflection Matrices

For a reflection over the:	Symbolized by:	Multiply the vertex matrix by:
x-axis	$R_{x\text{-axis}}$	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
y-axis	$R_{y\text{-axis}}$	$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$
line $y = x$	$R_{y = x}$	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Example 2 Real World Application

ANIMATION **To create an image that appears to be reflected in a mirror, an animator will use a matrix to reflect an image over the y-axis. Use a reflection matrix to find the coordinates of the vertices of a star reflected in a mirror (the y-axis) if the coordinates of the points connected to create the star are (−2, 4), (−3.5, 4), (−4, 5), (−4.5, 4), (−6, 4), (−5, 3), (−5, 1), (−4, 2), (−3, 1), and (−3, 3).**

First write the vertex matrix for the points used to define the star.

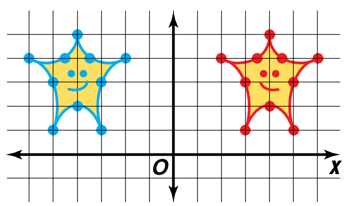
$$\begin{bmatrix} -2 & -3.5 & -4 & -4.5 & -6 & -5 & -5 & -4 & -3 & -3 \\ 4 & 4 & 5 & 4 & 4 & 3 & 1 & 2 & 1 & 3 \end{bmatrix}$$

Multiply by the y-axis reflection matrix.

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} -2 & -3.5 & -4 & -4.5 & -6 & -5 & -5 & -4 & -3 & -3 \\ 4 & 4 & 5 & 4 & 4 & 3 & 1 & 2 & 1 & 3 \end{bmatrix} =$$

$$\begin{bmatrix} 2 & 3.5 & 4 & 4.5 & 6 & 5 & 5 & 4 & 3 & 3 \\ 4 & 4 & 5 & 4 & 4 & 3 & 1 & 2 & 1 & 3 \end{bmatrix}$$

The vertices used to define the reflection are (2, 4), (3.5, 4), (4, 5), (4.5, 4), (6, 4), (5, 3), (5, 1), (4, 2), (3, 1), and (3, 3).



The preimage and the image under a rotation are congruent.

You may remember from geometry that a rotation of a figure on a coordinate plane can be achieved by a combination of reflections. For example, a 90° counterclockwise rotation can be found by first reflecting the image over the x-axis and then reflecting the reflected image over the line $y = x$. The **rotation matrix,** Rot_{90}, can be found by a composition of reflections. Since reflection matrices are applied using multiplication, the composition of two reflection matrices is a product. Remember that $[f \circ g](x)$ means that you find $g(x)$ first and then evaluate the result for $f(x)$. So, to define Rot_{90}, we use

$$Rot_{90} = R_{y=x} \circ R_{x\text{-axis}} \text{ or } Rot_{90} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \text{ or } \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Similarly, a rotation of 180° would be rotations of 90° twice or $Rot_{90} \circ Rot_{90}$. A rotation of 270° is a composite of Rot_{180} and Rot_{90}. The results of these composites are shown below.

Rotation Matrices		
For a counterclockwise rotation about the origin of:	**Symbolized by:**	**Multiply the vertex matrix by:**
90°	Rot_{90}	$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$
180°	Rot_{180}	$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$
270°	Rot_{270}	$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

Example 3 Real World Application

ANIMATION **Suppose a figure is animated to spin around a certain point. Numerous rotation images would be necessary to make a smooth movement image. If the image has key points at (1, 1), (−1, 4), (−2, 4), and, (−2, 3) and the rotation is about the origin, find the location of these points at the 90°, 180°, and 270° counterclockwise rotations.**

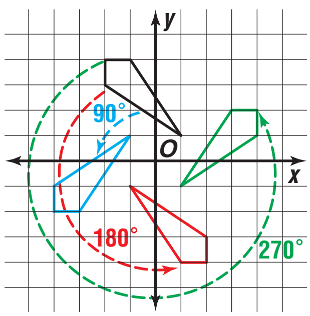
First write the vertex matrix. Then multiply it by each rotation matrix.

The vertex matrix is $\begin{bmatrix} 1 & -1 & -2 & -2 \\ 1 & 4 & 4 & 3 \end{bmatrix}$.

$$Rot_{90} \quad \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 & -2 & -2 \\ 1 & 4 & 4 & 3 \end{bmatrix} = \begin{bmatrix} -1 & -4 & -4 & -3 \\ 1 & -1 & -2 & -2 \end{bmatrix}$$

$$Rot_{180} \quad \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 & -2 & -2 \\ 1 & 4 & 4 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 2 & 2 \\ -1 & -4 & -4 & -3 \end{bmatrix}$$

$$Rot_{180} \quad \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 & -2 & -2 \\ 1 & 4 & 4 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 4 & 3 \\ -1 & 1 & 2 & 2 \end{bmatrix}$$



All of the transformations we have discussed have maintained the shape and size of the figure. However, a dilation changes the size of the figure. The dilated figure is similar to the original figure. Dilations using the origin as a center of projection can be achieved by multiplying the vertex matrix by the scale factor needed for the dilation. *All dilations in this lesson are with respect to the origin.*

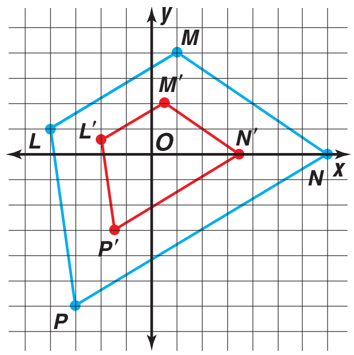
Example 4 **A trapezoid has vertices at $L(-4, 1)$, $M(1, 4)$, $N(7, 0)$, and $P(-3, -6)$. Find the coordinates of the dilated trapezoid $L'M'N'P'$ for a scale factor of 0.5. Describe the dilation.**

First write the coordinates of the vertices as a matrix. Then do a scalar multiplication using the scale factor.

$$0.5\begin{bmatrix} -4 & 1 & 7 & -3 \\ 1 & 4 & 0 & -6 \end{bmatrix} = \begin{bmatrix} -2 & 0.5 & 3.5 & -1.5 \\ 0.5 & 2 & 0 & -3 \end{bmatrix}$$

The vertices of the image are $L'(-2, 0.5)$, $M'(0.5, 2)$, $N'(3.5, 0)$, and $P'(-1.5, -3)$.

The image has sides that are half the length of the original figure.



CHECK FOR UNDERSTANDING

Communicating Mathematics

Read and study the lesson to answer each question.

1. **Name** all the transformations described in this lesson. Tell how the pre-image and image are related in each type of transformation.
2. **Explain** how 90°, 180°, and 270° counterclockwise rotations correspond to clockwise rotations.
3. *Math Journal* **Describe** a way that you can remember the elements of the reflection matrices if you forget where the 1s, −1s, and 0s belong.
4. **Match** each matrix with the phrase that best describes its type.

a. $\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}$

b. $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

c. $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

d. $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

(1) dilation of scale factor 2

(2) reflection over the y-axis

(3) reflection over the line $y = x$

(4) rotation of 90° counterclockwise about the origin

(5) rotation of 180° about the origin

(6) translation 1 unit left and 1 unit up

Guided Practice

Use matrices to perform each transformation. Then graph the pre-image and the image on the same coordinate grid.

5. Triangle JKL has vertices $J(-2, 5)$, $K(1, 3)$, and $L(0, -2)$. Use scalar multiplication to find the coordinates of the triangle after a dilation of scale factor 1.5.

6. Square $ABCD$ has vertices $A(-1, 3)$, $B(3, 3)$, $C(3, -1)$, and $D(-1, -1)$. Find the coordinates of the square after a translation of 1 unit left and 2 units down.

7. Square $ABCD$ has vertices at $(-1, 2)$, $(-4, 1)$, $(-3, -2)$, and $(0, -1)$. Find the image of the square after a reflection over the y-axis.

8. Triangle PQR is represented by the matrix $\begin{bmatrix} 3 & -1 & 1 \\ 2 & 4 & -2 \end{bmatrix}$. Find the image of the triangle after a rotation of 270° counterclockwise about the origin.

9. Find the image of $\triangle LMN$ after $Rot_{180} \circ R_{y\text{-axis}}$ if the vertices are $L(-6, 4)$, $M(-3, 2)$, and $N(-1, -2)$.

10. **Physics** The wind was blowing quite strongly when Jenny was baby-sitting. She was outside with the children, and they were throwing their large plastic ball up into the air. The wind blew the ball so that it landed approximately 3 feet east and 4 feet north of where it was thrown into the air.

   a. Make a drawing to demonstrate the original location of the ball and the translation of the ball to its landing spot.

   b. If $\begin{bmatrix} x \\ y \end{bmatrix}$ represents the original location of the ball, write a matrix that represents the location of the translated ball.

EXERCISES

Practice

Use scalar multiplication to determine the coordinates of the vertices of each dilated figure. Then graph the pre-image and the image on the same coordinate grid.

11. triangle with vertices $A(1, 1)$, $B(1, 4)$, and $C(5, 1)$; scale factor 3

12. triangle with vertices $X(0, 8)$, $Y(-5, 9)$, and $Z(-3, 2)$; scale factor $\frac{3}{4}$

13. quadrilateral $PQRS$ with vertex matrix $\begin{bmatrix} -3 & -2 & 1 & 4 \\ 0 & 2 & 3 & 2 \end{bmatrix}$; scale factor 2

14. Graph a square with vertices $A(-1, 0)$, $B(0, 1)$, $C(1, 0)$, and $D(0, -1)$ on two separate coordinate planes.

   a. On one of the coordinate planes, graph the dilation of square $ABCD$ after a dilation of scale factor 2. Label it $A'B'C'D'$. Then graph a dilation of $A'B'C'D'$ after a scale factor of 3.

   b. On the second coordinate plane, graph the dilation of square $ABCD$ after a dilation of scale factor 3. Label it $A'B'C'D'$. Then graph a dilation of $A'B'C'D'$ after a scale factor of 2.

   c. Compare the results of parts **a** and **b**. Describe what you observe.

Use matrices to determine the coordinates of the vertices of each translated figure. Then graph the pre-image and the image on the same coordinate grid.

15. triangle WXY with vertex matrix $\begin{bmatrix} -2 & 1 & 3 \\ 0 & 5 & -1 \end{bmatrix}$ translated 3 units right and 2 units down

16. quadrilateral with vertices $O(0, 0)$, $P(1, 5)$, $Q(4, 7)$, and $R(3, 2)$ translated 2 units left and 1 unit down

17. square $CDEF$ translated 3 units right and 4 units up if the vertices are $C(-3, 1)$, $D(1, 5)$, $E(5, 1)$, and $F(1, -3)$

18. Graph $\triangle FGH$ with vertices $F(4, 1)$, $G(0, 3)$, and $H(2, -1)$.

a. Graph the image of $\triangle FGH$ after a translation of 6 units left and 2 units down. Label the image $\triangle F'G'H'$.

b. Then translate $\triangle F'G'H'$ 1 unit right and 5 units up. Label this image $\triangle F''G''H''$.

c. What translation would move $\triangle FGH$ to $\triangle F''G''H''$ directly?

Use matrices to determine the coordinates of the vertices of each reflected figure. Then graph the pre-image and the image on the same coordinate grid.

19. $\triangle ABC$ with vertices $A(-1, -2)$, $B(0, -4)$, and $C(2, -3)$ reflected over the x-axis

20. $R_{y\text{-axis}}$ for a rectangle with vertices $D(2, 4)$, $E(6, 2)$, $F(3, -4)$, and $G(-1, -2)$

21. a trapezoid with vertices $H(-1, -2)$, $I(-3, 1)$, $J(-1, 5)$, and $K(2, 4)$ for a reflection over the line $y = x$

Use matrices to determine the coordinates of the vertices of each rotated figure. Then graph the pre-image and the image on the same coordinate grid.

22. Rot_{90} for $\triangle LMN$ with vertices $L(1, -1)$, $M(2, -2)$, and $N(3, -1)$

23. square with vertices $O(0, 0)$, $P(4, 0)$, $Q(4, 4)$, $R(0, 4)$ rotated 180°

24. pentagon $STUVW$ with vertices $S(-1, -2)$, $T(-3, -1)$, $U(-5, -2)$, $V(-4, -4)$, and $W(-2, -4)$ rotated 270° counterclockwise

25. **Proof** Suppose $\triangle ABC$ has vertices $A(1, 3)$, $B(-2, -1)$, and $C(-1, -3)$. Use each result of the given transformation of $\triangle ABC$ to show how the matrix for that reflection or rotation is derived.

a. $\begin{bmatrix} 1 & -2 & -1 \\ -3 & 1 & 3 \end{bmatrix}$ under $R_{x\text{-axis}}$

b. $\begin{bmatrix} -1 & 2 & 1 \\ 3 & -1 & -3 \end{bmatrix}$ under $R_{y\text{-axis}}$

c. $\begin{bmatrix} 3 & -1 & -3 \\ 1 & -2 & -1 \end{bmatrix}$ under $R_{y=x}$

d. $\begin{bmatrix} -3 & 1 & 3 \\ 1 & -2 & -1 \end{bmatrix}$ under Rot_{90}

e. $\begin{bmatrix} -1 & 2 & 1 \\ -3 & 1 & 3 \end{bmatrix}$ under Rot_{180}

f. $\begin{bmatrix} 3 & -1 & -3 \\ -1 & 2 & 1 \end{bmatrix}$ under Rot_{270}

Given $\triangle JKL$ with vertices $J(-6, 4)$, $K(-3, 2)$, and $L(-1, -2)$. Find the coordinates of each composite transformation. Then graph the pre-image and the image on the same coordinate grid.

26. rotation of 180° followed by a translation 2 units left 5 units up

27. $R_{y\text{-axis}} \circ R_{x\text{-axis}}$

28. $Rot_{90} \circ R_{y\text{-axis}}$

Applications and Problem Solving

Real World Application

29. Games Each of the pieces on the chess board has a specific number of spaces and direction it can move. Research the game of chess and describe the possible movements for each piece as a translation matrix.

a. bishop **b.** knight **c.** king

30. Critical Thinking Show that a dilation with scale factor of -1 is the same result as Rot_{180}.

31. Entertainment The Ferris Wheel first appeared at the 1893 Chicago Exposition. Its axle was 45 feet long. Spokes radiated from it that supported 36 wooden cars, which could hold 60 people each. The diameter of the wheel itself was 250 feet. Suppose the axle was located at the origin. Find the coordinates of the car located at the loading platform. Then find the location of the car at the 90° counterclockwise, 180°, and 270° counterclockwise rotation positions.

32. Critical Thinking $R_{y\text{-axis}}$ gives a matrix for reflecting a figure over the y-axis. Do you think a matrix that would represent a reflection over the line $y = 4$ exists? If so, make a conjecture and verify it.

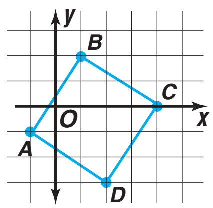
33. Animation Divide two sheets of grid paper into fourths by halving the length and width of the paper. Draw a simple figure on one of the pieces. On another piece, draw the figure dilated with a scale factor of 1.25. On a third piece, draw the original figure dilated with a scale factor of 1.5. On the fourth piece, draw the original figure dilated with a scale factor of 1.75. Continue dilating the original figure on each of the remaining pieces by an increase of 0.25 in scale factor each time. Put the pieces of paper in order and flip through them. What type of motion does the result of these repeated dilations animate?

34. Critical Thinking Write the vertex matrix for the figure graphed below.

a. Make a conjecture about the resulting figure if you multiply the vertex matrix by $\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$.

b. Copy the figure on grid paper and graph the resulting vertex matrix after the multiplication described.

c. How does the result compare with your conjecture? This is often called a *shear*. Why do you think it has this name?



Mixed Review

35. Find $A + B$ if $A = \begin{bmatrix} 3 & 8 \\ -2 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 5 \\ -2 & 8 \end{bmatrix}$. *(Lesson 2-3)*

36. Solve the system of equations. *(Lesson 2-2)*

$x - 2y = 4.6$

$y - z = -5.6$

$x + y + z = 1.8$

37. **Sales** The Grandview Library holds an annual book sale to raise funds and dispose of excess books. One customer bought 4 hardback books and 7 paperbacks for \$5.75. The next customer paid \$4.25 for 3 hardbacks and 5 paperbacks. What are the prices for hardbacks and for paperbacks? *(Lesson 2-1)*

38. Of (0, 0), (3, 2), (−4, 2), or (−2, 4), which satisfy $x + y \geq 3$? *(Lesson 1-8)*

39. Write the standard form of the equation of the line that is parallel to the graph of $y = 4x - 8$ and passes through (−2, 1). *(Lesson 1-5)*

40. Write the slope-intercept form of the equation of the line that passes through the point at (1, 6) and has a slope of 2. *(Lesson 1-3)*

41. If $f(x) = x^3$ and $g(x) = x^2 - 3x + 7$, find $(f \cdot g)(x)$ and $\left(\frac{f}{g}\right)(x)$. *(Lesson 1-2)*

42. **SAT/ACT Practice** If $2x + y = 12$ and $x + 2y = -6$, find the value of $2x + 2y$.

A 0 **B** 4 **C** 8 **D** 12 **E** 14

MID-CHAPTER QUIZ

1. Use graphing to solve the system of equations $\frac{1}{2}x + 5y = 17$ and $3x + 2y = 18$. (Lesson 2-1)

2. Solve the system of equations $4x + y = 8$ and $6x - 2y = -9$ algebraically. (Lesson 2-1)

3. **Sales** HomeMade Toys manufactures solid pine trucks and cars and usually sells four times as many trucks as cars. The net profit from each truck is \$6 and from each car is \$5. If the company wants a total profit of \$29,000, how many trucks and cars should they sell? (Lesson 2-1)

Solve each system of equations. (Lesson 2-2)

4. $2x + y + 4z = 13$
$3x - y - 2z = -1$
$4x + 2y + z = 19$

5. $x + y = 1$
$2x - y = -2$
$4x + y + z = 8$

6. Find the values of x and y for which the matrix equation $\begin{bmatrix} y - 3 \\ y \end{bmatrix} = \begin{bmatrix} x \\ 2x \end{bmatrix}$ is true. (Lesson 2-3)

Use matrices A and B to find each of the following. If the matrix does not exist, write *impossible*. (Lesson 2-3)

$A = \begin{bmatrix} 3 & 5 & -7 \\ -1 & 0 & 4 \end{bmatrix}$ $B = \begin{bmatrix} -2 & 8 & 6 \\ 5 & -9 & 10 \end{bmatrix}$

7. $A + B$ 8. BA 9. $B - 3A$

10. What is the result of reflecting a triangle with vertices at $A(a, d)$, $B(b, e)$, and $C(c, f)$ over the x-axis and then reflecting the image back over the x-axis? Use matrices to justify your answer. (Lesson 2-4)

Extra Practice See p. A28.

HISTORY of MATHEMATICS

MATRICES

Computers use matrices to solve many types of mathematical problems, but matrices have been around for a long time.

Early Evidence Around 300 B.C., as evidenced by clay tablets found by archaeologists, the Babylonians solved problems that now can be solved by using a system of linear equations. However, the exact method of solution used by the Babylonians has not been determined.

Margaret H. Wright

About 100 B.C.–50 B.C., in ancient China, Chapter 8 of the work *Jiuzhang suanshu (Nine Chapters of the Mathematical Art)* presented a similar problem and showed a solution on a counting board that resembles an augmented coefficient matrix.

There are three types of corn, of which three bundles of the first type, two of the second, and one of the third make 39 measures. Two of the first, three of the second, and one of the third make 34 measures. And one of the first, two of the second, and three of the third make 26 measures. How many measures of grain are contained in one bundle of each type?

Author's table

1	2	3
2	3	2
3	1	1
26	34	39

The Chinese author goes on to detail how each column can be operated on to determine the solution. This method was later credited to **Carl Friedrich Gauss.**

The Renaissance The concept of the determinant of a matrix, which you will learn about in the next lesson, appeared in Europe and Japan at almost identical times. However, **Seki** of Japan wrote about it first in 1683 with his *Method of Solving the Dissimulated Problems*. Seki's work contained matrices written in table form like those found in the ancient Chinese writings. Seki developed the pattern for determinants for 2×2, 3×3, 4×4, and 5×5 matrices and used them to solve equations, but not systems of linear equations.

In the same year in Hanover (now Germany), **Gottfried Leibniz** wrote to **Guillaume De l'Hôpital** who lived in Paris, France, about a method he had for solving a system of equations in the form $C + Ax + By = 0$. His method later became known as **Cramer's Rule.**

Modern Era In 1850, the word *matrix* was first used by **James Joseph Sylvester** to describe the tabular array of numbers. Sylvester actually was a lawyer who studied mathematics as a hobby. He shared his interests with **Arthur Cayley,** who is credited with the first published reference to the inverse of a matrix.

Today, computer experts like **Margaret H. Wright** use matrices to solve problems that involve thousands of variables. In her job as Distinguished Member of Technical Staff at a telecommunications company, she applies linear algebra for the solution of real-world problems.

ACTIVITIES

1. Solve the problem from the *Jiuzhang suanshu* by using a system of equations.
2. Research the types of problems solved by the Babylonians using a system of equations.
3. interNET CONNECTION Find out more about the personalities referenced in this article and others who contributed to the history of matrices. Visit **www.amc.glencoe.com**

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