2-6

Solving Systems of Linear Inequalities

OBJECTIVES
- Graph systems of inequalities.
- Find the maximum or minimum value of a function defined for a polygonal convex set.

Real World Application

SHIPPING Package delivery services add extra charges for oversized parcels or those requiring special handling. An oversize package is one in which the sum of the length and the *girth* exceeds 84 inches. The *girth* of a package is the distance around the package. For a rectangular package, its girth is the sum of twice the width and twice the height. A package requiring special handling is one in which the length is greater than 60 inches. What size packages qualify for both oversize and special handling charges?

The situation described in the problem above can be modeled by a **system of linear inequalities.** To solve a system of linear inequalities, you must find the ordered pairs that satisfy both inequalities. One way to do this is to graph both inequalities on the same coordinate plane. The intersection of the two graphs contains points with ordered pairs in the solution set. If the graphs of the inequalities do not intersect, then the system has no solution.

Example 1 Real World Application

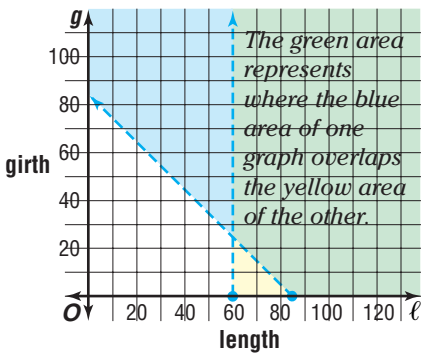
SHIPPING **What size packages qualify for both oversize and special handling charges when shipping?**

First write two inequalities that represent each type of charge. Let ℓ represent the length of a package and g represent its girth.

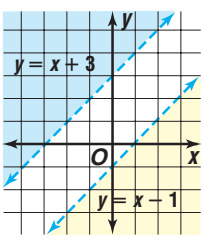
Oversize: $\ell + g > 84$

Special handling: $\ell > 60$

Neither of these inequalities includes the boundary line, so the lines are dashed. The graph of $\ell + g > 84$ is composed of all points above the line $\ell + g = 84$. The graph of $\ell > 60$ includes all points to the right of the line $\ell = 60$. The green area is the solution to the system of inequalities. That is, the ordered pair for any point in the green area satisfies both inequalities. For example, (90, 20) is a length greater than 90 inches and a girth of 20 inches which represents an oversize package that requires special handling.



Not every system of inequalities has a solution. For example, $y > x + 3$ and $y < x - 1$ are graphed at the right. Since the graphs have no points in common, there is no solution.



A system of more than two linear inequalities can have a solution that is a bounded set of points. A bounded set of all points on or inside a convex polygon graphed on a coordinate plane is called a **polygonal convex set.**

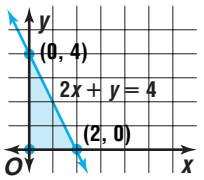
Example 2 **a. Solve the system of inequalities by graphing.**

$x \geq 0$
$y \geq 0$
$2x + y \leq 4$

b. Name the coordinates of the vertices of the polygonal convex set.

a. Since each inequality includes an equality, the boundary lines will be solid. The shaded region shows points that satisfy all three inequalities.

b. The region is a triangle whose vertices are the points at (0, 0), (0, 4) and (2, 0).



An expression whose value depends on two variables is a function of two variables. For example, the value of $6x + 7y - 9$ is a function of x and y and can be written $f(x, y) = 6x + 7y - 9$. The expression $f(3, 5)$ would then stand for the value of the function f when x is 3 and y is 5.

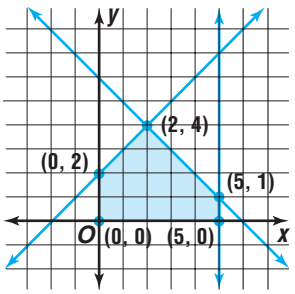
$$f(3, 5) = 6(3) + 7(5) - 9 \text{ or } 44.$$

Sometimes it is necessary to find the maximum or minimum value that a function has for the points in a polygonal convex set. Consider the function $f(x, y) = 5x - 3y$, with the following inequalities forming a polygonal convex set.

$$y \geq 0 \qquad -x + y \leq 2 \qquad 0 \leq x \leq 5 \qquad x + y \leq 6$$

You may need to use algebraic methods to determine the coordinates of the vertices of the convex set.

By graphing the inequalities and finding the intersection of the graphs, you can determine a polygonal convex set of points for which the function can be evaluated. The region shown at the right is the polygonal convex set determined by the inequalities listed above. Since the polygonal convex set has infinitely many points, it would be impossible to evaluate the function for all of them. However, according to the **Vertex Theorem,** a function such as $f(x, y) = 5x - 3y$ need only be evaluated for the coordinates of the vertices of the polygonal convex boundary in order to find the maximum and minimum values.



Vertex Theorem

The maximum or minimum value of f (x, y) = $ax + by + c$ on a polygonal convex set occurs at a vertex of the polygonal boundary.

The value of $f(x, y) = 5x - 3y$ at each vertex can be found as follows.

$f(x, y) = 5x - 3y$
$f(0, 0) = 5(0) - 3(0) = 0$
$f(0, 2) = 5(0) - 3(2) = -6$
$f(2, 4) = 5(2) - 3(4) = -2$
$f(5, 1) = 5(5) - 3(1) = 22$
$f(5, 0) = 5(5) - 3(0) = 25$

Therefore, the maximum value of $f(x, y)$ in the polygon is 25, and the minimum is -6. The maximum occurs at (5, 0), and the minimum occurs at (0, 2).

Example 3 **Find the maximum and minimum values of $f(x, y) = x - y + 2$ for the polygonal convex set determined by the system of inequalities.**

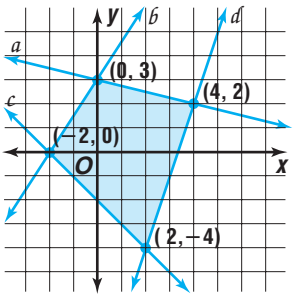
$x + 4y \leq 12$ $\quad 3x - 2y \geq -6$ $\quad x + y \geq -2$ $\quad 3x - y \leq 10$

First write each inequality in slope-intercept form for ease in graphing the boundaries.

Boundary ***a***	*Boundary* ***b***	*Boundary* ***c***	*Boundary* ***d***
$x + 4y \leq 2$	$3x - 2y \geq -6$	$x + y \geq -2$	$3x - y \leq 10$
$4y \leq -x + 12$	$-2y \geq -3x - 6$	$y \geq -x - 2$	$-y \leq -3x + 10$
$y \leq -\frac{1}{4}x + 3$	$y \leq \frac{3}{2}x + 3$		$y \geq 3x - 10$

You can use the matrix approach from Lesson 2-5 to find the coordinates of the vertices.

Graph the inequalities and find the coordinates of the vertices of the resulting polygon.



The coordinates of the vertices are $(-2, 0)$, $(2, -4)$, $(4, 2)$, $(0, 3)$.

Now evaluate the function $f(x, y) = x - y + 2$ at each vertex.

$f(-2, 0) = -2 - 0 + 2$ or 0

$f(2, -4) = 2 - (-4) + 2$ or 8

$f(4, 2) = 4 - 2 + 2$ or 4

$f(0, 3) = 0 - 3 + 2$ or -1

The maximum value of the function is 8, and the minimum value is -1.

CHECK FOR UNDERSTANDING

Communicating Mathematics

Read and study the lesson to answer each question.

1. Refer to the application at the beginning of the lesson.
   a. **Define** the *girth* of a rectangular package.
   b. **Name** some objects that might be shipped by a package delivery service and classified as oversized *and* requiring special handling.
2. **You Decide** Marcel says there is only one vertex that will yield a maximum for any given function. Tomas says that if the numbers are correct, there could be two vertices that yield the same maximum. Who is correct? Explain your answer.
3. **Determine** how many vertices of a polygonal convex set you might expect if the system defining the set contained five inequalities, no two of which are parallel.

Guided Practice

4. Solve the system of inequalities by graphing. $x + 2y \geq 4$ $\quad x - y \leq 3$
5. Solve the system of inequalities by graphing. Name the coordinates of the vertices of the polygonal convex set.
   $y \geq 0$ $\quad -1 \leq x \leq 7$ $\quad -x + y \leq 4$ $\quad x + 2y \leq 8$

Find the maximum and minimum values of each function for the polygonal convex set determined by the given system of inequalities.

6. $f(x, y) = 4x + 3y$
   $4y \leq x + 8$
   $x + y \geq 2$
   $y \geq 2x - 5$

7. $f(x, y) = 3x - 4y$
   $x - 2y \geq -7$
   $x + y \geq 8$
   $2x - y \leq 7$

8. **Business** Gina Chuez has considered starting her own custom greeting card business. With an initial start-up cost of $1500, she figures it will cost $0.45 to produce each card. In order to remain competitive with the larger greeting card companies, Gina must sell her cards for no more than $1.70 each. To make a profit, her income must exceed her costs. How many cards must she sell before making a profit?

EXERCISES

Practice

Solve each system of inequalities by graphing.

9. $y + x \geq 1$
$y - x \leq -1$

10. $y > 1$
$y < -3x + 3$
$y > -3x + 1$

11. $2x + 5y < 25$
$y < 3x - 2$
$5x - 7y < 14$

12. Determine if $(3, -2)$ belongs to the solution set of the system of inequalities $y < \frac{1}{3}x + 5$ and $y < 2x + 1$. Verify your answer.

Solve each system of inequalities by graphing. Name the coordinates of the vertices of the polygonal convex set.

13. $y \geq -0.5x + 1$
$y \leq -3x + 5$
$y \leq 2x + 2$

14. $x \leq 0$
$y + 3 \geq 0$
$x \geq y$

15. $y \geq 0$
$y - 5 \leq 0$
$y + x \leq 7$
$5x + 3y \geq 20$

16. Find the maximum and minimum values of $f(x, y) = 8x + y$ for the polygonal convex set having vertices at $(0, 0)$, $(4, 0)$, $(3, 5)$, and $(0, 5)$.

Find the maximum and minimum values of each function for the polygonal convex set determined by the given system of inequalities.

17. $f(x, y) = 3x + y$
$x \leq 5$
$y \geq 2$
$2x - 5y \geq -10$

18. $f(x, y) = y - x$
$y \leq 4 - 2x$
$x + 2 \geq 2$
$y \geq 0$

19. $f(x, y) = x + y$
$y \leq 6$
$4x - 5y \geq -10$
$2x - 5y \leq -10$

20. $f(x , y) = 4x + 2y + 7$
$x \geq 0$
$y \geq 1$
$x + y \leq 4$

21. $f(x, y) = 2x - y$
$y \leq 4x + 6$
$x + 4y \leq 7$
$2x + y \leq 7$
$x - 6y \leq 10$

22. $f(x, y) = -2x + y + 5$
$2 \leq y \leq 8$
$x \geq 1$
$2x + y + 2 \leq 16$
$y \geq 5 - x$

Applications and Problem Solving

23. **Geometry** Find the system of inequalities that will define a polygonal convex set that includes all points in the interior of a square whose vertices are $A(4, 4)$, $B(4, -4)$, $C(-4, -4)$, and $D(-4, 4)$.

24. **Critical Thinking** Write a system of more than two linear inequalities whose set of solutions is not bounded.

25. **Critical Thinking** A polygonal convex set is defined by the following system of inequalities.

$y \leq 16 - x$
$0 \leq 2y \leq 17$
$3y \geq -2x + 11$
$y \leq 3x + 1$
$y \geq 2x - 13$
$y \geq 7 - 2x$

a. Determine which lines intersect and solve pairs of equations to determine the coordinates of each vertex.

b. Find the maximum and minimum values for $f(x, y) = 5x + 6y$ in the set.

26. **Business** Christine's Butter Cookies sells large tins of butter cookies and small tins of butter cookies. The factory can prepare at most 200 tins of cookies a day. Each large tin of cookies requires 2 pounds of butter, and each small tin requires 1 pound of butter, with a maximum of 300 pounds of butter available each day. The profit from each day's cookie production can be estimated by the function $f(x, y) = \$6.00x + \$4.80y$, where x represents the number of large tins sold and y the number of small tins sold. Find the maximum profit that can be expected in a day.

27. **Fund-raising** The Band Boosters want to open a craft bazaar to raise money for new uniforms. Two sites are available. A Main Street site costs \$10 per square foot per month. The other site on High Street costs \$20 per square foot per month. Both sites require a minimum rental of 20 square feet. The Main Street site has a potential of 30 customers per square foot, while the High Street site could see 40 customers per square foot. The budget for rental space is \$1200 per month. The Band Boosters are studying their options for renting space at both sites.

   a. Graph the polygonal convex region represented by the cost of renting space.

   b. Determine what function would represent the possible number of customers per square foot at both locations.

   c. If space is rented at both sites, how many square feet of space should the Band Boosters rent at each site to maximize the number of potential customers?

   d. Suppose you were president of the Band Boosters. Would you rent space at both sites or select one of the sites? Explain your answer.

28. **Culinary Arts** A gourmet restaurant sells two types of salad dressing, garlic and raspberry, in their gift shop. Each batch of garlic dressing requires 2 quarts of oil and 2 quarts of vinegar. Each batch of raspberry dressing requires 3 quarts of oil and 1 quart of vinegar. The chef has 18 quarts of oil and 10 quarts of vinegar on hand for making the dressings that will be sold in the gift shop that week. If x represents the number of batches of garlic dressing sold and y represents the batches of raspberry dressing sold, the total profits from dressing sold can be expressed by the function $f(x, y) = 3x + 2y$.

   a. What do you think the 3 and 2 in the function $f(x, y) = 3x + 2y$ represent?

   b. How many batches of each types of dressing should the chef make to maximize the profit on sales of the dressing?

Mixed Review

29. Find the inverse of $\begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix}$. *(Lesson 2-5)*

30. Graph $y < -2x + 8$. *(Lesson 1-8)*

31. **Scuba Diving** Graph the equation $d + 33 = 33p$, which relates atmospheres of pressure p to ocean depth d in feet. *(Lesson 1-3)*

32. State the domain and range of the relation $\{(16, -4), (16, 4)\}$. Is this relation a function? Explain. *(Lesson 1-1)*

33. **SAT Practice** **Grid-In** What is the sum of four integers whose mean is 15?

Extra Practice See p. A29.