2-7

Linear Programming

OBJECTIVES

- Use linear programming procedures to solve applications.
- Recognize situations where exactly one solution to a linear programming application may not exist.

Real World Application

MILITARY SCIENCE When the U.S. Army needs to determine how many soldiers or officers to put in the field, they turn to mathematics. A system called the *Manpower Long-Range Planning System* (MLRPS) enables the army to meet the personnel needs for 7- to 20-year planning periods. Analysts are able to effectively use the MLRPS to simulate gains, losses, promotions, and reclassifications. This type of planning requires solving up to 9,060 inequalities with 28,730 variables! However, with a computer, a problem like this can be solved in less than five minutes.

The Army's MLRPS uses a procedure called **linear programming.** Many practical applications can be solved by using this method. The nature of these problems is that certain **constraints** exist or are placed upon the variables, and some function of these variables must be maximized or minimized. The constraints are often written as a system of linear inequalities.

The following procedure can be used to solve linear programming applications.

Linear Programming Procedure

1. Define variables.
2. Write the constraints as a system of inequalities.
3. Graph the system and find the coordinates of the vertices of the polygon formed.
4. Write an expression whose value is to be maximized or minimized.
5. Substitute values from the coordinates of the vertices into the expression.
6. Select the greatest or least result.

In Lesson 2-6, you found the maximum and minimum values for a given function in a defined polygonal convex region. In linear programming, you must use your reasoning abilities to determine the function to be maximized or minimized and the constraints that form the region.

Example 1

Real World Application

MANUFACTURING **Suppose a lumber mill can turn out 600 units of product each week. To meet the needs of its regular customers, the mill must produce 150 units of lumber and 225 units of plywood. If the profit for each unit of lumber is \$30 and the profit for each unit of plywood is \$45, how many units of each type of wood product should the mill produce to maximize profit?**

Define variables. Let x = the units of lumber produced.
Let y = the units of plywood produced.

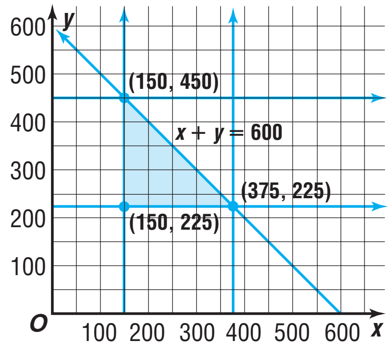
Write inequalities.

$x \geq 150$ — *There cannot be less than 150 units of lumber produced.*

$y \geq 225$ — *There cannot be less than 225 units of plywood produced.*

$x + y \leq 600$ — *The maximum number of units produced is 600.*

Graph the system.



The vertices are at (150, 225), (375, 225), and (150, 450).

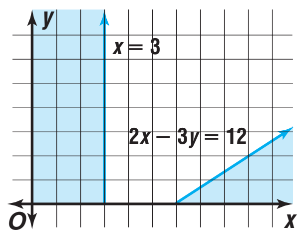
Write an expression. Since profit is \$30 per unit of lumber and \$45 per unit of plywood, the profit function is $P(x, y) = 30x + 45y$.

Substitute values.

$P(150, 225) = 30(150) + 45(225)$ or 14,625
$P(375, 225) = 30(375) + 45(225)$ or 21,375
$P(150, 450) = 30(150) + 45(450)$ or 24,750

Answer the problem. The maximum profit occurs when 150 units of lumber are produced and 450 units of plywood are produced.

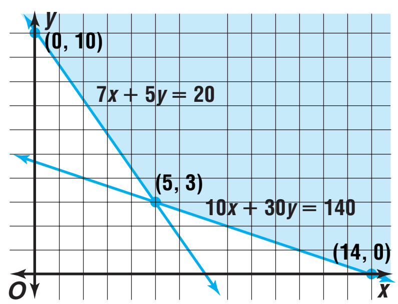
In certain circumstances, the use of linear programming is not helpful because a polygonal convex set is not defined. Consider the graph at the right, based on the following constraints.



$x \geq 0$
$y \geq 0$
$x \leq 3$
$2x - 3y \geq 12$

The constraints do not define a region with any points in common in Quadrant I. When the constraints of a linear programming application cannot be satisfied simultaneously, then the problem is said to be **infeasible.**

Sometimes the region formed by the inequalities in a linear programming application is **unbounded.** In that case, an optimal solution for the problem may not exist. Consider the graph at the right. A function like $f(x, y) = x + 2y$ has a *minimum* value at (5, 3), but it is not possible to find a maximum value.



It is also possible for a linear programming application to have two or more optimal solutions. When this occurs, the problem is said to have **alternate optimal solutions.** This usually occurs when the graph of the function to be maximized or minimized is parallel to one side of the polygonal convex set.

Example 2 Real World Application

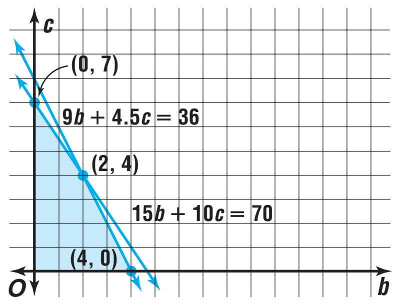
SMALL BUSINESS **The Woodell Carpentry Shop makes bookcases and cabinets. Each bookcase requires 15 hours of woodworking and 9 hours of finishing. The cabinets require 10 hours of woodworking and 4.5 hours of finishing. The profit is $60 on each bookcase and $40 on each cabinet. There are 70 hours available each week for woodworking and 36 hours available for finishing. How many of each item should be produced in order to maximize profit?**

Define variables. Let b = the number of bookcases produced.
Let c = the number of cabinets produced.

Write inequalities.

$b \geq 0,\ c \geq 0$	*There cannot be less than 0 bookcases or cabinets.*
$15b + 10c \leq 70$	*No more than 70 hours of woodworking are available.*
$9b + 4.5c \leq 36$	*No more than 36 hours of finishing are available.*

Graph the system.



The vertices are at (0, 7), (2, 4), (4, 0), and (0, 0).

Write an expression. Since profit on each bookcase is $60 and the profit on each cabinet is $40, the profit function is $P(b, c) = 60b + 40c$.

Substitute values.

$P(0, 0) = 60(0) + 40(0)$ or 0
$P(0, 7) = 60(0) + 40(7)$ or 280
$P(2, 4) = 60(2) + 40(4)$ or 280
$P(4, 0) = 60(4) + 40(0)$ or 240

Answer the problem. The problem has alternate optimal solutions. The shop will make the same profit if they produce 2 bookcases and 4 cabinets as it will from producing 7 cabinets and no bookcases.

Check for Understanding

Communicating Mathematics

Read and study the lesson to answer each question.

1. **Explain** why the inequalities $x \geq 0$ and $y \geq 0$ are usually included as constraints in linear programming applications.
2. **Discuss** the difference between the graph of the constraints when a problem is infeasible and a graph whose constraints yield an unbounded region.
3. **Write,** in your own words, the steps of the linear programming procedure.

Guided Practice

4. Graph the system of inequalities. In a problem asking you to find the maximum value of $f(x, y)$, state whether this situation is *infeasible,* has *alternate optimal solutions,* or is *unbounded.* Assume that $x \geq 0$ and $y \geq 0$.
   $0.5x + 1.5y \geq 7$
   $3x + 9y \leq 2$
   $f(x, y) = 30x + 20y$
5. **Transportation** A package delivery service has a truck that can hold 4200 pounds of cargo and has a capacity of 480 cubic feet. The service handles two types of packages: small, which weigh up to 25 pounds each and are no more than 3 cubic feet each; and large, which are 25 to 50 pounds each and are 3 to 5 cubic feet each. The delivery service charges \$5 for each small package and \$8 for each large package. Let x be the number of small packages and y be the number of large packages in the truck.
   a. Write an inequality to represent the weight of the packages in pounds the truck can carry.
   b. Write an inequality to represent the volume, in cubic feet, of packages the truck can carry.
   c. Graph the system of inequalities.
   d. Write a function that represents the amount of money the delivery service will make on each truckload.
   e. Find the number of each type of package that should be placed on a truck to maximize revenue.
   f. What is the maximum revenue per truck?
   g. In this situation, is maximizing the revenue necessarily the best thing for the company to do? Explain.

Solve each problem, if possible. If not possible, state whether the problem is *infeasible*, has *alternate optimal solutions,* or is *unbounded.*

6. **Business** The manager of a gift store is printing brochures and fliers to advertise sale items. Each brochure costs 8¢ to print, and each flier costs 4¢ to print. A brochure requires 3 pages, and a flier requires 2 pages. The manager does not want to use more than 600 pages, and she needs at least 50 brochures and 150 fliers. How many of each should she print to minimize the cost?
7. **Manufacturing** Woodland Bicycles makes two models of off-road bicycles: the Explorer, which sells for \$250, and the Grande Expedition, which sells for \$350. Both models use the same frame, but the painting and assembly time required for the Explorer is 2 hours, while the time is 3 hours for the Grande Expedition. There are 375 frames and 450 hours of labor available for production. How many of each model should be produced to maximize revenue?

8. **Business** The Grainery Bread Company makes two types of wheat bread, light whole wheat and regular whole wheat. A loaf of light whole wheat bread requires 2 cups of flour and 1 egg. A loaf of regular whole wheat uses 3 cups of flour and 2 eggs. The bakery has 90 cups of flour and 80 eggs on hand. The profit on the light bread is \$1 per loaf and on the regular bread is \$1.50 per loaf. In order to maximize profits, how many of each loaf should the bakery make?

EXERCISES

Practice

Graph each system of inequalities. In a problem asking you to find the maximum value of $f(x, y)$, state whether the situation is *infeasible*, has *alternate optimal solutions*, or is *unbounded*. In each system, assume that $x \geq 0$ and $y \geq 0$ unless stated otherwise.

9. $y \geq 6$
   $5x + 3y \leq 15$
   $f(x, y) = 12x + 3y$

10. $2x + y \geq 48$
    $x + 2y \geq 42$
    $f(x, y) = 2x + y$

11. $4x + 3y \geq 12$
    $y \leq 3$
    $x \leq 4$
    $f(x, y) = 3 + 3y$

Applications and Problem Solving

Real World Application

12. **Veterinary Medicine** Dr. Chen told Miranda that her new puppy needs a diet that includes at least 1.54 ounces of protein and 0.56 ounce of fat each day to grow into a healthy dog. Each cup of Good Start puppy food contains 0.84 ounce of protein and 0.21 ounce of fat. Each cup of Sirius puppy food contains 0.56 ounce of protein and 0.49 ounce of fat. If Good Start puppy food costs 36¢ per cup and Sirius costs 22¢ per cup, how much of each food should Miranda use in order to satisfy the dietary requirements at the minimum cost?
    a. Write an inequality to represent the ounces of protein required.
    b. Write an inequality to represent the ounces of fat required.
    c. Graph the system of inequalities.
    d. Write a function to represent the daily cost of puppy food.
    e. How many cups of each type of puppy food should be used in order to minimize the cost?
    f. What is the minimum cost?

13. **Management** Angela's Pizza is open from noon to midnight each day. Employees work 8-hour shifts from noon to 8 P.M. or 4 P.M. to midnight. The store manager estimates that she needs at least 5 employees from noon to 4 P.M., at least 14 employees from 4 P.M. to 8 P.M., and 6 employees from 8 P.M. to midnight. Employees are paid \$5.50 per hour for hours worked between noon and 4 P.M. The hourly pay between 4 P.M. and midnight is \$7.50.
    a. Write inequalities to represent the number of day-shift workers, the number of night-shift workers, and the total number of workers needed.
    b. Graph the system of inequalities.
    c. Write a function to represent the daily cost of payroll.
    d. Find the number of day-shift workers and night-shift workers that should be scheduled to minimize the cost.
    e. What is the minimal cost?

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Solve each problem, if possible. If not possible, state whether the problem is infeasible, has alternate optimal solutions, or is unbounded.

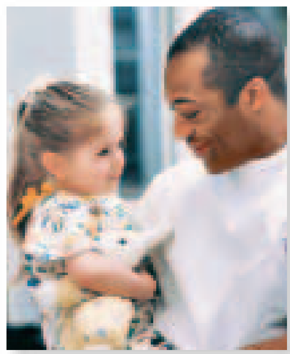
14. **Agriculture** The county officials in Chang Qing County, China used linear programming to aid the farmers in their choices of crops and other forms of agricultural production. This led to a 12% increase in crop profits, a 54% increase in animal husbandry profits, while improving the region's ecology. Suppose an American farmer has 180 acres on which to grow corn and soybeans. He is planting at least 40 acres of corn and 20 acres of soybeans. Based on his calculations, he can earn $150 per acre of corn and $250 per acre of soybeans.
    a. If the farmer plants at least 2 acres of corn for every acre of soybeans, how many acres of each should he plant to earn the greatest profit?
    b. What is the farmer's maximum profit?

15. **Education** Ms. Carlyle has written a final exam for her class that contains two different sections. Questions in section I are worth 10 points each, and questions in section II are worth 15 points each. Her students will have 90 minutes to complete the exam. From past experience, she knows that on average questions from section I take 6 minutes to complete and questions from section II take 15 minutes. Ms. Carlyle requires her students to answer at least 2 questions from section II. Assuming they answer correctly, how many questions from each section will her students need to answer to get the highest possible score?

16. **Manufacturing** Newline Recyclers processes used aluminum into food or drink containers. The recycling plant processes up to 1200 tons of aluminum per week. At least 300 tons must be processed for food containers, while at least 450 tons must be processed for drink containers. The profit is $17.50 per ton for processing food containers and $20 per ton for processing drink containers. What is the profit if the plant maximizes processing?

17. **Investment** Diego wants to invest up to $11,000 in certificates of deposit at First Bank and City Bank. He does not want to deposit more than $7,500 at First Bank. He will deposit at least $1,000 but not more than $7,000 at City Bank. First Bank offers 6% simple interest on deposits, while City Bank offers $6\frac{1}{2}$% simple interest. How much should Diego deposit into each account so he can earn the most interest possible in one year?

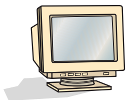
18. **Human Resources** Memorial Hospital wants to hire nurses and nurse's aides to meet patient needs at minimum cost. The average annual salary is $35,000 for a nurse and $18,000 for a nurse's aide. The hospital can hire up to 50 people, but needs to hire at least 20 to function properly. The head nurse wants at least 12 aides, but the number of nurses must be at least twice the number of aides to meet state regulations. How many nurses and nurse's aides should be hired to minimize salary costs?



19. **Manufacturing** A potato chip company makes chips to fill snack-size bags and family-size bags. In one week, production cannot exceed 2,400 units, of which at least 600 units must be for snack-size bags and at least 900 units must be for family size. The profit on a unit of snack-size bags is $12, and the profit on a unit of family-size bags is $18. How much of each type of bag must be processed to maximize profits?

20. **Crafts** Shelly is making soap and shampoo for gifts. She has 48 ounces of lye and 76 ounces of coconut oil and an ample supply of the other needed ingredients. She plans to make as many batches of soap and shampoo as possible. A batch of soap requires 12 ounces of lye and 20 ounces of coconut oil. Each batch of shampoo needs 6 ounces of lye and 8 ounces of coconut oil. What is the maximum number of batches of *both* soap and shampoo possible?

21. **Manufacturing** An electronics plant makes standard and large computer monitors on three different machines. The profit is \$40 on each monitor. Use the table below to determine how many of each monitor the plant should make to maximize profits.



Manufacturing Analysis

Machine	Hours Needed to Make		Total Hours Available
	Small Monitor	Large Monitor	
A	1	2	16
B	1	1	9
C	1	4	24

22. **Critical Thinking** Find the area enclosed by the polygonal convex set defined by the system of inequalities $y \geq 0$, $x \leq 12$, $2x + 6y \leq 84$, $2x - 3y \leq -3$, and $8x + 3y \geq 33$.

23. **Critical Thinking** Mr. Perez has an auto repair shop. He offers two bargain maintenance services, an oil change and a tune-up. His profit is \$12 on an oil change and \$20 on a tune-up. It takes Mr. Perez 30 minutes to do an oil change and 1 hour to do a tune-up. He wants to do at least 25 oil changes per week and no more than 10 tune-ups per week. He can spend up to 30 hours each week on these two services.

   a. What is the most Mr. Perez can earn performing these services?

   b. After seeing the results of his linear program, Mr. Perez decides that he must make a larger profit to keep his business growing. How could the constraints be modified to produce a larger profit?

Mixed Review

24. A polygonal convex set is defined by the inequalities $y \leq -x + 5$, $y \leq x + 5$, and $y + 5x \geq -5$. Find the minimum and maximum values for $f(x, y) = \frac{1}{3}x - \frac{1}{2}y$ within the set. *(Lesson 2-6)*

25. Find the values of x and y for which $\begin{bmatrix} 4x + y \\ x \end{bmatrix} = \begin{bmatrix} 6 \\ 2y - 12 \end{bmatrix}$ is true. *(Lesson 2-3)*

26. Graph $y = 3|x - 2|$. *(Lesson 1-7)*

27. **Budgeting** Martina knows that her monthly telephone charge for local calls is \$13.65 (excluding tax) for service allowing 30 local calls plus \$0.15 for each call after 30. Write an equation to calculate Martina's monthly telephone charges and find the cost if she made 42 calls. *(Lesson 1-4)*

28. **SAT/ACT Practice** If $\frac{2x - 3}{x} = \frac{3 - x}{2}$, which could be a value for x?

A -3 **B** -1 **C** 37 **D** 5 **E** 15

Extra Practice See p. A29.
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