# 3-1 Symmetry and Coordinate Graphs

**OBJECTIVES**
- Use algebraic tests to determine if the graph of a relation is symmetrical.
- Classify functions as even or odd.

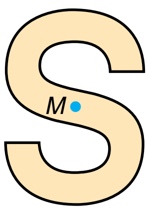
**PHARMACOLOGY** Designing a drug to treat a disease requires an understanding of the molecular structures of the substances involved in the disease process. The substances are isolated in crystalline form, and X rays are passed through the symmetrically-arranged atoms of the crystals. The existence of symmetry in crystals causes the X rays to be diffracted in regular patterns. These symmetrical patterns are used to determine and visualize the molecular structure of the substance. *A related problem is solved in Example 4.*

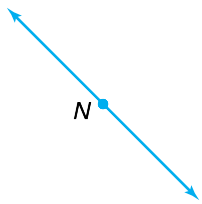
Like crystals, graphs of certain functions display special types of symmetry. For some functions with symmetrical graphs, knowledge of symmetry can often help you sketch and analyze the graphs. One type of symmetry a graph may have is **point symmetry.**

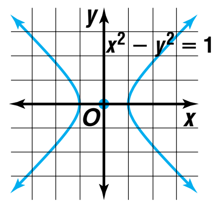
**Point Symmetry** — Two distinct points $P$ and $P'$ are symmetric with respect to point $M$ if and only if $M$ is the midpoint of $\overline{PP'}$. Point $M$ is symmetric with respect to itself.

When the definition of point symmetry is extended to a set of points, such as the graph of a function, then each point $P$ in the set must have an **image point** $P'$ that is also in the set. A figure that is symmetric with respect to a given point can be rotated 180° about that point and appear unchanged. Each of the figures below has point symmetry with respect to the labeled point.

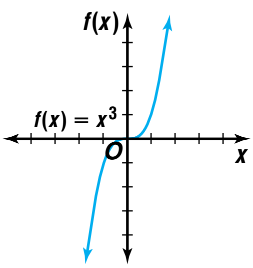
*Symmetry with respect to a given point M can be expressed as symmetry about point M.*



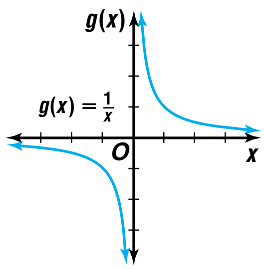




The origin is a common point of symmetry. Observe that the graphs of $f(x) = x^3$ and $g(x) = \frac{1}{x}$ exhibit symmetry with respect to the origin. Look for patterns in the table of function values beside each graph.




| $f(x) = x^3$ | | | |
|---|---|---|---|
| $x$ | $f(x)$ | $f(-x)$ | $-f(x)$ |
| 1 | 1 | −1 | −1 |
| 2 | 8 | −8 | −8 |
| 3 | 27 | −27 | −27 |
| 4 | 64 | −64 | −64 |

*Note that $f(-x) = -f(x)$.*




| $g(x) = \frac{1}{x}$ | | | |
|---|---|---|---|
| $x$ | $g(x)$ | $g(-x)$ | $-g(x)$ |
| 1 | 1 | −1 | −1 |
| 2 | $\frac{1}{2}$ | $-\frac{1}{2}$ | $-\frac{1}{2}$ |
| 3 | $\frac{1}{3}$ | $-\frac{1}{3}$ | $-\frac{1}{3}$ |
| 4 | $\frac{1}{4}$ | $-\frac{1}{4}$ | $-\frac{1}{4}$ |

*Note that $g(-x) = -g(x)$.*

The values in the tables suggest that $f(-x) = -f(x)$ whenever the graph of a function is symmetric with respect to the origin.

**Symmetry with Respect to the Origin**

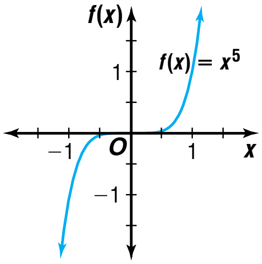
The graph of a relation $S$ is symmetric with respect to the origin if and only if $(a, b) \in S$ implies that $(-a, -b) \in S$. A function has a graph that is symmetric with respect to the origin if and only if $f(-x) = -f(x)$ for all $x$ in the domain of $f$.

*$(a, b) \in S$ means the ordered pair $(a, b)$ belongs to the solution set $S$.*

Example 1 demonstrates how to algebraically test for symmetry about the origin.

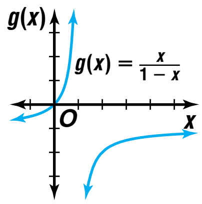
**Example 1** **Determine whether each graph is symmetric with respect to the origin.**

**a.** $f(x) = x^5$




The graph of $f(x) = x^5$ appears to be symmetric with respect to the origin.

**b.** $g(x) = \frac{x}{1 - x}$




The graph of $g(x) = \frac{x}{1 - x}$ does not appear to be symmetric with respect to the origin.

We can verify these conjectures algebraically by following these two steps.

1. Find $f(-x)$ and $-f(x)$.
2. If $f(-x) = -f(x)$, the graph has point symmetry.

**a.** $f(x) = x^5$

Find $f(-x)$.

$f(-x) = (-x)^5$ *Replace $x$ with $-x$.*

$f(-x) = -x^5$ *$(-x)^5 = (-1)^5x^5 = -1x^5$ or $-x^5$*

Find $-f(x)$.

$-f(x) = -x^5$ *Determine the opposite of the function.*

The graph of $f(x) = x^5$ is symmetric with respect to the origin because $f(-x) = -f(x)$.

**b.** $g(x) = \frac{x}{1 - x}$

Find $g(-x)$.

$$g(-x) = \frac{-x}{1 - (-x)} = \frac{-x}{1 + x}$$ *Replace $x$ with $-x$.*

Find $-g(x)$.

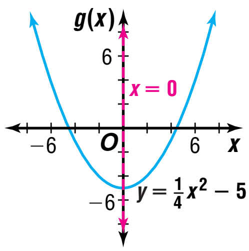
$$-g(x) = -\frac{x}{1 - x} = \frac{-x}{1 - x}$$ *Determine the opposite of the function.*

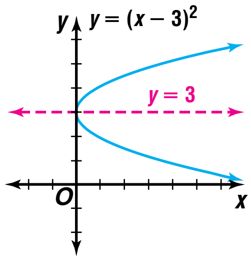
The graph of $g(x) = \frac{x}{1 - x}$ is not symmetric with respect to the origin because $g(-x) \neq -g(x)$.

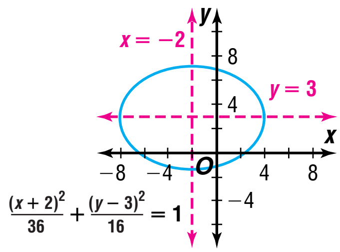
Another type of symmetry is **line symmetry.**

| **Line Symmetry** | Two distinct points $P$ and $P'$ are symmetric with respect to a line $\ell$ if and only if $\ell$ is the perpendicular bisector of $\overline{PP'}$. A point $P$ is symmetric to itself with respect to line $\ell$ if and only if $P$ is on $\ell$. |
|---|---|

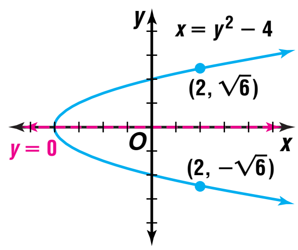
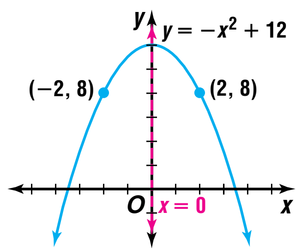
Each graph below has line symmetry. The equation of each line of symmetry is given. Graphs that have line symmetry can be folded along the line of symmetry so that the two halves match exactly. Some graphs, such as the graph of an ellipse, have more than one line of symmetry.



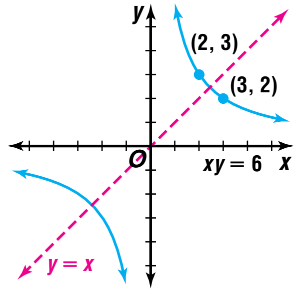
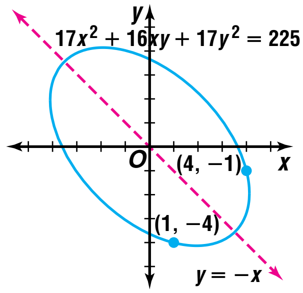




Some common lines of symmetry are the $x$-axis, the $y$-axis, the line $y = x$, and the line $y = -x$. The following table shows how the coordinates of symmetric points are related for each of these lines of symmetry. Set notation is often used to define the conditions for symmetry.

| Symmetry with Respect to the: | Definition and Test | Example |
|---|---|---|
| $x$-axis | $(a, -b) \in S$ if and only if $(a, b) \in S$.<br>*Example:* $(2, \sqrt{6})$ and $(2, -\sqrt{6})$ are on the graph.<br>*Test:* Substituting $(a, b)$ and $(a, -b)$ into the equation produces equivalent equations. |  |
| $y$-axis | $(-a, b) \in S$ if and only if $(a, b) \in S$.<br>*Example:* $(2, 8)$ and $(-2, 8)$ are on the graph.<br>*Test:* Substituting $(a, b)$ and $(-a, b)$ into the equation produces equivalent equations. |  |

*(continued on the next page)*

| Symmetry with Respect to the Line: | Definition and Test | Example |
|---|---|---|
| $y = x$ | $(b, a) \in S$ if and only if $(a, b) \in S$. *Example:* (2, 3) and (3, 2) are on the graph. *Test:* Substituting $(a, b)$ and $(b, a)$ into the equation produces equivalent equations. |   |
| $y = -x$ | $(-b, -a) \in S$ if and only if $(a, b) \in S$. *Example:* (4, −1) and (1, −4) are on the graph. *Test:* Substituting $(a, b)$ and $(-b, -a)$ into the equation produces equivalent equations. |   |

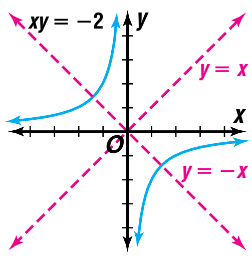
You can determine whether the graph of an equation has line symmetry without actually graphing the equation.

**Example 2** **Determine whether the graph of $xy = -2$ is symmetric with respect to the $x$-axis, $y$-axis, the line $y = x$, the line $y = -x$, or none of these.**

Substituting $(a, b)$ into the equation yields $ab = -2$. Check to see if each test produces an equation equivalent to $ab = -2$.

| | | |
|---|---|---|
| $x$-axis | $a(-b) = -2$ | *Substitute $(a, -b)$ into the equation.* |
| | $-ab = -2$ | *Simplify.* |
| | $ab = 2$ | *Not equivalent to $ab = -2$* |
| $y$-axis | $(-a)b = -2$ | *Substitute $(-a, b)$ into the equation.* |
| | $-ab = -2$ | *Simplify.* |
| | $ab = 2$ | *Not equivalent to $ab = -2$* |
| $y = x$ | $(b)(a) = -2$ | *Substitute $(b, a)$ into the equation.* |
| | $ab = -2$ | *Equivalent to $ab = -2$* |
| $y = -x$ | $(-b)(-a) = -2$ | *Substitute $(-b, -a)$ into the equation.* |
| | $ab = -2$ | *Equivalent to $ab = -2$* |

Therefore, the graph of $xy = -2$ is symmetric with respect to the line $y = x$ and the line $y = -x$. A sketch of the graph verifies the algebraic tests.



You can use information about symmetry to draw the graph of a relation.

**Example 3** **Determine whether the graph of $|y| = 2 - |2x|$ is symmetric with respect to the $x$-axis, the $y$-axis, both, or neither. Use the information about the equation's symmetry to graph the relation.**

Substituting $(a, b)$ into the equation yields $|b| = 2 - |2a|$. Check to see if each test produces an equation equivalent to $|b| = 2 - |2a|$.

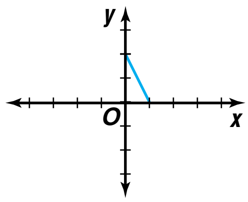
$x$-axis $|-b| = 2 - |2a|$ *Substitute (a, −b) into the equation.*
$|b| = 2 - |2a|$ *Equivalent to $|b| = 2 - |2a|$ since $|-b| = |b|$.*

$y$-axis $|b| = 2 - |-2a|$ *Substitute (−a, b) into the equation.*
$|b| = 2 - |2a|$ *Equivalent to $|b| = 2 - |2a|$, since $|-2a| = |2a|$.*

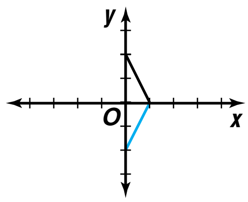
Therefore, the graph of $|y| = 2 - |2x|$ is symmetric with respect to both the $x$-axis and the $y$-axis.

To graph the relation, let us first consider ordered pairs where $x \geq 0$ and $y \geq 0$. The relation $|y| = 2 - |2x|$ contains the same points as $y = 2 - 2x$ in the first quadrant.

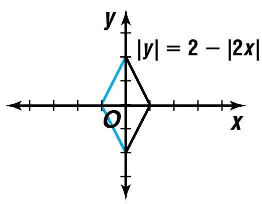
Therefore, in the first quadrant, the graph of $|y| = 2 - |2x|$ is the same as the graph of $y = 2 - 2x$.




Since the graph is symmetric with respect to the $x$-axis, every point in the first quadrant has a corresponding point in the fourth quadrant.

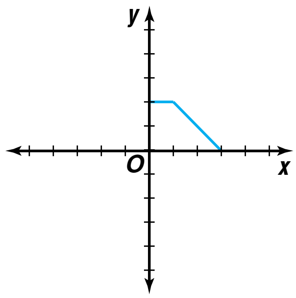



Since the graph is symmetric with respect to the $y$-axis, every point in the first and fourth quadrants has a corresponding point on the other side of the $y$-axis.



**Example 4** **CRYSTALLOGRAPHY** **A crystallographer can model a cross-section of a crystal with mathematical equations. After sketching the outline on a graph, she notes that the crystal has both $x$-axis and $y$-axis symmetry. She uses the piecewise function $y = \begin{cases} 2 \text{ if } 0 \le x \le 1 \\ 3 - x \text{ if } 1 \le x \le 3 \end{cases}$ to model the first quadrant portion of the cross-section. Write piecewise equations for the remaining sides.**

*Real World Application*



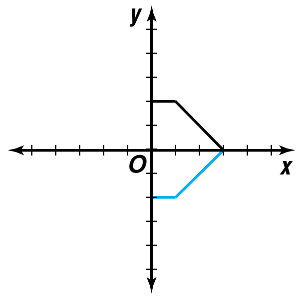

**Look Back**
Refer to Lesson 1-7 for more about piecewise functions.

Since the graph has $x$-axis symmetry, substitute $(x, -y)$ into the original equation to produce the equation for the fourth quadrant portion.

$$y = \begin{cases} 2 \text{ if } 0 \le x \le 1 \\ 3 - x \text{ if } 1 \le x \le 3 \end{cases}$$

$$-y = \begin{cases} 2 \text{ if } 0 \le x \le 1 \\ 3 - x \text{ if } 1 \le x \le 3 \end{cases}$$ *Substitute $(x, -y)$ into the equation.*

$$y = \begin{cases} -2 \text{ if } 0 \le x \le 1 \\ 3 + x \text{ if } 1 \le x \le 3 \end{cases}$$ *Solve for y.*




The equation $y = \begin{cases} -2 \text{ if } 0 \le x \le 1 \\ x - 3 \text{ if } 1 \le x \le 3 \end{cases}$ models the fourth quadrant portion of the cross section.

Since the graph has $y$-axis symmetry, substitute $(-x, y)$ into the first and fourth quadrant equations to produce the equations for the second and third quadrants.

$$y = \begin{cases} 2 \text{ if } 0 \le x \le 1 \\ 3 - x \text{ if } 1 \le x \le 3 \end{cases}$$ *Start with the first quadrant equation.*

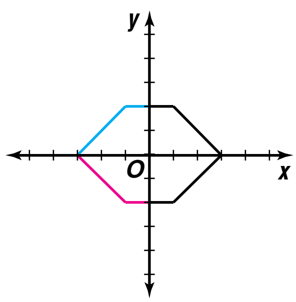
$$y = \begin{cases} 2 \text{ if } 0 \le -x \le 1 \\ 3 - (-x) \text{ if } 1 \le -x \le 3 \end{cases}$$ *Substitute $(-x, y)$ into the equation.*

$$y = \begin{cases} 2 \text{ if } 0 \ge x \ge -1 \\ 3 + x \text{ if } -1 \ge x \ge -3 \end{cases}$$ *This is the second quadrant equation.*

$$y = \begin{cases} -2 \text{ if } 0 \le x \le 1 \\ x - 3 \text{ if } 1 \le x \le 3 \end{cases}$$ *Start with the fourth quadrant equation.*

$$y = \begin{cases} -2 \text{ if } 0 \le -x \le 1 \\ -(-x) - 3 \text{ if } 1 \le -x \le 3 \end{cases}$$ *Substitute $(-x, y)$ into the equation.*

$$y = \begin{cases} -2 \text{ if } 0 \ge x \ge -1 \\ x - 3 \text{ if } -1 \ge x \ge -3 \end{cases}$$ *This is the third quadrant equation.*

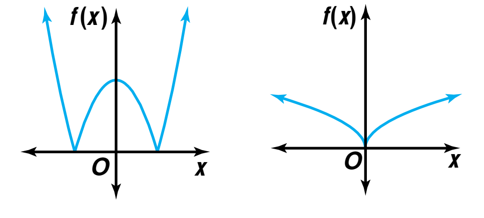
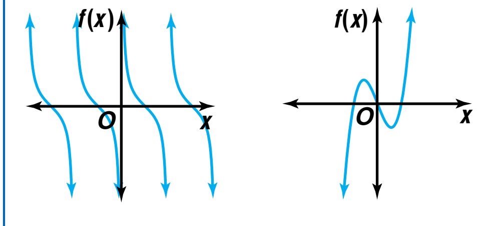



The equations $y = \begin{cases} 2 \text{ if } -1 \le x \le 0 \\ 3 + x \text{ if } -3 \le x \le -1 \end{cases}$ and $y = \begin{cases} -2 \text{ if } -1 \le x \le 0 \\ x - 3 \text{ if } -3 \le x \le -1 \end{cases}$ model the second and third quadrant portions of the cross section respectively.

**interNET CONNECTION**

**Graphing Calculator Programs**
For a graphing calculator program that determines whether a function is even, odd, or neither, visit **www.amc.glencoe.com**

Functions whose graphs are symmetric with respect to the $y$-axis are **even functions.** Functions whose graphs are symmetric with respect to the origin are **odd functions.** Some functions are neither even nor odd. From Example 1, $f(x) = x^5$ is an odd function, and $g(x) = \frac{x}{1 - x}$ is neither even nor odd.

| even functions | odd functions |
|---|---|
| $f(-x) = f(x)$ | $f(-x) = -f(x)$ |
|  |  |
| symmetric with respect to the $y$-axis | symmetric with respect to the origin |

## GRAPHING CALCULATOR EXPLORATION

You can use the **TRACE** function to investigate the symmetry of a function.

- Graph the function.
- Use TRACE to observe the relationship between points of the graph having opposite $x$-coordinates.
- Use this information to determine the relationship between $f(x)$ and $f(-x)$.

**TRY THESE** **Graph each function to determine how $f(x)$ and $f(-x)$ are related.**

**1.** $f(x) = x^8 - 3x^4 + 2x^2 + 2$
**2.** $f(x) = x^7 + 4x^5 - x^3$

**WHAT DO YOU THINK?**

**3.** Identify the functions in Exercises 1 and 2 as *odd, even,* or *neither* based on your observations of their graphs.

**4.** Verify your conjectures algebraically.

**5.** How could you use symmetry to help you graph an even or odd function? Give an example.

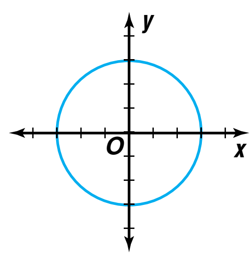
**6.** Explain how you could use the **ASK** option in **TBLSET** to determine the relationship between $f(x)$ and $f(-x)$ for a given function.

## CHECK FOR UNDERSTANDING

**Communicating Mathematics**

**Read and study the lesson to answer each question.**

1. Refer to the tables on pages 129–130. **Identify** each graph as an even function, an odd function, or neither. Explain.
2. **Explain** how rotating a graph of an odd function 180° will affect its appearance. Draw an example.
3. Consider the graph at the right.
   a. **Determine** four lines of symmetry for the graph.
   b. How many other lines of symmetry does this graph possess?
   c. What other type of symmetry does this graph possess?




4. **Write** an explanation of how to test for symmetry with respect to the line $y = -x$.

5. **You Decide** Alicia says that any graph that is symmetric to the origin and to the $y$-axis must also be symmetric to the $x$-axis. Chet disagrees. Who is correct? Support your answer graphically and algebraically.

**Guided Practice**

**Determine whether the graph of each function is symmetric with respect to the origin.**

6. $f(x) = x^6 + 9x$

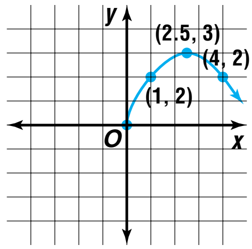
7. $f(x) = \frac{1}{5x} - x^{19}$

**Determine whether the graph of each equation is symmetric with respect to the $x$-axis, $y$-axis, the line $y = x$, the line $y = -x$, or none of these.**

8. $6x^2 = y - 1$

9. $x^3 + y^3 = 4$

10. Copy and complete the graph at the right so that it is the graph of an even function.

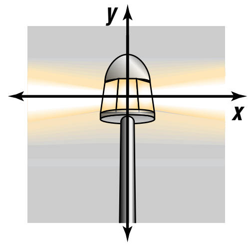



**Determine whether the graph of each equation is symmetric with respect to the $x$-axis, the $y$-axis, both, or neither. Use the information about symmetry to graph the relation.**

11. $y = \sqrt{2 - x^2}$

12. $|y| = x^3$

13. **Physics** Suppose the light pattern from a fog light can be modeled by the equation $\frac{x^2}{25} - \frac{y^2}{9} = 1$. One of the points on the graph of this equation is at $\left(6, \frac{3\sqrt{11}}{5}\right)$, and one of the $x$-intercepts is $-5$. Find the coordinates of three additional points on the graph and the other $x$-intercept.




## EXERCISES

**Practice**

**Determine whether the graph of each function is symmetric with respect to the origin.**

14. $f(x) = 3x$

15. $f(x) = x^3 - 1$

16. $f(x) = 5x^2 + 6x + 9$

17. $f(x) = \frac{1}{4x^7}$

18. $f(x) = -7x^5 + 8x$

19. $f(x) = \frac{1}{x} - x^{100}$

20. Is the graph of $g(x) = \frac{x^2 - 1}{x}$ symmetric with respect to the origin? Explain how you determined your answer.

**Determine whether the graph of each equation is symmetric with respect to the $x$-axis, $y$-axis, the line $y = x$, the line $y = -x$, or none of these.**

21. $xy = -5$

22. $x + y^2 = 1$

23. $y = -8x$

24. $y = \frac{1}{x^2}$

25. $x^2 + y^2 = 4$

26. $y^2 = \frac{4x^2}{9} - 4$

27. Which line(s) are lines of symmetry for the graph of $x^2 = \frac{1}{y^2}$?



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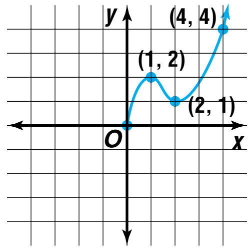
CONTENTS

**For Exercises 28–30, refer to the graph.**




**28.** Complete the graph so that it is the graph of an odd function.

**29.** Complete the graph so that it is the graph of an even function.

**30.** Complete the graph so that it is the graph of a function that is neither even nor odd.

**Determine whether the graph of each equation is symmetric with respect to the *x*-axis, the *y*-axis, both, or neither. Use the information about symmetry to graph the relation.**

**31.** $y^2 = x^2$ **32.** $|x| = -3y$ **33.** $y^2 + 3x = 0$

**34.** $|y| = 2x^2$ **35.** $x = \pm\sqrt{12 - 8y^2}$ **36.** $|y| = xy$

**37.** Graph the equation $|y| = x^3 - x$ using information about the symmetry of the graph.

## Applications and Problem Solving

Real World Application

**38. Physics** The path of a comet around the Sun can be modeled by a transformation of the equation $\frac{x^2}{8} + \frac{y^2}{10} = 1$.

a. Determine the symmetry in the graph of the comet's path.

b. Use symmetry to graph the equation $\frac{x^2}{8} + \frac{y^2}{10} = 1$.

c. If it is known that the comet passes through the point at $(2, \sqrt{5})$, name the coordinates of three other points through which it must pass.

**39. Critical Thinking** Write the equation of a graph that is symmetric with respect to the *x*-axis.

**40. Geometry** Draw a diagram composed of line segments that exhibits both *x*- and *y*-axis symmetry. Write equations for the boundaries.

**41. Communication** Radio waves emitted from two different radio towers interfere with each other's signal. The path of interference can be modeled by the equation $\frac{y^2}{12} - \frac{x^2}{16} = 1$, where the origin is the midpoint of the line segment between the two towers and the positive *y*-axis represents north. Juana lives on an east-west road 6 miles north of the *x*-axis and cannot receive the radio station at her house. At what coordinates might Juana live relative to the midpoint between the two towers?

**42. Critical Thinking** Must the graph of an odd function contain the origin? Explain your reasoning and illustrate your point with the graph of a specific function.

## Mixed Review

**43. Manufacturing** A manufacturer makes a profit of \$6 on a bicycle and \$4 on a tricycle. Department A requires 3 hours to manufacture the parts for a bicycle and 4 hours to manufacture parts for a tricycle. Department B takes 5 hours to assemble a bicycle and 2 hours to assemble a tricycle. How many bicycles and tricycles should be produced to maximize the profit if the total time available in department A is 450 hours and in department B is 400 hours? *(Lesson 2-7)*

**Extra Practice** See p. A30.

44. Find $AB$ if $A = \begin{bmatrix} 4 & 3 \\ 7 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 8 & 5 \\ 9 & 6 \end{bmatrix}$. *(Lesson 2-3)*

45. Solve the system of equations, $2x + y + z = 0$, $3x - 2y - 3z = -21$, and $4x + 5y + 3z = -2$. *(Lesson 2-2)*

46. State whether the system, $4x - 2y = 7$ and $-12x + 6y = -21$, is *consistent and independent, consistent and dependent,* or *inconsistent.* *(Lesson 2-1)*

47. Graph $0 \le x - y \le 2$. *(Lesson 1-8)*

48. Write an equation in slope-intercept form for the line that passes through $A(0, 2)$ and $B(-2, 16)$. *(Lesson 1-4)*

49. If $f(x) = -2x + 11$ and $g(x) = x - 6$, find $[f \circ g](x)$ and $[g \circ f](x)$. *(Lesson 1-2)*

50. **SAT/ACT Practice** What is the product of $75^3$ and $75^7$?

**A** $75^5$ **B** $75^{10}$ **C** $150^{10}$ **D** $5625^{10}$ **E** $75^{21}$

# CAREER CHOICES

## Biomedical Engineering

Would you like to help people live better lives? Are you interested in a career in the field of health? If you answered yes, then biomedical engineering may be the career for you. Biomedical engineers apply engineering skills and life science knowledge to design artificial parts for the human body and devices for investigating and repairing the human body. Some examples are artificial organs, pacemakers, and surgical lasers.

In biomedical engineering, there are three primary work areas: research, design, and teaching. There are also many specialty areas in this field. Some of these are bioinstrumentation, biomechanics, biomaterials, and rehabilitation engineering.

The graph shows an increase in the number of outpatient visits over the number of hospital visits. This is due in part to recent advancements in biomedical engineering.

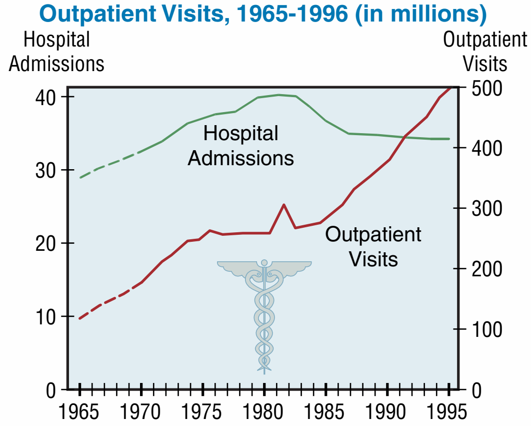
### Career Overview

**Degree Preferred:**
bachelor's degree in biomedical engineering

**Related Courses:**
biology, chemistry, mathematics

**Outlook:**
number of jobs expected to increase through the year 2006

**Hospital Vital Signs**
**Total Number of Hospital Admissions and Outpatient Visits, 1965-1996 (in millions)**




Source: *The Wall Street Journal Almanac*

**interNET CONNECTION** For more information on careers in biomedical engineering, visit: **www.amc.glencoe.com**