3-2 Families of Graphs

OBJECTIVES

- Identify transformations of simple graphs.
- Sketch graphs of related functions.

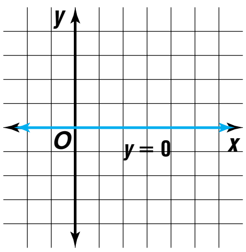
Real World Application

ENTERTAINMENT At some circuses, a human cannonball is shot out of a special cannon. In order to perform this death-defying feat safely, the maximum height and distance of the performer must be calculated accurately. Quadratic functions can be used to model the height of a projectile like a human cannonball at any time during its flight. The quadratic equation used to model height versus time is closely related to the equation of $y = x^2$. *A problem related to this is solved in Example 5.*

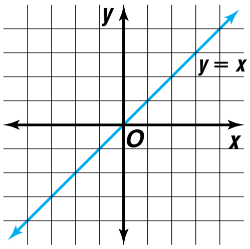
All parabolas are related to the graph of $y = x^2$. This makes $y = x^2$ the **parent graph** of the family of parabolas. Recall that a *family of graphs* is a group of graphs that displays one or more similar characteristics.

A parent graph is a basic graph that is transformed to create other members in a family of graphs. Some different types are shown below. Notice that with the exception of the constant function, the coefficient of x in each equation is 1.

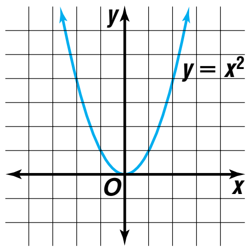
constant function

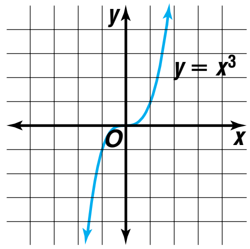


identity function

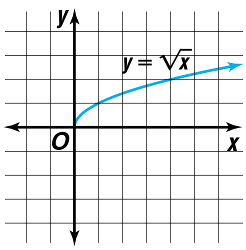


polynomial functions

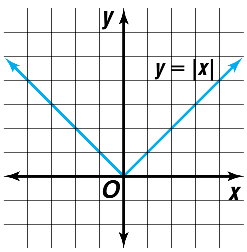




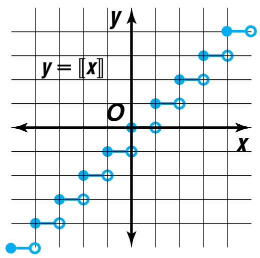
square root function



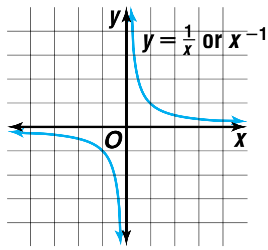
absolute value function



greatest integer function



rational function



Look Back
Refer to Lesson 2-4 for more about reflections and translations.

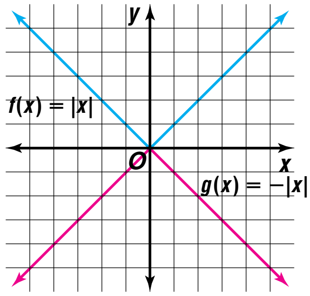
Reflections and translations of the parent function can affect the appearance of the graph. The transformed graph may appear in a different location, but it will resemble the parent graph. A reflection flips a figure over a line called the *axis of symmetry*. *The axis of symmetry is also called the line of symmetry.*

Graphing Calculator Tip

You can use a graphing calculator to check your sketch of any function in this lesson.

Example 1 **Graph $f(x) = |x|$ and $g(x) = -|x|$. Describe how the graphs of $f(x)$ and $g(x)$ are related.**

x	$f(x) = \|x\|$	$g(x) = -\|x\|$
-2	2	-2
-1	1	-1
0	0	0
1	1	-1
2	2	-2



To graph both equations on the same axis, let $y = f(x)$ and $y = g(x)$.

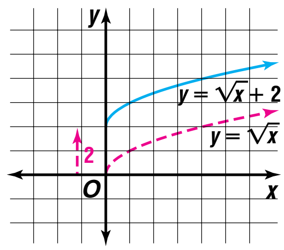
The graph of $g(x)$ is a reflection of the graph of $f(x)$ over the x-axis. The symmetric relationship can be stated algebraically by $g(x) = -f(x)$, or $f(x) = -g(x)$. *Notice that the effect of multiplying a function by -1 is a reflection over the x-axis.*

When a constant c is added to or subtracted from a parent function, the result, $f(x) \pm c$, is a translation of the graph up or down. When a constant c is added or subtracted from x before evaluating a parent function, the result, $f(x \pm c)$, is a translation left or right.

Example 2 **Use the parent graph $y = \sqrt{x}$ to sketch the graph of each function.**

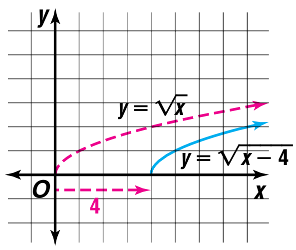
a. $y = \sqrt{x} + 2$

This function is of the form $y = f(x) + 2$. Since 2 is added to the parent function $y = \sqrt{x}$, the graph of the parent function moves up 2 units.



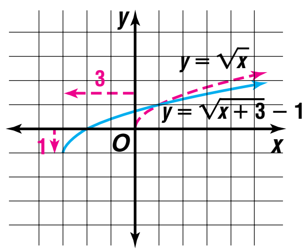
b. $y = \sqrt{x - 4}$

This function is of the form $y = f(x - 4)$. Since 4 is being subtracted from x before being evaluated by the parent function, the graph of the parent function $y = \sqrt{x}$ slides 4 units right.



c. $y = \sqrt{x + 3} - 1$

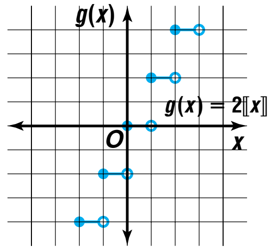
This function is of the form $y = f(x + 3) - 1$. The addition of 3 indicates a slide of 3 units left, and the subtraction of 1 moves the parent function $y = \sqrt{x}$ down 1 unit.



Remember that a dilation has the effect of shrinking or enlarging a figure. Likewise, when the leading coefficient of x is not 1, the function is expanded or compressed.

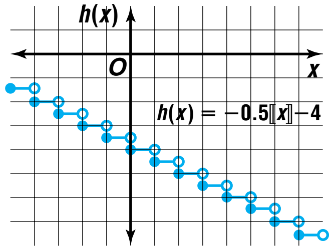
Example 3 Graph each function. Then describe how it is related to its parent graph.

a. $g(x) = 2[\![x]\!]$



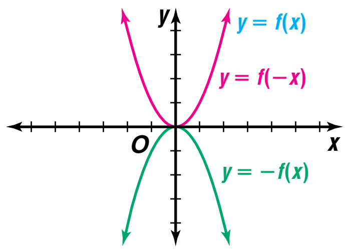
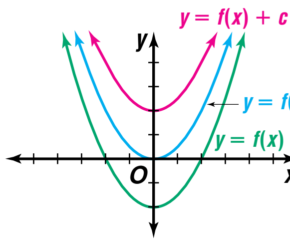
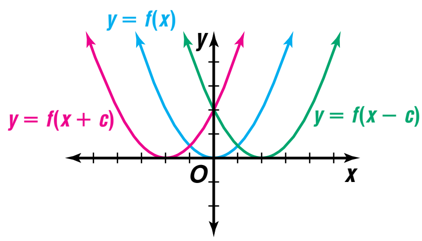
The parent graph is the greatest integer function, $f(x) = [\![x]\!]$. $g(x) = 2[\![x]\!]$ is a vertical expansion by a factor of 2. The vertical distance between the steps is 2 units.

b. $h(x) = -0.5[\![x]\!] - 4$



$h(x) = -0.5[\![x]\!] - 4$ reflects the parent graph over the x-axis, compresses it vertically by a factor of 0.5, and shifts the graph down 4 units. *Notice that multiplying by a positive number less than 1 compresses the graph vertically.*

The following chart summarizes the relationships in families of graphs. The parent graph may differ, but the transformations of the graphs have the same effect. Remember that more than one transformation may affect a parent graph.

Change to the Parent Function $y = f(x)$, $c > 0$	Change to Parent Graph	Examples
Reflections $y = -f(x)$ $y = f(-x)$	Is reflected over the x-axis. Is reflected over the y-axis.	
Translations $y = f(x) + c$ $y = f(x) - c$	Translates the graph c units up. Translates the graph c units down.	
$y = f(x + c)$ $y = f(x - c)$	Translates the graph c units left. Translates the graph c units right.	

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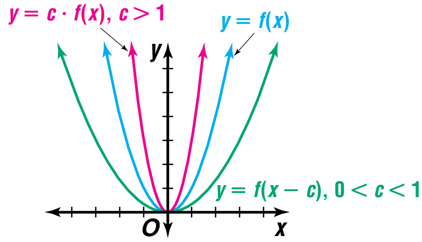
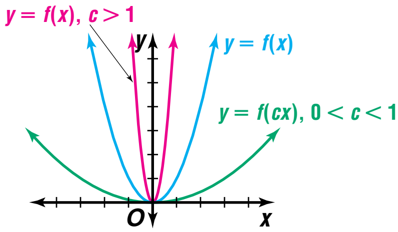
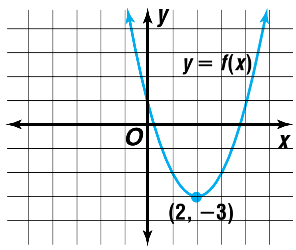
Change to the Parent Function $y = f(x)$, $c > 0$	Change to Parent Graph	Examples
Dilations		
$y = c \cdot f(x)$, $c > 1$ $y = c \cdot f(x)$, $0 < c < 1$	Expands the graph vertically. Compresses the graph vertically.	
$y = f(cx)$, $c > 1$ $y = f(cx)$, $0 < c < 1$	Compresses the graph horizontally. Expands the graph horizontally.	

Other transformations may affect the appearance of the graph. We will look at two more transformations that change the shape of the graph.

Example 4 **Observe the graph of each function. Describe how the graphs in parts b and c relate to the graph in part a.**

a. $f(x) = (x - 2)^2 - 3$

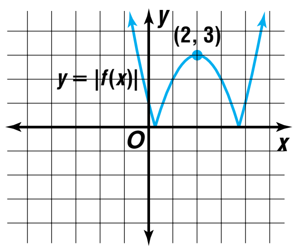
The graph of $y = f(x)$ is a translation of $y = x^2$. The parent graph has been translated right 2 units and down 3 units.



b. $y = |f(x)|$

$|f(x)| = |(x - 2)^2 - 3|$

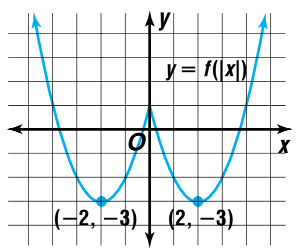
This transformation reflects any portion of the parent graph that is below the x-axis so that it is above the x-axis.



c. $y = f(|x|)$

$f(|x|) = (|x| - 2)^2 - 3$

This transformation results in the portion of the parent graph on the left of the y-axis being replaced by a reflection of the portion on the right of the y-axis.



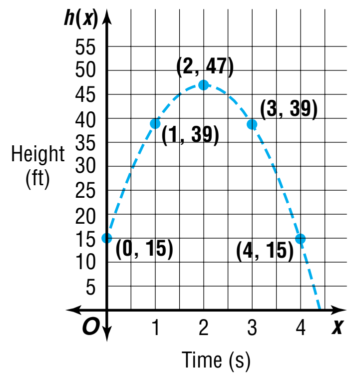
Example 5 Real World Application

ENTERTAINMENT **A traveling circus invites local schools to send math and science teams to its Science Challenge Day. One challenge is to write an equation that most accurately predicts the height of the flight of a human cannonball performer at any given time. Students collect data by witnessing a performance and examining time-lapse photographs of the flight. Using the performer's initial height of 15 feet and the photographs, one team records the data at the right. Write the equation of the related parabola that models the data.**

Time (seconds)	Height (feet)
0	15
1	39
2	47
3	39
4	15

A graph of the data reveals that a parabola is the best model for the data. The parent graph of a parabola is the graph of the equation $y = x^2$. To write the equation of the related parabola that models the data, we need to compare points located near the vertex of each graph. An analysis of the transformation these points have undergone will help us determine the equation of the transformed parabola.

From the graph, we can see that parent graph has been turned upside-down, indicating that the equation for this parabola has been multiplied by some negative constant c. Through further inspection of the graph and its data points, we can see that the vertex of the parent graph has been translated to the point (2, 47). Therefore, an equation that models the data is $y = c(x - 2)^2 + 47$.



To find c, compare points near the vertex of the graphs of the parent function $f(x)$ and the graph of the data points. Look at the relationship between the differences in the y-coordinates for each set of points.

x	$f(x) = x^2$	
−2	4	
−1	1	−3
0	0	−1
1	1	−1
2	4	−3

x Time (seconds)	$h(x)$ Height (feet)	
0	15	
1	39	24
2	47	8
3	39	8
4	15	24

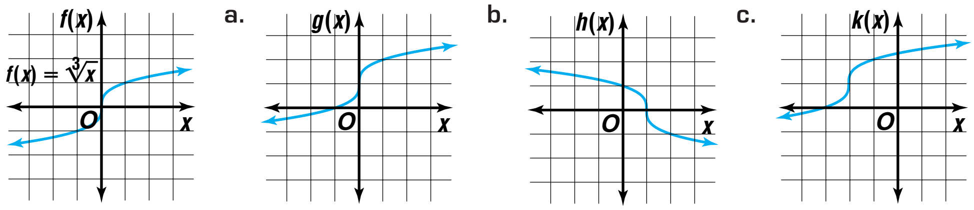
These differences are in a ratio of 1 to −8. This means that the graph of the parent graph has been expanded by a factor of −8. Thus, an equation that models the data is $y = -8(x - 2)^2 + 47$.

CHECK FOR UNDERSTANDING

Communicating Mathematics

Read and study the lesson to answer each question.

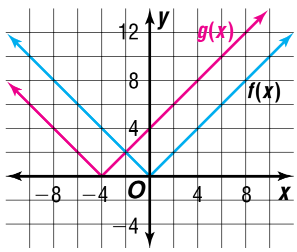
1. **Write** the equation of the graph obtained when the parent graph $y = x^3$ is translated 4 units left and 7 units down.
2. **Explain** the difference between the graphs of $y = (x + 3)^2$ and $y = x^2 + 3$.
3. **Name** two types of transformations for which the pre-image and the image are congruent figures.
4. **Describe** the differences between the graphs of $y = f(x)$ and $y = f(cx)$ for $c > 0$.
5. **Write** equations for the graphs of $g(x)$, $h(x)$, and $k(x)$ if the graph of $f(x) = \sqrt[3]{x}$ is the parent graph.



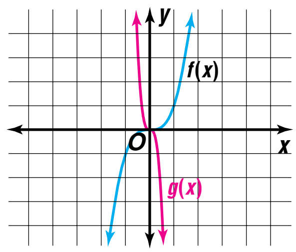
Guided Practice

Describe how the graphs of $f(x)$ and $g(x)$ are related.

6. $f(x) = |x|$ and $g(x) = |x + 4|$



7. $f(x) = x^3$ and $g(x) = -(3x)^3$



Use the graph of the given parent function to describe the graph of each related function.

8. $f(x) = x^2$
   a. $y = (0.2x)^2$
   b. $y = (x - 5)^2 - 2$
   c. $y = 3x^2 + 6$

9. $f(x) = x^3$
   a. $y = |x^3 + 3|$
   b. $y = -(2x)^3$
   c. $y = 0.75(x + 1)^3$

Sketch the graph of each function.

10. $f(x) = 2(x - 3)^3$

11. $g(x) = (0.5x)^2 - 1$

12. **Consumer Costs** The cost of labor for servicing cars at B & B Automotive is \$50 for each whole hour or for any fraction of an hour.
    a. Graph the function that describes the cost for x hours of labor.
    b. Graph the function that would show a \$25 additional charge if you decide to also get the oil changed and fluids checked.
    c. What would be the cost of servicing a car that required 3.45 hours of labor if the owner requested that the oil be changed and the fluids be checked?

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CONTENTS

EXERCISES

Practice

Describe how the graphs of $f(x)$ and $g(x)$ are related.

13. $f(x) = x$ and $g(x) = x + 6$

14. $f(x) = x^2$ and $g(x) = \frac{3}{4}x^2$

15. $f(x) = |x|$ and $g(x) = |5x|$

16. $f(x) = x^3$ and $g(x) = (x - 5)^3$

17. $f(x) = \frac{1}{x}$ and $g(x) = \frac{3}{x}$

18. $f(x) = [\![x]\!] + 1$ and $g(x) = -[\![x]\!] - 1$

19. Describe the relationship between the graphs of $f(x) = \sqrt{x}$ and $g(x) = -\sqrt{0.4x} + 3$.

Use the graph of the given parent function to describe the graph of each related function.

20. $f(x) = x^2$
- **a.** $y = -(1.5x)^2$
- **b.** $y = 4(x - 3)^2$
- **c.** $y = \frac{1}{2}x^2 - 5$

21. $f(x) = |x|$
- **a.** $y = |0.2x|$
- **b.** $y = 7|x| - 0.4$
- **c.** $y = -9|x + 1|$

22. $f(x) = x^3$
- **a.** $y = (x + 2)^3 - 5$
- **b.** $y = -(0.8x)^3$
- **c.** $y = \left(\frac{5}{3}x\right)^3 + 2$

23. $f(x) = \sqrt{x}$
- **a.** $y = \frac{1}{3}\sqrt{x + 2}$
- **b.** $y = \sqrt{-x} - 7$
- **c.** $y = 4 + 2\sqrt{x - 3}$

24. $f(x) = \frac{1}{x}$
- **a.** $y = \frac{1}{0.5x}$
- **b.** $y = \frac{1}{6x} + 8$
- **c.** $y = \frac{1}{|x|}$

25. $f(x) = [\![x]\!]$
- **a.** $y = \left[\!\left[\frac{5}{2}x\right]\!\right] - 3$
- **b.** $y = -0.75[\![x]\!]$
- **c.** $y = [\![\,|x| - 4]\!]$

26. Name the parent graph of $m(x) = |-9 + (0.75x)^2|$. Then sketch the graph of $m(x)$.

27. Write the equation of the graph obtained when the graph of $y = \frac{1}{x}$ is compressed vertically by a factor of 0.25, translated 4 units right, and then translated 3 units up.

Sketch the graph of each function.

28. $f(x) = -(x + 4) + 5$

29. $g(x) = |x^2 - 4|$

30. $h(x) = (0.5x - 1)^3$

31. $n(x) = -2.5[\![x]\!] + 3$

32. $q(x) = -4|x - 2| - 1$

33. $k(x) = -\frac{1}{2}(x - 3)^2 - 4$

34. Graph $y = f(x)$ and $y = f(|x|)$ on the same set of axes if $f(x) = (x + 3)^2 - 8$.

Graphing Calculator

Use a graphing calculator to graph each set of functions on the same screen. Name the x-intercept(s) of each function.

35.
- **a.** $y = x^2$
- **b.** $y = (4x - 2)^2$
- **c.** $y = (2x + 3)^2$

36.
- **a.** $y = x^3$
- **b.** $y = (3x - 2)^3$
- **c.** $y = (4x + 1)^3$

37.
- **a.** $y = \sqrt{x}$
- **b.** $y = \sqrt{2x + 5}$
- **c.** $y = \sqrt{5x - 3}$

Applications and Problem Solving

38. Technology Transformations can be used to create a series of graphs that appear to move when shown sequentially on a video screen. Suppose you start with the graph of $y = f(x)$. Describe the effect of graphing the following functions in succession if n has integer values from 1 to 100.

a. $y_n = f(x + 2n) - 3n$.

b. $y_n = (-1)^n f(x - n)$.

39. Critical Thinking Study the coordinates of the x-intercepts you found in the related graphs in Exercises 35–37. Make a conjecture about the x-intercept of $y = (ax + b)^n$ if $y = x^n$ is the parent function.

40. Business The standard cost of a taxi fare is \$1.50 for the first unit and 25 cents for each additional unit. A unit is composed of distance (one unit equals 0.2 mile) and/or wait time (one unit equals 75 seconds). As the cab moves at more than 9.6 miles per hour, the taxi's meter clocks distance. When the cab is stopped or moving at less than 9.6 miles per hour, the meter clocks time. Thus, traveling 0.1 mile and then waiting at a stop light for 37.5 seconds generates one unit and a 25-cent charge.

a. Assuming that the cab meter rounds up to the nearest unit, write a function that would determine the cost for x units of cab fare, where $x > 0$.

b. Graph the function found in part a.

41. Geometry Suppose $f(x) = 5 - |x - 6|$.

a. Sketch the graph of $f(x)$ and calculate the area of the triangle formed by $f(x)$ and the positive x-axis.

b. Sketch the graph of $y = 2f(x)$ and calculate the area of the new triangle formed by $2f(x)$ and the positive x-axis. How do the areas of part a and part b compare? Make a conjecture about the area of the triangle formed by $y = c \cdot f(x)$ in the first quadrant if $c \geq 0$.

c. Sketch the graph of $y = f(x - 3)$ and recalculate the area of the triangle formed by $f(x - 3)$ and the positive x-axis. How do the areas of part a and part c compare? Make a conjecture about the area of the triangle formed by $y = f(x - c)$ in the first quadrant if $c \geq 0$.

42. Critical Thinking Study the parent graphs at the beginning of this lesson.

a. Select a parent graph or a modification of a parent graph that meets each of the following specifications.
(1) positive at its leftmost points and positive at its rightmost points
(2) negative at its leftmost points and positive at its rightmost points
(3) negative at its leftmost points and negative at its rightmost points
(4) positive at its leftmost points and negative at its rightmost points

b. Sketch the related graph for each parent graph that is translated 3 units right and 5 units down.

c. Write an equation for each related graph.

43. **Critical Thinking** Suppose a reflection, a translation, or a dilation were applied to an even function.

a. Which transformations would result in another even function?

b. Which transformations would result in a function that is no longer even?

Mixed Review

44. Is the graph of $f(x) = x^{17} - x^{15}$ symmetric with respect to the origin? Explain. *(Lesson 3-1)*

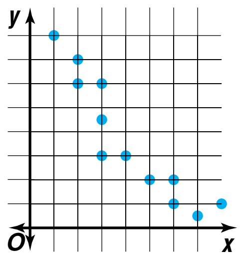
45. **Child Care** Elisa Dimas is the manager for the Learning Loft Day Care Center. The center offers all day service for preschool children for $18 per day and after school only service for $6 per day. Fire codes permit only 50 children in the building at one time. State law dictates that a child care worker can be responsible for a maximum of 3 preschool children and 5 school-age children at one time. Ms. Dimas has ten child care workers available to work at the center during the week. How many children of each age group should Ms. Dimas accept to maximize the daily income of the center? *(Lesson 2-7)*

46. **Geometry** Triangle ABC is represented by the matrix $\begin{bmatrix} 5 & 1 & -2 \\ -4 & 3 & -1 \end{bmatrix}$. Find the image of the triangle after a 90° counterclockwise rotation about the origin. *(Lesson 2-4)*

47. Find the values of x, y, and z for which $\begin{bmatrix} x^2 & 7 & 9 \\ 5 & 12 & 6 \end{bmatrix} = \begin{bmatrix} 25 & 7 & y \\ 5 & 2z & 6 \end{bmatrix}$ is true. *(Lesson 2-2)*

48. Solve the system of equations algebraically. *(Lesson 2-1)*
$6x + 5y = -14$
$5x + 2y = -3$

49. Describe the linear relationship implied in the scatter plot at the right. *(Lesson 1-6)*



50. Find the slope of a line perpendicular to a line whose equation is $3x - 4y = 0$. *(Lesson 1-5)*

51. **Fund-Raising** The Band Boosters at Palermo High School are having their annual doughnut sale to raise money for new equipment. The equation $5d - 2p = 500$ represents the amount of profit p in dollars the band will make selling d boxes of doughnuts. What is the p-intercept of the line represented by this equation? *(Lesson 1-3)*

52. Find $[f \circ g](x)$ and $[g \circ f](x)$ if $f(x) = \frac{2}{3}x - 2$ and $g(x) = x^2 - 6x + 9$. *(Lesson 1-2)*

53. **SAT/ACT Practice** If $d = m - \frac{50}{m}$ and m is a positive number that increases in value, then d

A increases in value.
B increases, then decreases.
C remains unchanged.
D decreases in value.
E decreases, then increases.

Extra Practice See p. A30.