3-3

Graphs of Nonlinear Inequalities

OBJECTIVES
- Graph polynomial, absolute value, and radical inequalities in two variables.
- Solve absolute value inequalities.

Real World Application

PHARMACOLOGY Pharmacists label medication as to how much and how often it should be taken. Because oral medication requires time to take effect, the amount of medication in your body varies with time. Suppose the equation $m(x) = 0.5x^4 + 3.45x^3 - 96.65x^2 + 347.7x$ for $0 < x \le 6$ models the number of milligrams of a certain pain reliever in the bloodstream x hours after taking 400 milligrams of it. The medicine is to be taken every 4 hours. At what times during the first 4-hour period is the level of pain reliever in the bloodstream above 300 milligrams? *This problem will be solved in Example 5.*

Problems like the one above can be solved by graphing inequalities. Graphing inequalities in two variables identifies all ordered pairs that will satisfy the inequality.

Example 1 **Determine whether (3, −4), (4, 7), (1, 1), and (−1, 6) are solutions for the inequality $y \le (x - 2)^2 - 3$.**

Substitute the x-value and y-value from each ordered pair into the inequality.

$$\begin{aligned} y &\le (x-2)^2 - 3 \\ -4 &\overset{?}{\le} (3-2)^2 - 3 \qquad (x, y) = (3, -4) \\ -4 &\le -2 \checkmark \qquad\qquad\quad \text{true} \end{aligned}$$

$$\begin{aligned} y &\le (x-2)^2 - 3 \\ 7 &\overset{?}{\le} (4-2)^2 - 3 \qquad (x, y) = (4, 7) \\ 7 &\le 1 \qquad\qquad\qquad\quad \text{false} \end{aligned}$$

$$\begin{aligned} y &\le (x-2)^2 - 3 \\ 1 &\overset{?}{\le} (1-2)^2 - 3 \qquad (x, y) = (1, 1) \\ 1 &\le -2 \qquad\qquad\qquad \text{false} \end{aligned}$$

$$\begin{aligned} y &\le (x-2)^2 - 3 \\ 6 &\overset{?}{\le} (-1-2)^2 - 3 \qquad (x, y) = (-1, 6) \\ 6 &\le 6 \checkmark \qquad\qquad\qquad \text{true} \end{aligned}$$

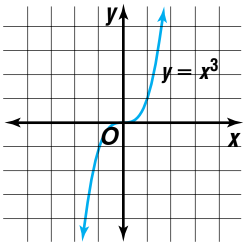
Of these ordered pairs, (3, −4) and (−1, 6) are solutions for $y \le (x - 2)^2 - 3$.

Look Back Refer to Lesson 1-8 for more about graphing linear inequalities.

Similar to graphing linear inequalities, the first step in graphing nonlinear inequalities is graphing the boundary. You can use concepts from Lesson 3-2 to graph the boundary.

Example 2 **Graph $y \ge (x - 4)^3 - 2$.**

The boundary of the inequality is the graph of $y = (x - 4)^3 - 2$. To graph the boundary curve, start with the parent graph $y = x^3$. Analyze the boundary equation to determine how the boundary relates to the parent graph.



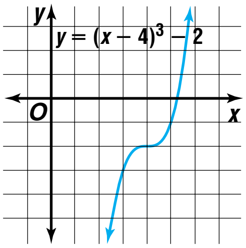
$$y = (x - \underset{\uparrow}{4})^3 - \underset{\uparrow}{2}$$

move 4 units right *move 2 units down*

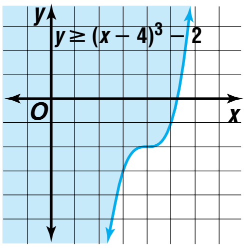
Graphing Calculator Appendix

For keystroke instruction on how to graph inequalities see pages A13-A15.

Since the boundary is included in the inequality, the graph is drawn as a solid curve.



The inequality states that the y-values of the solution are greater than the y-values on the graph of $y = (x - 4)^3 - 2$. For a particular value of x, all of the points in the plane that lie above the curve have y-values greater than $y = (x - 4)^3 - 2$. So this portion of the graph should be shaded.



To verify numerically, you can test a point not on the boundary. *It is common to test (0, 0) whenever it is not on the boundary.*

$y \geq (x - 4)^3 - 2$

$0 \overset{?}{\geq} (0 - 4)^3 - 2$ *Replace (x, y) with (0, 0).*

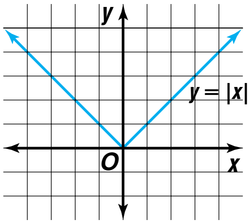
$0 \geq -66$ ✓ *True*

Since (0, 0) satisfies the inequality, the correct region is shaded.

The same process used in Example 2 can be used to graph inequalities involving absolute value.

Example ③ **Graph $y > 3 - |x + 2|$.**

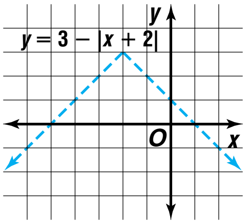
Begin with the parent graph $y = |x|$.



It is easier to sketch the graph of the given inequality if you rewrite it so that the absolute value expression comes first.

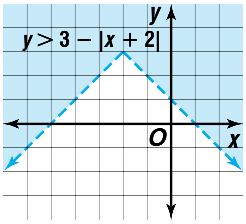
$$y = 3 - |x + 2| \quad \rightarrow \quad y = -|x + 2| + 3$$

This more familiar form tells us the parent graph is reflected over the x-axis and moved 2 units left and three units up. The boundary is not included, so draw it as a dashed line.



The y-values of the solution are greater than the y-values on the graph of $y = 3 - |x + 2|$, so shade above the graph of $y = 3 - |x + 2|$.

Verify by substituting (0, 0) in the inequality to obtain $0 > 1$. Since this statement is false, the part of the graph containing (0, 0) should not be shaded. Thus, the graph is correct.



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To solve absolute value inequalities algebraically, use the definition of absolute value to determine the solution set. That is, if $a < 0$, then $|a| = -a$, and if $a \geq 0$, then $|a| = a$.

Example **4** **Solve $|x - 2| - 5 < 4$.**

There are two cases that must be solved. In one case, $x - 2$ is negative, and in the other, $x - 2$ is positive.

Case 1

$$\begin{aligned} (x - 2) &< 0 \\ |x - 2| - 5 &< 4 \\ -(x - 2) - 5 &< 4 \quad |x - 2| = -(x - 2) \\ -x + 2 - 5 &< 4 \\ -x &< 7 \\ x &> -7 \end{aligned}$$

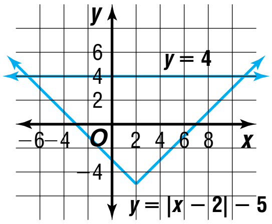
Case 2

$$\begin{aligned} (x - 2) &> 0 \\ |x - 2| - 5 &< 4 \\ x - 2 - 5 &< 4 \quad |x - 2| = (x - 2) \\ x - 7 &< 4 \\ x &< 11 \end{aligned}$$

The solution set is $\{x \mid -7 < x < 11\}$. *$\{x \mid -7 < x < 11\}$ is read "the set of all numbers x such that x is between −7 and 11."*

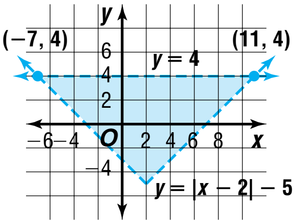
Verify this solution by graphing.

First, graph $y = |x - 2| - 5$. Since we are solving $|x - 2| - 5 < 4$ and $|x - 2| - 5 = y$, we are looking for a region in which $y = 4$. Therefore, graph $y = 4$ and graph it on the same set of axes.



Identify the points of intersection of the two graphs. By inspecting the graph, we can see that they intersect at $(-7, 4)$ and $(11, 4)$.

Now shade the region where the graphs of the inequalities $y > |x - 2| - 5$ and $y < 4$ intersect. This occurs in the region of the graph where $-7 < x < 11$. Thus, the solution to $|x - 2| - 5 < 4$ is the set of x-values such that $-7 < x < 11$.



Nonlinear inequalities have applications to many real-world situations, including business, education, and medicine.

Example **5** **PHARMACOLOGY** **Refer to the application at the beginning of the lesson. At what times during the first 4-hour period is the amount of pain reliever in the bloodstream above 300 milligrams?**

Real World Application

Since we need to know when the level of pain reliever in the bloodstream is above 300 milligrams, we can write an inequality.

$0.5x^4 + 3.45x^3 - 96.65x^2 + 347.7x > 300$ for $0 < x \leq 6$

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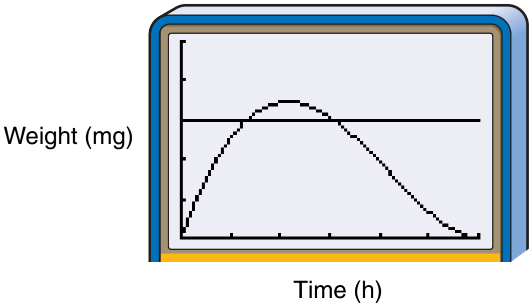
Let $y = 0.5x^4 + 3.45x^3 - 96.65x^2 + 347.7x$ and $y = 300$. Graph both equations on the same set of axes using a graphing calculator.

Calculating the points of intersection, we find that the two equations intersect at about (1.3, 300) and (3.1, 300). Therefore, when $1.3 < x < 3.1$, the amount of pain reliever in the bloodstream is above 300 milligrams. That is, the amount exceeds 300 milligrams between about 1 hour 18 minutes and 3 hours 6 minutes after taking the medication.



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CHECK FOR UNDERSTANDING

Communicating Mathematics

Read and study the lesson to answer each question.

1. **Describe** how knowledge of transformations can help you graph the inequality $y \leq 5 + \sqrt{x - 2}$.
2. **State** the two cases considered when solving a one-variable absolute value inequality algebraically.
3. **Write** a procedure for determining which region of the graph of an inequality should be shaded.
4. *Math Journal* **Sketch** the graphs of $y = |x - 3| + 2$ and $y = 1$ on the same set of axes. Use your sketch to solve the inequality $|x - 3| + 2 < 1$. If no solution exists, write *no solution.* Write a paragraph to explain your answer.

Guided Practice

Determine whether the ordered pair is a solution for the given inequality. Write *yes* or *no*.

5. $y \geq -5x^4 + 7x^3 + 8$, $(-1, -3)$
6. $y < |3x - 4| - 1$, $(0, 3)$

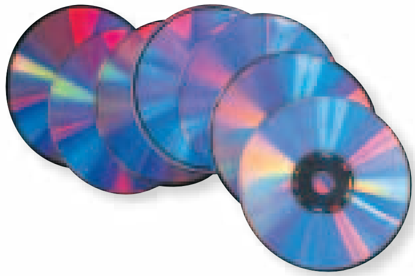
Graph each inequality.

7. $y \leq (x + 1)^3$
8. $y \leq 2(x - 3)^2$
9. $y > -|x - 4| + 2$

Solve each inequality.

10. $|x + 6| > 4$
11. $|3x - 4| \leq x$

12. **Manufacturing** The AccuData Company makes compact disks measuring 12 centimeters in diameter. The diameters of the disks can vary no more than 50 micrometers or 5×10^{-3} centimeter.
    a. Write an absolute value inequality to represent the range of diameters of compact disks.
    b. What are the largest and smallest diameters that are allowable?



EXERCISES

Practice

Determine whether the ordered pair is a solution for the given inequality. Write *yes* or *no*.

13. $y < x^3 - 4x^2 + 2$, $(1, 0)$

14. $y < |x - 2| + 7$, $(3, 8)$

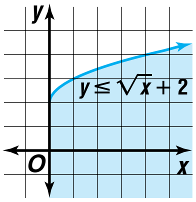
15. $y > -\sqrt{x + 11} + 1$, $(-2, -1)$

16. $y < -0.2x^2 + 9x - 7$, $(10, 63)$

17. $y \leq \frac{x^2 - 6}{x}$, $(-6, -9)$

18. $y \geq 2|x|^3 - 7$, $(0, 0)$

19. Which of the ordered pairs, $(0, 0)$, $(1, 4)$, $(1, 1)$, $(-1, 0)$, and $(1, -1)$, is a solution for $y \leq \sqrt{x} + 2$? How can you use these results to determine if the graph at the right is correct?



Graph each inequality.

20. $y \leq x^2 - 4$

21. $y > \sqrt{0.5x}$

22. $y < |x - 9|$

23. $y > |2x| + 3$

24. $y < (x - 5)^2$

25. $y \geq -x^3$

26. $y > -(0.4x)^2$

27. $y \leq |3(x - 4)|$

28. $y < \sqrt{x + 3} + 5$

29. $y \geq (x - 1)^2 - 3$

30. $y \geq (2x + 1)^3 + 2$

31. $y \leq -3|x - 2| + 4$

32. Sketch the graph of the inequality $y \geq x^3 - 6x^2 + 12x - 8$.

Solve each inequality.

33. $|x + 4| > 5$

34. $|3x + 12| \geq 42$

35. $|7 - 2x| - 8 < 3$

36. $|5 - x| \leq x$

37. $|5x - 8| < 0$

38. $|2x + 9| - 2x \geq 0$

39. Find all values of x that satisfy $-\frac{2}{3}|x + 5| \geq -8$.

Applications and Problem Solving

40. Chemistry Katie and Wes worked together on a chemistry lab. They determined the quantity of the unknown in their sample to be 37.5 ± 1.2 grams. If the actual quantity of unknown is x, write their results as an absolute value inequality. Solve for x to find the range of possible values of x.

41. Critical Thinking Solve $3|x - 7| < |x - 1|$.

42. Critical Thinking Find the area of the region described by $y \geq 2|x - 3| + 4$ and $x - 2y \geq -20$.

43. Education Amanda's teacher calculates grades using a weighted average. She counts homework as 10%, quizzes as 15%, projects as 20%, tests as 40%, and the final exam as 15% of the final grade. Going into the final, Amanda has scores of 90 for homework, 75 for quizzes, 76 for projects, and 80 for tests. What grade does Amanda need on the final exam if she wants to get an overall grade of at least an 80?

44. Critical Thinking Consider the equation $|(x - 3)^2 - 4| = b$. Determine the value(s) of b so that the equation has

a. no solution.

b. one solution.

c. two solutions.

d. three solutions.

e. four solutions.

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45. **Business** After opening a cookie store in the mall, Paul and Carol Mason hired an consultant to provide them with information on increasing their profit. The consultant told them that their profit P depended on the number of cookies x that they sold according to the relation $P(x) \leq -0.005(x - 1200)^2 + 400$. They typically sell between 950 and 1000 cookies in a given day.

a. Sketch a graph to model this situation.

b. Explain the significance of the shaded region.

Mixed Review

46. How are the graphs of $f(x) = x^3$ and $g(x) = -2x^3$ related? *(Lesson 3-2)*

47. Determine whether the graph of $y = -\frac{1}{x^4}$ is symmetric with respect to the x-axis, y-axis, the line $y = x$, the line $y = -x$, or none of these. *(Lesson 3-1)*

48. Find the inverse of $\begin{bmatrix} 8 & -3 \\ 4 & -5 \end{bmatrix}$. *(Lesson 2-5)*

49. Multiply $\begin{bmatrix} 8 & -7 \\ -4 & 0 \end{bmatrix}$ by $\frac{3}{4}$. *(Lesson 2-3)*

50. Graph $y = 3|x| + 5$. *(Lesson 1-7).*

51. **Criminal Justice** The table shows the number of states with teen courts over a period of several years. Make a scatter plot of the data. *(Lesson 1-6)*

Year	States with Teen Courts
1976	2
1991	14
1994	17
1997	36
1999	47*

Source: American Probation and Parole Association.

*Includes District of Columbia

52. Find $[f \circ g](4)$ and $[g \circ f](4)$ for $f(x) = 5x + 9$ and $g(x) = 0.5x - 1$. *(Lesson 1-2)*

53. **SAT Practice** **Grid-In** Student A is 15 years old. Student B is one-third older. How many years ago was student B twice as old as student A?

MID-CHAPTER QUIZ

Determine whether each graph is symmetric with respect to the *x*-axis, the *y*-axis, the line *y* = *x*, the line *y* = −*x*, the origin, or none of these. (Lesson 3-1)

1. $x^2 + y^2 - 9 = 0$

2. $5x^2 + 6x - 9 = y$

3. $x = \frac{7}{y}$

4. $y = |x| + 1$

Use the graph of the given parent function to describe the graph of each related function. (Lesson 3-2)

5. $f(x) = [\![x]\!]$
    - a. $y = [\![x]\!] - 2$
    - b. $y = -[\![x - 3]\!]$
    - c. $y = \frac{1}{4}[\![x]\!] + 1$

6. $f(x) = x^3$
    - a. $y = 3x^3$
    - b. $y = (0.5x)^3 - 1$
    - c. $y = (x + 1)^3 + 4$

7. Sketch the graph of $g(x) = -0.5(x - 2)^2 + 3$. (Lesson 3-2)

8. Graph the inequality $y \geq \left(\frac{1}{3}x\right)^2 + 2$. (Lesson 3-3)

9. Find all values of x that satisfy $|2x - 7| < 15$. (Lesson 3-3)

10. **Technology** In September of 1999, a polling organization reported that 64% of Americans were "not very" or "not at all" concerned about the Year-2000 computer bug, with a margin of error of 3%. Write and solve an absolute value inequality to describe the range of the possible percent of Americans who were relatively unconcerned about the "Y2K bug." (Lesson 3-3)

Source: The Gallup Organization

Extra Practice See p. A30.

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