3-4 Inverse Functions and Relations

OBJECTIVES
- Determine inverses of relations and functions.
- Graph functions and their inverses.

Real World Application

METEOROLOGY The hottest temperature ever recorded in Montana was 117° F on July 5, 1937. To convert this temperature to degrees Celsius C, subtract 32° from the Fahrenheit temperature F and then multiply the result by $\frac{5}{9}$. The formula for this conversion is $C = \frac{5}{9}(F - 32)$. The coldest temperature ever recorded in Montana was −57° C on January 20, 1954. To convert this temperature to Fahrenheit, multiply the Celsius temperature by $\frac{9}{5}$ and then add 32°. The formula for this conversion is $F = \frac{9}{5}C + 32$.

The temperature conversion formulas are examples of **inverse functions.** Relations also have inverses, and these inverses are themselves relations.

Inverse Relations	Two relations are inverse relations if and only if one relation contains the element (b, a) whenever the other relation contains the element (a, b).

If $f(x)$ denotes a function, then $f^{-1}(x)$ denotes the inverse of $f(x)$. However, $f^{-1}(x)$ may not necessarily be a function. To graph a function or relation and its inverse, you switch the x- and y-coordinates of the ordered pairs of the function. This results in graphs that are symmetric to each other with respect to the line $y = x$.

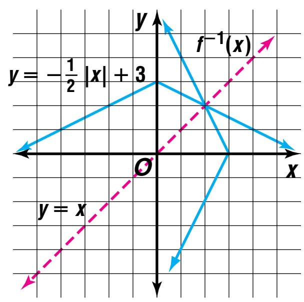
Example 1 **Graph $f(x) = -\frac{1}{2}|x| + 3$ and its inverse.**

To graph the function, let $y = f(x)$. To graph $f^{-1}(x)$, interchange the x- and y-coordinates of the ordered pairs of the function.

Note that the domain of one relation or function is the range of the inverse and vice versa.

$f(x) = -\frac{1}{2}\|x\| + 3$	
x	$f(x)$
−3	1.5
−2	2
−1	2.5
0	3
1	2.5
2	2
3	1.5

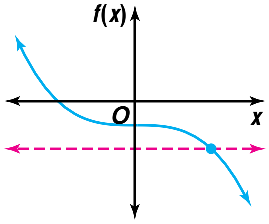
$f^{-1}(x)$	
x	$f^{-1}(x)$
1.5	−3
2	−2
2.5	−1
3	0
2.5	1
2	2
1.5	3



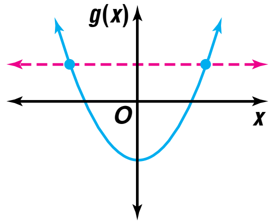
The graph of $f^{-1}(x)$ is the reflection of $f(x)$ over the line $y = x$.

Note that the inverse in Example 1 is not a function because it fails the vertical line test.

You can use the **horizontal line test** to determine if the inverse of a relation will be a function. If every horizontal line intersects the graph of the relation in at most one point, then the inverse of the relation is a function.



The inverse of f(x) is a function.



The inverse of g(x) is not a function.

You can find the inverse of a relation algebraically. First, let $y = f(x)$. Then interchange x and y. Finally, solve the resulting equation for y.

Example 2 **Consider $f(x) = (x + 3)^2 - 5$.**

a. Is the inverse of $f(x)$ a function?

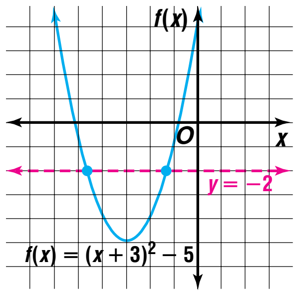
b. Find $f^{-1}(x)$.

c. Graph $f(x)$ and $f^{-1}(x)$ using a graphing calculator.

Graphing Calculator Tip

You can differentiate the appearance of your graphs by highlighting the symbol in front of each equation in the **Y=** list and pressing ENTER to select line (\), thick (\), or dot (·.).

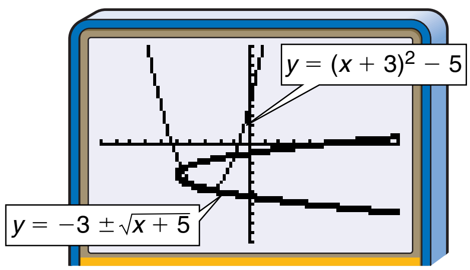
a. Since the line $y = -2$ intersects the graph of $f(x)$ at more than one point, the function fails the horizontal line test. Thus, the inverse of $f(x)$ is not a function.



b. To find $f^{-1}(x)$, let $y = f(x)$ and interchange x and y. Then, solve for y.

$y = (x + 3)^2 - 5$	*Let $y = f(x)$.*
$x = (y + 3)^2 - 5$	*Interchange x and y.*
$x + 5 = (y + 3)^2$	*Isolate the expression containing y.*
$\pm\sqrt{x + 5} = y + 3$	*Take the square root of each side.*
$y = -3 \pm \sqrt{x + 5}$	*Solve for y.*
$f^{-1}(x) = -3 \pm \sqrt{x + 5}$	*Replace y with $f^{-1}(x)$.*

c. To graph $f(x)$ and its inverse, enter the equations $y = (x + 3)^2 - 5$, $y = -3 + \sqrt{x + 5}$, and $y = -3 - \sqrt{x + 5}$ in the same viewing window.

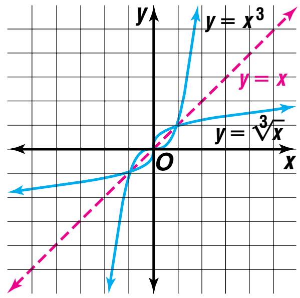


[−15.16, 15.16] scl: 2 by [−10, 10] scl: 2

You can graph a function by using the parent graph of an inverse function.

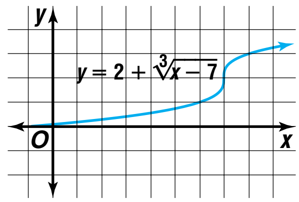
Example 3 **Graph $y = 2 + \sqrt[3]{x - 7}$.**

The parent function is $y = \sqrt[3]{x}$ which is the inverse of $y = x^3$.



To graph $y = \sqrt[3]{x}$, start with the graph of $y = x^3$. Reflect the graph over the line $y = x$.

To graph $y = 2 + \sqrt[3]{x - 7}$, translate the reflected graph 7 units to the right and 2 units up.



To find the inverse of a function, you use an **inverse process** to solve for y after switching variables. This inverse process can be used to solve many real-world problems.

Example 4 Real World Application

FINANCE **When the Garcias decided to begin investing, their financial advisor instructed them to set a goal. Their net pay is about 65% of their gross pay. They decided to subtract their monthly food allowance from their monthly net pay and then invest 10% of the remainder.**

a. Write an equation that gives the amount invested I as a function of their monthly gross pay G given that they allow \$450 per month for food.

b. Determine the equation for the inverse process and describe the real-world situation it models.

c. Determine the gross pay needed in order to invest \$100 per month.

a. One model for the amount they will invest is as follows.

investment	equals	10%	of	(65% of gross pay	less	food allowance)
↓	↓	↓	↓	↓	↓	↓
I	$=$	0.10	$\cdot$	$(0.65G$	$-$	$450)$

b. Solve for G.

$$I = 0.10(0.65G - 450)$$

$$10I = 0.65G - 450 \quad \textit{Multiply each side by 10.}$$

$$10I + 450 = 0.65G \quad \textit{Add 450 to each side.}$$

$$\frac{10I + 450}{0.65} = G \quad \textit{Divide each side by 0.65.}$$

This equation models the gross pay needed to meet a monthly investment goal I with the given conditions.

c. Substituting 100 for I gives $G = \frac{10I + 450}{0.65}$, or about \$2231. So the Garcias need to earn a monthly gross pay of about \$2231 in order to invest \$100 per month.

Look Back
Refer to Lesson 1-2 to review the composition of two functions.

If the inverse of a function is also a function, then a composition of the function and its inverse produces a unique result.

Consider $f(x) = 3x - 2$ and $f^{-1}(x) = \frac{x+2}{3}$. You can find functions $[f \circ f^{-1}](x)$ and $[f^{-1} \circ f](x)$ as follows.

$$
\begin{aligned}
[f \circ f^{-1}](x) &= f\left(\frac{x+2}{3}\right) && \textit{f}^{-1}(x) = \frac{x+2}{3} \\
&= 3\left(\frac{x+2}{3}\right) - 2 && f(x) = 3x - 2 \\
&= x
\end{aligned}
$$

$$
\begin{aligned}
[f^{-1} \circ f](x) &= f^{-1}(3x - 2) && f(x) = 3x - 2 \\
&= \frac{(3x-2)+2}{3} && f^{-1}(x) = \frac{x+2}{3} \\
&= x
\end{aligned}
$$

This leads to the formal definition of inverse functions.

Inverse Functions

Two functions, f and f^{-1}, are inverse functions if and only if $[f \circ f^{-1}](x) = [f^{-1} \circ f](x) = x$.

Example 5 **Given $f(x) = 4x - 9$, find $f^{-1}(x)$, and verify that f and f^{-1} are inverse functions.**

$$
\begin{aligned}
y &= 4x - 9 && f(x) = y \\
x &= 4y - 9 && \textit{Interchange } x \textit{ and } y. \\
x + 9 &= 4y && \textit{Solve for } y. \\
\frac{x+9}{4} &= y \\
\frac{x+9}{4} &= f^{-1}(x) && \textit{Replace } y \textit{ with } f^{-1}(x).
\end{aligned}
$$

Now show that $[f \circ f^{-1}](x) = [f^{-1} \circ f](x) = x$.

$$
\begin{aligned}
[f \circ f^{-1}](x) &= f\left(\frac{x+9}{4}\right) \\
&= 4\left(\frac{x+9}{4}\right) - 9 \\
&= x
\end{aligned}
$$

$$
\begin{aligned}
[f^{-1} \circ f](x) &= f^{-1}(4x - 9) \\
&= \frac{(4x-9)+9}{4} \\
&= x
\end{aligned}
$$

Since $[f \circ f^{-1}](x) = [f^{-1} \circ f](x) = x$, f and f^{-1} are inverse functions.

CHECK FOR UNDERSTANDING

Communicating Mathematics

Read and study the lesson to answer each question.

1. **Write** an explanation of how to determine the equation for the inverse of the relation $y = \pm\sqrt{x - 3}$.
2. **Determine** the values of n for which $f(x) = x^n$ has an inverse that is a function. Assume that n is a whole number.

3. **Find a counterexample** to this statement: The inverse of a function is also a function.

4. **Show** how you know whether the inverse of a function is also a function without graphing the inverse.

5. **You Decide** Nitayah says that the inverse of $y = 3 \pm \sqrt{x+2}$ cannot be a function because $y = 3 \pm \sqrt{x+2}$ is not a function. Is she right? Explain.

Guided Practice

Graph each function and its inverse.

6. $f(x) = |x| + 1$
7. $f(x) = x^3 + 1$
8. $f(x) = -(x-3)^2 + 1$

Find $f^{-1}(x)$. Then state whether $f^{-1}(x)$ is a function.

9. $f(x) = -3x + 2$
10. $f(x) = \frac{1}{x^3}$
11. $f(x) = (x+2)^2 + 6$

12. Graph the equation $y = 3 \pm \sqrt{x+1}$ using the parent graph $p(x) = x^2$.

13. Given $f(x) = \frac{1}{2}x - 5$, find $f^{-1}(x)$. Then verify that f and f^{-1} are inverse functions.

14. **Finance** If you deposit \$1000 in a savings account with an interest rate of r compounded annually, then the balance in the account after 3 years is given by the function $B(r) = 1000(1 + r)^3$, where r is written as a decimal.
    a. Find a formula for the interest rate, r, required to achieve a balance of B in the account after 3 years.
    b. What interest rate will yield a balance of \$1100 after 3 years?

EXERCISES

Practice

Graph each function and its inverse.

15. $f(x) = |x| + 2$
16. $f(x) = |2x|$
17. $f(x) = x^3 - 2$
18. $f(x) = x^5 - 10$
19. $f(x) = [\![x]\!]$
20. $f(x) = 3$
21. $f(x) = x^2 + 2x + 4$
22. $f(x) = -(x+2)^2 - 5$
23. $f(x) = (x+1)^2 - 4$

24. For $f(x) = x^2 + 4$, find $f^{-1}(x)$. Then graph $f(x)$ and $f^{-1}(x)$.

Find $f^{-1}(x)$. Then state whether $f^{-1}(x)$ is a function.

25. $f(x) = 2x + 7$
26. $f(x) = -x - 2$
27. $f(x) = \frac{1}{x}$
28. $f(x) = -\frac{1}{x^2}$
29. $f(x) = (x-3)^2 + 7$
30. $f(x) = x^2 - 4x + 3$
31. $f(x) = \frac{1}{x+2}$
32. $f(x) = \frac{1}{(x-1)^2}$
33. $f(x) = -\frac{2}{(x-2)^3}$

34. If $g(x) = \frac{3}{x^2 + 2x}$, find $g^{-1}(x)$.

Graph each equation using the graph of the given parent function.

35. $f(x) = \sqrt{x+5}$, $p(x) = x^2$
36. $y = 1 \pm \sqrt{x-2}$, $p(x) = x^2$
37. $f(x) = -2 - \sqrt[3]{x+3}$, $p(x) = x^3$
38. $y = 2\sqrt[5]{x-4}$, $p(x) = x^5$

Given $f(x)$, find $f^{-1}(x)$. Then verify that f and f^{-1} are inverse functions.

39. $f(x) = -\frac{2}{3}x + \frac{1}{6}$
40. $f(x) = (x-3)^3 + 4$

Applications and Problem Solving

Real World Application

41. **Analytic Geometry** The function $d(x) = |x - 4|$ gives the distance between x and 4 on the number line.
    a. Graph $d^{-1}(x)$.
    b. Is $d^{-1}(x)$ a function? Why or why not?
    c. Describe what $d^{-1}(x)$ represents. Then explain how you could have predicted whether $d^{-1}(x)$ is a function without looking at a graph.

42. **Fire Fighting** The velocity v and maximum height h of water being pumped into the air are related by the equation $v = \sqrt{2gh}$ where g is the acceleration due to gravity (32 feet/second^2).
    a. Determine an equation that will give the maximum height of the water as a function of its velocity.
    b. The Mayfield Fire Department must purchase a pump that is powerful enough to propel water 80 feet into the air. Will a pump that is advertised to project water with a velocity of 75 feet/second meet the fire department's needs? Explain.

43. **Critical Thinking**
    a. Give an example of a function that is its own inverse.
    b. What type of symmetry must the graph of the function exhibit?
    c. Would all functions with this type of symmetry be their own inverses? Justify your response.

44. **Consumer Costs** A certain long distance phone company charges callers 10 cents for every minute or part of a minute that they talk. Suppose that you talk for x minutes, where x is any real number greater than 0.
    a. Sketch the graph of the function $C(x)$ that gives the cost of an x-minute call.
    b. What are the domain and range of $C(x)$?
    c. Sketch the graph of $C^{-1}(x)$.
    d. What are the domain and range of $C^{-1}(x)$?
    e. What real-world situation is modeled by $C^{-1}(x)$?

45. **Critical Thinking** Consider the parent function $y = x^2$ and its inverse $y = \pm\sqrt{x}$. If the graph of $y = x^2$ is translated 6 units right and 5 units down, what must be done to the graph of $y = \pm\sqrt{x}$ to get a graph of the inverse of the translated function? Write an equation for each of the translated graphs.

46. **Physics** The formula for the kinetic energy of a particle as a function of its mass m and velocity v is $KE = \frac{1}{2}mv^2$.
    a. Find the equation for the velocity of a particle based on its mass and kinetic energy.
    b. Use your equation from part a to find the velocity in meters per second of a particle of mass 1 kilogram and kinetic energy 15 joules.
    c. Explain why the velocity of the particle is not a function of its mass and kinetic energy.

47. **Cryptography** One way to encode a message is to assign a numerical value to each letter of the alphabet and encode the message by assigning each number to a new value using a mathematical relation.

a. Does the encoding relation have to be a function? Explain.

b. Why should the graph of the encoding function pass the horizontal line test?

c. Suppose a value was assigned to each letter of the alphabet so that $1 = \text{A}$, $2 = \text{B}$, $3 = \text{C}$, …, $26 = \text{Z}$, and a message was encoded using the relation $c(x) = -2 + \sqrt{x + 3}$. What function would decode the message?

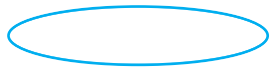
d. Try this decoding function on the following message:

1	2.899	2.123	0.449	2.796	1.464	2.243	2.123	2.690
0	2.583	0.828	1	2.899	2.123			

Mixed Review

48. Solve the inequality $|2x + 4| \leq 6$. *(Lesson 3-3)*

49. State whether the figure at the right has point symmetry, line symmetry, neither, or both. *(Lesson 3-1)*



50. **Retail** Arturo Alvaré, a sales associate at a paint store, plans to mix as many gallons as possible of colors A and B. He has 32 units of blue dye and 54 units of red dye. Each gallon of color A requires 4 units of blue dye and 1 unit of red dye. Each gallon of color B requires 1 unit of blue dye and 6 units of red dye. Use linear programming to answer the following questions. *(Lesson 2-7)*

a. Let a be the number of gallons of color A and let b be the number of gallons of color B. Write the inequalities that describe this situation.

b. Find the maximum number of gallons possible.

51. Solve the system of equations $4x + 2y = 10$, $y = 6 - x$ by using a matrix equation. *(Lesson 2-5)*

52. Find the product $\frac{1}{2}\begin{bmatrix} 9 & -3 \\ -6 & 6 \end{bmatrix}$. *(Lesson 2-3)*

53. Graph $y < -2x + 8$. *(Lesson 1-8)*

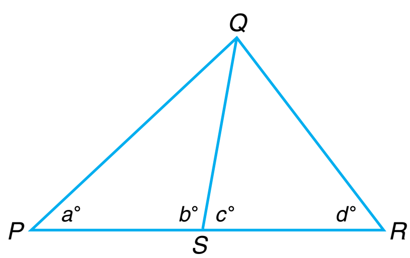
54. Line ℓ_1 has a slope of $\frac{1}{4}$, and line ℓ_2 has a slope of 4. Are the lines *parallel*, *perpendicular*, or *neither*? *(Lesson 1-5)*

55. Write the slope-intercept form of the equation of the line that passes through points at (0, 7) and (5, 2). *(Lesson 1-4)*

56. **SAT/ACT Practice** In the figure at the right, if $\overline{PQ}$ is perpendicular to $\overline{QR}$, then $a + b + c + d = ?$

A 180 **B** 225 **C** 270

D 300 **E** 360



Extra Practice See p. A30.

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