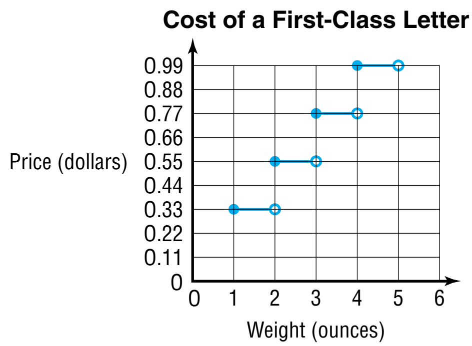
3-5 Continuity and End Behavior

OBJECTIVES

- Determine whether a function is continuous or discontinuous.
- Identify the end behavior of functions.
- Determine whether a function is increasing or decreasing on an interval.

Real World Application

POSTAGE On January 10, 1999, the United States Postal Service raised the cost of a first-class stamp. After the change, mailing a letter cost \$0.33 for the first ounce and \$0.22 for each additional ounce or part of an ounce. The graph summarizes the cost of mailing a first-class letter. *A problem related to this is solved in Example 2.*

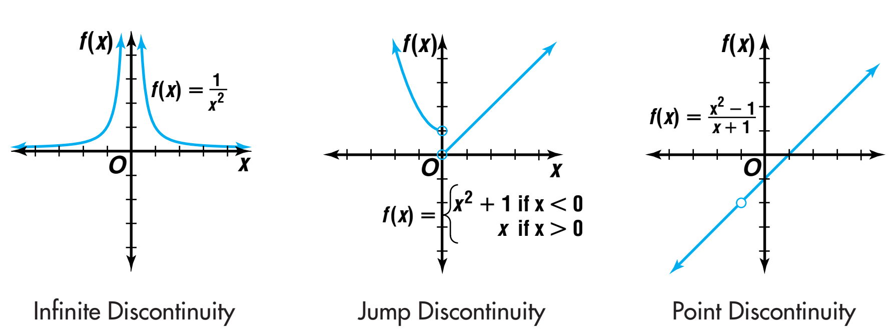


Most graphs we have studied have been smooth, continuous curves. However, a function like the one graphed above is a **discontinuous** function. That is, you cannot trace the graph of the function without lifting your pencil.

There are many types of discontinuity. Each of the functions graphed below illustrates a different type of discontinuity. That is, each function is discontinuous at some point in its domain.

The postage graph exhibits jump discontinuity.

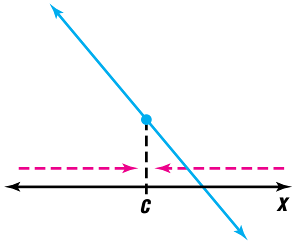
Graphs of Discontinuous Functions



- **Infinite discontinuity** means that $|f(x)|$ becomes greater and greater as the graph approaches a given x-value.
- **Jump discontinuity** indicates that the graph stops at a given value of the domain and then begins again at a different range value for the same value of the domain.
- When there is a value in the domain for which the function is undefined, but the pieces of the graph match up, we say the function has **point discontinuity.**

There are functions that are impossible to graph in the real number system. Some of these functions are said to be **everywhere discontinuous.** An example of such a function is $f(x) = \begin{cases} 1 \text{ if } x \text{ is rational} \\ -1 \text{ if } x \text{ is irrational} \end{cases}$.

If a function is not discontinuous, it is said to be **continuous**. That is, a function is continuous at a number c if there is a point on the graph with x-coordinate c and the graph passes through that point without a break.



Linear and quadratic functions are continuous at all points. If we only consider x-values less than c as x approaches c, then we say x is approaching c from the left. Similarly, if we only consider x-values greater than c as x approaches c, then we say x is approaching c from the right.

Continuity Test

A function is continuous at $x = c$ if it satisfies the following conditions:
(1) the function is defined at c; in other words, $f(c)$ exists;
(2) the function approaches the same y-value on the left and right sides of $x = c$; and
(3) the y-value that the function approaches from each side is $f(c)$.

Example 1 **Determine whether each function is continuous at the given x-value.**

a. $f(x) = 3x^2 + 7;\ x = 1$

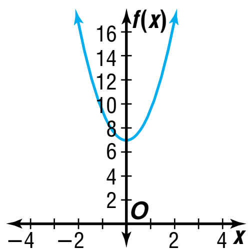
Check the three conditions in the continuity test.

(1) The function is defined at $x = 1$. In particular, $f(1) = 10$.

(2) The first table below suggests that when x is less than 1 and x approaches 1, the y-values approach 10. The second table suggests that when x is greater than 1 and x approaches 1, the y-values approach 10.

x	$y = f(x)$
0.9	9.43
0.99	9.9403
0.999	9.994003

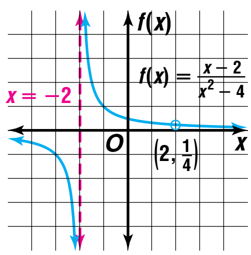
x	$y = f(x)$
1.1	10.63
1.01	10.0603
1.001	10.006003



(3) Since the y-values approach 10 as x approaches 1 from both sides and $f(1) = 10$, the function is continuous at $x = 1$. This can be confirmed by examining the graph.

b. $f(x) = \frac{x - 2}{x^2 - 4};\ x = 2$

Start with the first condition in the continuity test. The function is not defined at $x = 2$ because substituting 2 for x results in a denominator of zero. So the function is discontinuous at $x = 2$.
This function has point discontinuity at $x = 2$.



CONTENTS

c. $f(x) = \begin{cases} \frac{1}{x} \text{ if } x > 1 \\ x \text{ if } x \leq 1 \end{cases}; x = 1$

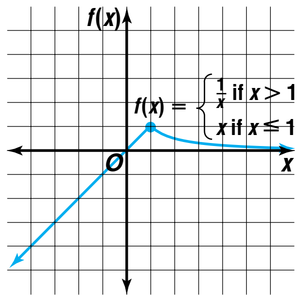
The function is defined at $x = 1$.
Using the second formula we find $f(1) = 1$.

The first table suggests that $f(x)$ approaches 1 as x approaches 1 from the left. We can see from the second table that $f(x)$ seems to approach 1 as x approaches 1 from the right.

x	$f(x)$
0.9	0.9
0.99	0.99
0.999	0.999

x	$f(x)$
1.1	0.9091
1.01	0.9901
1.001	0.9990

Since the $f(x)$-values approach 1 as x approaches 1 from both sides and $f(1) = 1$, the function is continuous at $x = 1$.



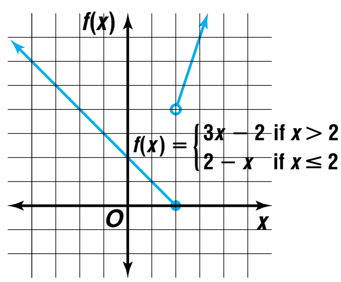
A function may have a discontinuity at one or more x-values but be continuous on an interval of other x-values. For example, the function $f(x) = \frac{1}{x^2}$ is continuous for $x > 0$ and $x < 0$, but discontinuous at $x = 0$.

Continuity on an Interval	A function $f(x)$ is continuous on an interval if and only if it is continuous at each number x in the interval.

In Chapter 1, you learned that a piecewise function is made from several functions over various intervals. The piecewise function

$f(x) = \begin{cases} 3x - 2 \text{ if } x > 2 \\ 2 - x \text{ if } x \leq 2 \end{cases}$ is continuous for $x > 2$

and $x < 2$ but is discontinuous at $x = 2$. The graph has a jump discontinuity. This function fails the second part of the continuity test because the values of $f(x)$ approach 0 as x approaches 2 from the left, but the $f(x)$-values approach 4 as x approaches 2 from the right.



Example 2 **POSTAGE** **Refer to the application at the beginning of the lesson.**

Real World Application

a. Use the continuity test to show that the step function is discontinuous.

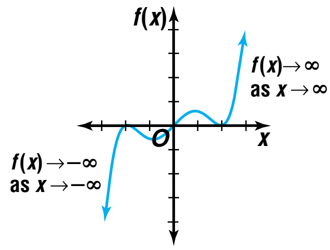
b. Explain why a continuous function would not be appropriate to model postage costs.

a. The graph of the postage function is discontinuous at each integral value of w in its domain because the function does not approach the same value from the left and the right. For example, as w approaches 1 from the left, $C(w)$ approaches 0.33 but as w approaches 1 from the right, $C(w)$ approaches 0.55.

b. A continuous function would have to achieve all real y-values (greater than or equal to 0.33.) This would be an inappropriate model for this situation since the weight of a letter is rounded to the nearest ounce and postage costs are rounded to the nearest cent.

$x \to \infty$ is read as "x approaches infinity."

Another tool for analyzing functions is **end behavior.** The end behavior of a function describes what the y-values do as $|x|$ becomes greater and greater. When x becomes greater and greater, we say that x approaches infinity, and we write $x \to \infty$. Similarly, when x becomes more and more negative, we say that x approaches negative infinity, and we write $x \to -\infty$. The same notation can also be used with y or $f(x)$ and with real numbers instead of infinity.



Example 3 **Describe the end behavior of $f(x) = -2x^3$ and $g(x) = -x^3 + x^2 - x + 5$.**

Use your calculator to create a table of function values so you can investigate the behavior of the y-values.

$f(x) = -2x^3$	
x	$f(x)$
−10,000	2×10^{12}
−1000	2×10^9
−100	2,000,000
−10	2000
0	0
10	−2000
100	−2,000,000
1000	-2×10^9
10,000	-2×10^{12}

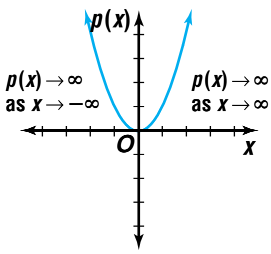
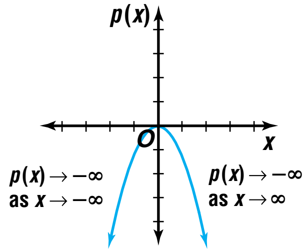
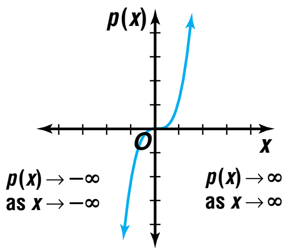
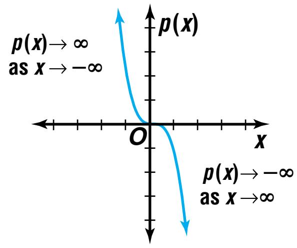
$g(x) = x^3 + x^2 - x + 5$	
x	$g(x)$
−10,000	1.0001×10^{12}
−1000	1,001,001,005
−100	1,010,105
−10	1115
0	5
10	−905
100	−990,095
1000	−999,000,995
10,000	-9.999×10^{11}

Notice that both polynomial functions have y-values that become very large in absolute value as x gets very large in absolute value. The end behavior of $f(x)$ can be summarized by stating that as $x \to \infty$, $f(x) \to -\infty$ and as $x \to -\infty$, $f(x) \to \infty$. The end behavior of $g(x)$ is the same. *You may wish to graph these functions on a graphing calculator to verify this summary.*

In general, the end behavior of any polynomial function can be modeled by the function comprised solely of the term with the highest power of x and its coefficient. Suppose for $n \geq 0$

$$p(x) = a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_2x^2 + a_1x + a_0.$$

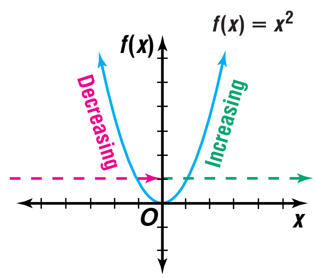
Then $f(x) = a_nx^n$ has the same end behavior as $p(x)$. The following table organizes the information for such functions and provides an example of a function displaying each type of end behavior.

<table>
<tr><th colspan="2">End Behavior of Polynomial Functions
$p(x) = a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_2x^2 + a_1x + a_0,\ n > 0$</th></tr>
<tr><th>a_n: positive, n: even</th><th>a_n: negative, n: even</th></tr>
<tr><td>$p(x) = x^2$

</td><td>$p(x) = -x^2$

</td></tr>
<tr><th>a_n: positive, n: odd</th><th>a_n: negative, n: odd</th></tr>
<tr><td>$p(x) = x^3$

</td><td>$p(x) = -x^3$

</td></tr>
</table>

Another characteristic of functions that can help in their analysis is the **monotonicity** of the function. A function is said to be monotonic on an interval I if and only if the function is increasing on I or decreasing on I.

Whether a graph is increasing or decreasing is always judged by viewing a graph from left to right.

The graph of $f(x) = x^2$ shows that the function is decreasing for $x < 0$ and increasing for $x > 0$.



Increasing, Decreasing, and Constant Functions

A function f is increasing on an interval I if and only if for every a and b contained in I, $f(a) < f(b)$ whenever $a < b$.
A function f is decreasing on an interval I if and only if for every a and b contained in I, $f(a) > f(b)$ whenever $a < b$.
A function f remains constant on an interval I if and only if for every a and b contained in I, $f(a) = f(b)$ whenever $a < b$.

Points in the domain of a function where the function changes from increasing to decreasing or vice versa are special points called *critical points.* You will learn more about these special points in Lesson 3-6. Using a graphing calculator can help you determine where the direction of the function changes.

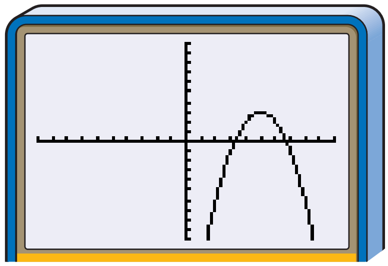
Graphing Calculator Tip

By watching the x- and y-values while using the **TRACE** function, you can determine approximately where a function changes from increasing to decreasing and vice versa.

Example 4 **Graph each function. Determine the interval(s) on which the function is increasing and the interval(s) on which the function is decreasing.**

a. $f(x) = 3 - (x - 5)^2$

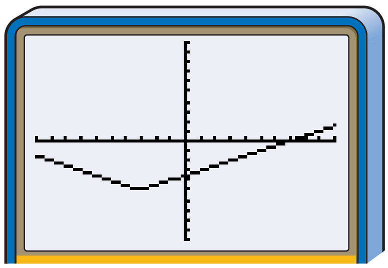
The graph of this function is obtained by transforming the parent graph $p(x) = x^2$. The parent graph has been reflected over the x-axis, translated 5 units to the right, and translated up 3 units. The function is increasing for $x < 5$ and decreasing for $x > 5$. At $x = 5$, there is a critical point.



[−10, 10] scl:1 by [−10, 10] scl:1

b. $f(x) = \frac{1}{2}|x + 3| - 5$

The graph of this function is obtained by transforming the parent graph $p(x) = |x|$. The parent graph has been vertically compressed by a factor of $\frac{1}{2}$, translated 3 units to the left, and translated down 5 units. This function is decreasing for $x < -3$ and increasing for $x > -3$. There is a critical point when $x = -3$.



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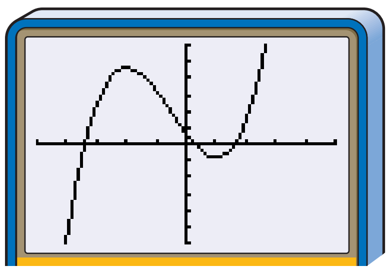
c. $f(x) = 2x^3 + 3x^2 - 12x + 3$

This function has more than one critical point. It changes direction at $x = -2$ and $x = 1$. The function is increasing for $x < -2$. The function is also increasing for $x > 1$. When $-2 < x < 1$, the function is decreasing.



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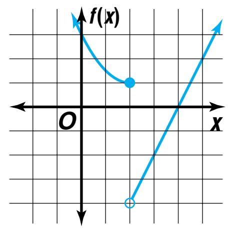
Communicating Mathematics

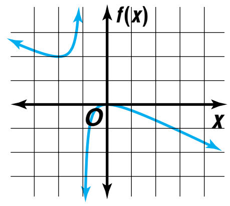
Read and study the lesson to answer each question.

1. **Explain** why the function whose graph is shown at the right is discontinuous at $x = 2$.



2. **Summarize** the end behavior of polynomial functions.

3. **State** whether the graph at the right has infinite discontinuity, jump discontinuity, or point discontinuity, or is continuous. Then describe the end behavior of the function.



4. *Math Journal* **Write** a paragraph that compares the monotonicity of $f(x) = x^2$ with that of $g(x) = -x^2$. In your paragraph, make a conjecture about the monotonicity of the reflection over the x-axis of any function as compared to that of the original function.

Guided Practice

Determine whether each function is continuous at the given x-value. Justify your answer using the continuity test.

5. $y = \frac{x - 5}{x + 3}$; $x = -3$

6. $f(x) = \begin{cases} x^2 + 2 \text{ if } x < -2 \\ 3x \text{ if } x \geq -2 \end{cases}$; $x = -2$

Describe the end behavior of each function.

7. $y = 4x^5 + 2x^4 - 3x - 1$

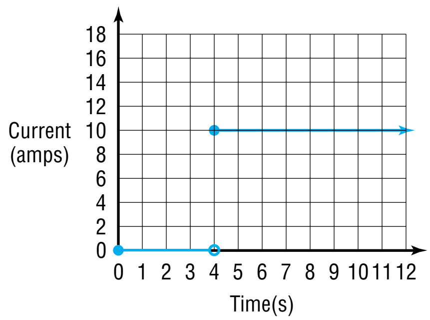
8. $y = -x^6 + x^4 - 5x^2 + 4$

Graph each function. Determine the interval(s) for which the function is increasing and the interval(s) for which the function is decreasing.

9. $f(x) = (x + 3)^2 - 4$

10. $y = \frac{x}{x^2 + 1}$

11. **Electricity** A simple electric circuit contains only a power supply and a resistor. When the power supply is off, there is no current in the circuit. When the power supply is turned on, the current almost instantly becomes a constant value. This situation can be modeled by a graph like the one shown at the right. I represents current in amps, and t represents time in seconds.



a. At what t-value is this function discontinuous?

b. When was the power supply turned on?

c. If the person who turned on the power supply left and came back hours later, what would he or she measure the current in the circuit to be?

EXERCISES

Practice

Determine whether each function is continuous at the given *x*-value. Justify your answer using the continuity test.

12. $y = x^3 - 4;\ x = 1$

13. $y = \frac{x+1}{x-2};\ x = 2$

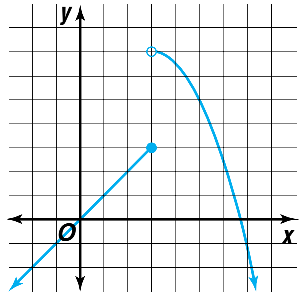
14. $f(x) = \frac{x+3}{(x-3)^2};\ x = -3$

15. $y = \left[\!\left[\frac{1}{2}x\right]\!\right];\ x = 3$

16. $f(x) = \begin{cases} 3x + 5 \text{ if } x \leq -4 \\ -x + 2 \text{ if } x > -4 \end{cases};\ x = -4$

17. $f(x) = \begin{cases} 2x + 1 \text{ if } x \geq 1 \\ 4 - x^2 \text{ if } x < 1 \end{cases};\ x = 1$

18. Determine whether the graph at the right has infinite discontinuity, jump discontinuity, or point discontinuity, or is continuous.



19. Find a value of x at which the function $g(x) = \frac{x-4}{x^2 - 3x}$ is discontinuous. Use the continuity test to justify your answer.

Describe the end behavior of each function.

20. $y = x^3 + 2x^2 + x - 1$

21. $y = 8 - x^3 - 2x^4$

22. $f(x) = x^{10} - x^9 + 5x^8$

23. $g(x) = \left|(x-3)^2 - 1\right|$

24. $y = \frac{1}{x^2}$

25. $f(x) = -\frac{1}{x^3} + 2$

Graphing Calculator

Graph each function. Determine the interval(s) for which the function is increasing and the interval(s) for which the function is decreasing.

26. $y = x^3 + 3x^2 - 9x$

27. $y = -x^3 - 2x + 1$

28. $f(x) = \frac{1}{x+1} - 4$

29. $g(x) = \frac{x^2 + 5}{x - 2}$

30. $y = \left|x^2 - 4\right|$

31. $y = (2|x| - 3)^2 + 1$

Applications and Problem Solving

Real World Application

32. Physics The gravitational potential energy of an object is given by $U(r) = -\frac{GmM_e}{r}$, where G is Newton's gravitational constant, m is the mass of the object, M_e is the mass of Earth, and r is the distance from the object to the center of Earth. What happens to the gravitational potential energy of the object as it is moved farther and farther away from Earth?

33. Critical Thinking A function $f(x)$ is increasing when $0 < x < 2$ and decreasing when $x > 2$. The function has a jump discontinuity when $x = 3$ and $f(x) \to -\infty$ as $x \to \infty$.

a. If $f(x)$ is an even function, then describe the behavior of $f(x)$ for $x < 0$. Sketch a graph of such a function.

b. If $f(x)$ is an odd function, then describe the behavior of $f(x)$ for $x < 0$. Sketch a graph of such a function.

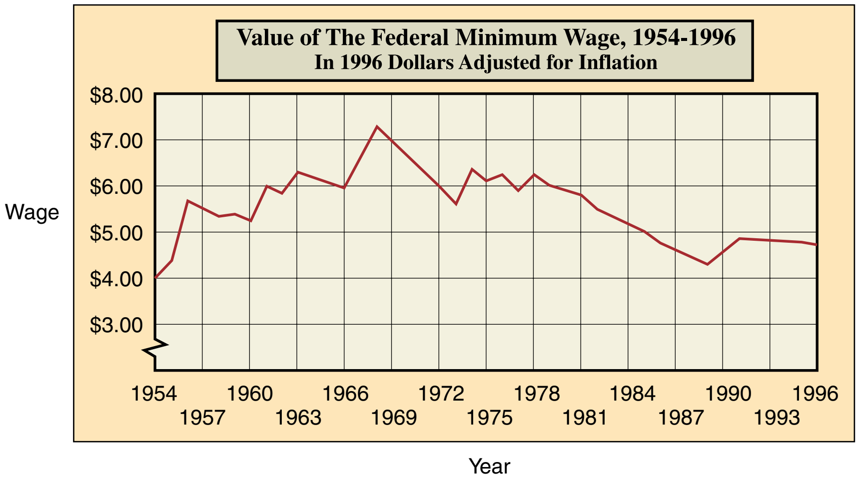
34. **Biology** One model for the population P of bacteria in a sample after t days is given by $P(t) = 1000 - 19.75t + 20t^2 - \frac{1}{3}t^3$.

a. What type of function is $P(t)$?

b. When is the bacteria population increasing?

c. When is it decreasing?

35. **Employment** The graph shows the minimum wage over a 43-year period in 1996 dollars adjusted for inflation.



Source: Department of Labor

*inter*NET CONNECTION

Data Update For the latest information about the minimum wage, visit **www.amc.glencoe.com.**

a. During what time intervals was the adjusted minimum wage increasing?

b. During what time intervals was the adjusted minimum wage decreasing?

36. **Analytic Geometry** A line is *secant* to the graph of a function if it intersects the graph in at least two distinct points. Consider the function $f(x) = -(x - 4)^2 - 3$.

a. On what interval(s) is $f(x)$ increasing?

b. Choose two points in the interval from part a. Determine the slope of the secant line that passes through those two points.

c. Make a conjecture about the slope of any secant line that passes through two points contained in an interval where a function is increasing. Explain your reasoning.

d. On what interval(s) is $f(x)$ decreasing?

e. Extend your hypothesis from part c to describe the slope of any secant line that passes through two points contained in an interval where the function is decreasing. Test your hypothesis by choosing two points in the interval from part d.

37. **Critical Thinking** Suppose a function is defined for all x-values and its graph passes the horizontal line test.

a. What can be said about the monotonicity of the function?

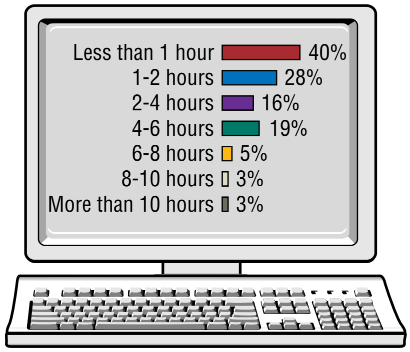
b. What can be said about the monotonicity of the inverse of the function?

38. **Computers** The graph at the right shows the amount of school computer usage per week for students between the ages of 12 and 18.

a. Use this set of data to make a graph of a step function. On each line segment in your graph, put the open circle at the right endpoint.

b. On what interval(s) is the function continuous?

Student Computer Usage



Source: Consumer Electronics Manufacturers Association

39. **Critical Thinking** Determine the values of a and b so that f is continuous.

$$f(x) = \begin{cases} x^2 + a \text{ if } x \geq 2 \\ bx + a \text{ if } -2 < x < 2 \\ \sqrt{-b - x} \text{ if } x \leq -2 \end{cases}$$

Mixed Review

40. Find the inverse of the function $f(x) = (x + 5)^2$. *(Lesson 3-4)*

41. Describe how the graphs of $f(x) = |x|$ and $g(x) = |x + 2| - 4$ are related. *(Lesson 3-2)*

42. Find the maximum and minimum values of $f(x, y) = x + 2y$ if it is defined for the polygonal convex set having vertices at (0, 0), (4, 0), (3, 5), and (0, 5). *(Lesson 2-6)*

43. Find the determinant of $\begin{bmatrix} 5 & -4 \\ 8 & 2 \end{bmatrix}$. *(Lesson 2-5)*

44. **Consumer Costs** Mario's Plumbing Service charges a fee of \$35 for every service call they make. In addition, they charge \$47.50 for every hour they work on each job. *(Lesson 1-4)*

a. Write an equation to represent the cost c of a service call that takes h hours to complete.

b. Find the cost of a $2\frac{1}{4}$-hour service call.

45. Find $f(-2)$ if $f(x) = 2x^2 - 2x + 8$. *(Lesson 1-1)*

46. **SAT Practice** One box is a cube with side of length x. Another box is a rectangular solid with sides of lengths $x + 1$, $x - 1$, and x. If $x > 1$, how much greater is the volume of the cube than that of the other box?

A x
B $x^2 - 1$
C $x - 1$
D 1
E 0

Extra Practice See p. A31.

GRAPHING CALCULATOR EXPLORATION

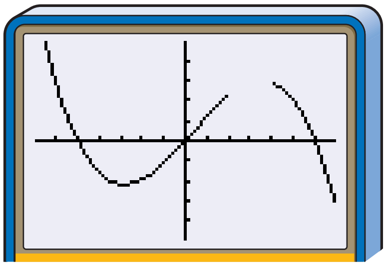
3-5B Gap Discontinuities

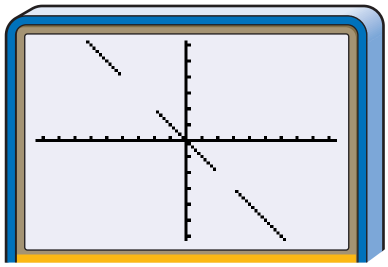
An Extension of Lesson 3-5

OBJECTIVE

- Construct and graph functions with gap discontinuities.

A function has a *gap discontinuity* if there is some interval of real numbers for which it is not defined. The graphs below show two types of gap discontinuities. The first function is undefined for $2 < x < 4$, and the second is undefined for $-4 \le x \le -2$ and for $2 \le x \le 3$.





These are also called Boolean operators.

The relational and logical operations on the **TEST** menu are primarily used for programming. Recall that the calculator delivers a value of 1 for a true equation or inequality and a value of 0 for a false equation or inequality. Expressions that use the logical connectives ("and", "or", "not", and so on) are evaluated according to the usual truth-table rules. Enter each of the following expressions and press ENTER to confirm that the calculator displays the value shown.

Expression	Value
$3 \le 7$	1
$-2 > -5$	1
$-4 > 6$	0

Expression	Value
$(-1 < 8)$ and $(8 < 9)$	1
$(2 > 4)$ or $(-7 \ge -3)$	0
$(4 < 3)$ and $(1 < 12)$	0

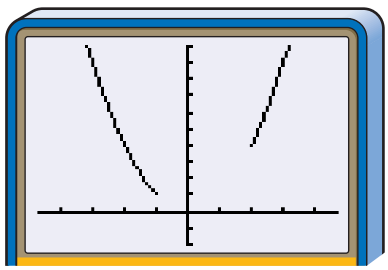
Relational and logical operations are also useful in defining functions that have point and gap discontinuities.

Example **Graph $y = x^2$ for $x \le -1$ or $x \ge 2$.**

Enter the following expression as **Y1** on the Y= list.

$$X^2/((X \le -1) \text{ or } (X \ge 2))$$

The function is defined as a quotient. The denominator is a Boolean statement, which has a value of 1 or 0. If the x-value for which the numerator is being evaluated is a member of the interval defined in the denominator, the denominator has a value of 1. Therefore, $\frac{f(x)}{1} = f(x)$ and that part of the function appears on the screen.



[−4.7, 4.7] scl:1 by [−2, 10] scl:1

(continued on the next page)

If the x-value is not part of the interval, the value of the denominator is 0. At these points, $\frac{f(x)}{0}$ would be undefined. Thus no graph appears on the screen for this interval.

When you use relational and logical operations to define functions, be careful how you use parentheses. Omitting parentheses can easily lead to an expression that the calculator may interpret in a way you did not intend.

TRY THESE

Graph each function and state its domain. You may need to adjust the window settings.

1. $y = \frac{x^2 - 2}{(x > 3)}$

2. $y = \frac{0.5x + 1}{((x \geq -2) \text{ and } (x \leq 4))}$

3. $y = \frac{-2x}{((x < -3) \text{ or } (x \geq 1))}$

4. $y = \frac{-0.2x^3 + 0.3x^2 - x}{((x \leq -3) \text{ or } (x > -2))}$

5. $y = \frac{|x|}{(|x| > 1)}$

6. $y = \frac{|x - 1| - |x - 3|}{(|x + 4| > 2)}$

7. $y = \frac{1.5x}{([\![x]\!] \neq 3)}$

8. $y = \frac{0.5x^2}{(([\![x]\!] \neq -2) \text{ and } ([\![x]\!] \neq 1))}$

Relational and logical operations are not the only tools available for defining and graphing functions with gap discontinuities. The square root function can easily be used for such functions. Graph each function and state its domain.

9. $y = \sqrt{(x - 1)(x - 2)(x - 3)(x - 4)}$

10. $y = \frac{x}{\frac{\sqrt{(x - 1)(x - 2)}}{\sqrt{(x - 1)(x - 2)}}}$

WHAT DO YOU THINK?

11. Suppose you want to construct a function whose graph is like that of $y = x^2$ except for "bites" removed for the values between 2 and 5 and the values between 7 and 8. What equation could you use for the function?

12. Is it possible to use the functions on the MATH NUM menu to take an infinite number of "interval bites" from the graph of a function? Justify your answer.

13. Is it possible to write an equation for a function whose graph looks just like the graph of $y = x^2$ for $x \leq -2$ and just like the graph of $y = 2x - 4$ for $x \geq 4$, with no points on the graph for values of x between -2 and 4? Justify your answer.

14. Use what you have learned about gap discontinuities to graph the following piecewise functions.

a. $f(x) = \begin{cases} -2x \text{ if } x < 0 \\ -x^4 + 2x^3 + 3x^2 + 3x \text{ if } x \geq 0 \end{cases}$

b. $g(x) = \begin{cases} (x + 4)^3 - 2 \text{ if } x < -2 \\ x^2 + 2 \text{ if } -2 \leq x \leq 2 \\ -(x - 4)^3 - 2 \text{ if } x > 2 \end{cases}$