3-6 Critical Points and Extrema

OBJECTIVE
- Find the extrema of a function.

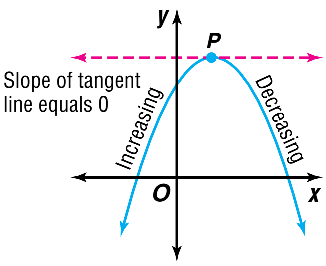
Real World Application

BUSINESS America's 23 million small businesses employ more than 50% of the private workforce. Owning a business requires good management skills. Business owners should always look for ways to compete and improve their businesses. Some business owners hire an analyst to help them identify strengths and weaknesses in their operation. Analysts can collect data and develop mathematical models that help the owner increase productivity, maximize profit, and minimize waste. *A problem related to this will be solved in Example 4.*

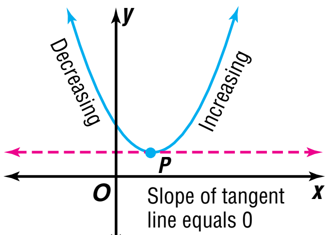
Optimization is an application of mathematics where one searches for a maximum or a minimum quantity given a set of constraints. When maximizing or minimizing quantities, it can be helpful to have an equation or a graph of a mathematical model for the quantity to be optimized.

Recall from geometry that a line is tangent to a curve if it intersects a curve in exactly one point.

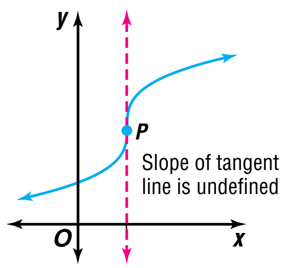
Critical points are those points on a graph at which a line drawn tangent to the curve is horizontal or vertical. A polynomial may possess three types of critical points. A critical point may be a **maximum**, a **minimum**, or a **point of inflection**. When the graph of a function is increasing to the left of $x = c$ and decreasing to the right of $x = c$, then there is a maximum at $x = c$. When the graph of a function is decreasing to the left of $x = c$ and increasing to the right of $x = c$, then there is a minimum at $x = c$. A point of inflection is a point where the graph changes its curvature as illustrated below.



Maximum at *P*



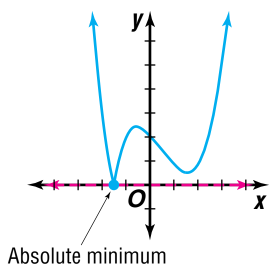
Minimum at *P*

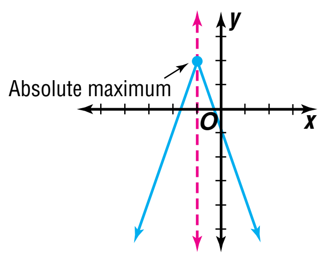


Point of inflection at *P*

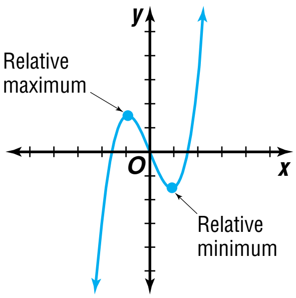
Maxima is the plural of maximum and minima is the plural of minimum. Extrema is the plural of extremum.

The graph of a function can provide a visual clue as to when a function has a maximum or a minimum value. The greatest value that a function assumes over its domain is called the **absolute maximum**. Likewise the least value of a function is the **absolute minimum**. The general term for maximum or minimum is **extremum**. The functions graphed below have absolute extrema.



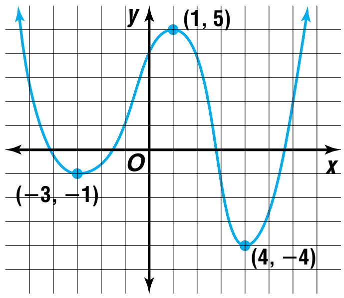


Functions can also have **relative extrema**. A **relative maximum** value of a function may not be the greatest value of f on the domain, but it is the greatest y-value on some interval of the domain. Similarly, a **relative minimum** is the least y-value on some interval of the domain. The function graphed at the right has both a relative maximum and a relative minimum.



Note that extrema are *values of the function;* that is, they are the y-coordinates of each maximum and minimum point.

Example 1 **Locate the extrema for the graph of $y = f(x)$. Name and classify the extrema of the function.**



The function has a relative minimum at $(-3, -1)$.

The function has a relative maximum at $(1, 5)$.

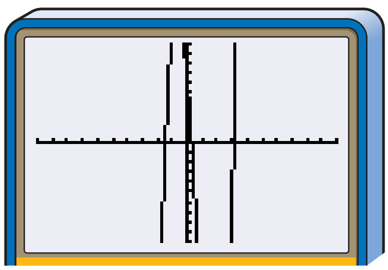
The function has a relative minimum at $(4, -4)$.

Since the point $(4, -4)$ is the lowest point on the graph, the function appears to have an absolute minimum of -4 when $x = 4$. This function appears to have no absolute maximum values, since the graph indicates that the function increases without bound as $x \to \infty$ and as $x \to -\infty$.

A branch of mathematics called calculus can be used to locate the critical points of a function. You will learn more about this in Chapter 15. A graphing calculator can also help you locate the critical points of a polynomial function.

Example 2 **Use a graphing calculator to graph $f(x) = 5x^3 - 10x^2 - 20x + 7$ and to determine and classify its extrema.**

Use a graphing calculator to graph the function in the standard viewing window. Notice that the x-intercepts of the graph are between -2 and -1, 0 and 1, and 3 and 4. Relative maxima and minima will occur somewhere between pairs of x-intercepts.



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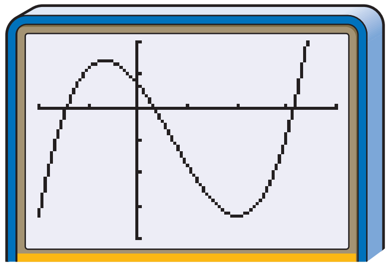
For a better view of the graph of the function, we need to change the window to encompass the observed x-intercepts more closely.

Graphing Calculator Tip

Xmin and **Xmax** define the *x*-axis endpoints. Likewise, **Ymin** and **Ymax** define the *y*-axis endpoints.

One way to do this is to change the x-axis view to $-2 \le x \le 4$. Since the top and bottom of the graph are not visible, you will probably want to change the y-axis view as well. The graph at the right shows $-40 \le y \le 20$. From the graph, we can see there is a relative maximum in the interval $-1 < x < 0$ and a relative minimum in the interval $1 < x < 3$.



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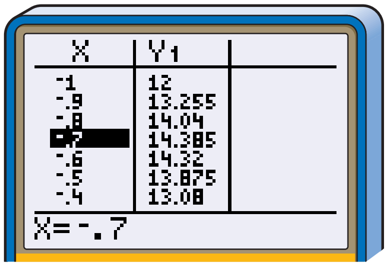
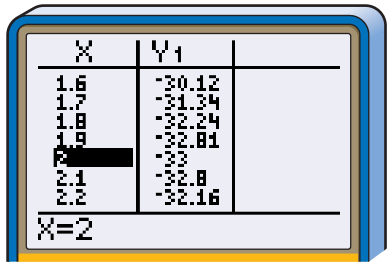
There are several methods you can use to locate these extrema more accurately.

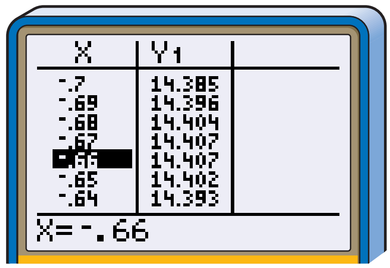
Method 1: Use a table of values to locate the approximate greatest and least value of the function. (*Hint:* Revise **TBLSET** to begin at -2 in intervals of 0.1.)



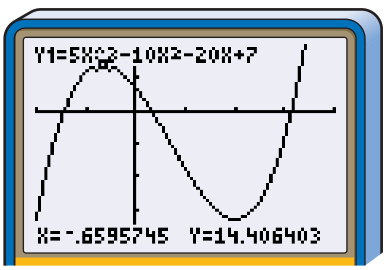


There seems to be a relative maximum of approximately 14.385 at $x = -0.7$ and a relative minimum of -33 at $x = 2$.

You can adjust the **TBLSET** increments to hundredths to more closely estimate the x-value for the relative maximum. A relative maximum of about 14.407 appears to occur somewhere between the x-values -0.67 and -0.66. A fractional estimation of the x-value might be $x = -\frac{2}{3}$.



Method 2: Use the **TRACE** function to approximate the relative maximum and minimum.



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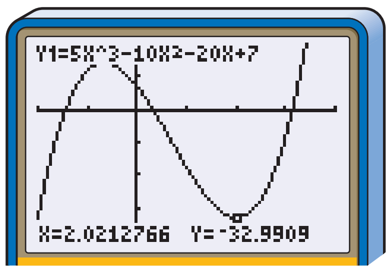

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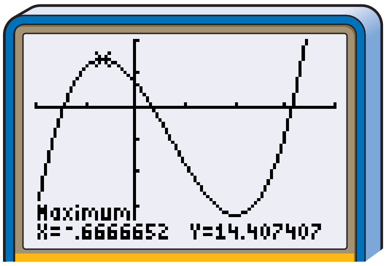
There seems to be a maximum at $x \approx -0.66$ and a minimum at $x \approx 2.02$.

Method 3: Use **3:minimum** and **4:maximum** options on the **CALC** menu to locate the approximate relative maximum and minimum.



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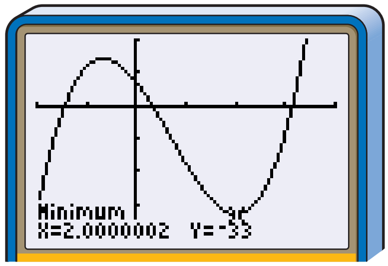

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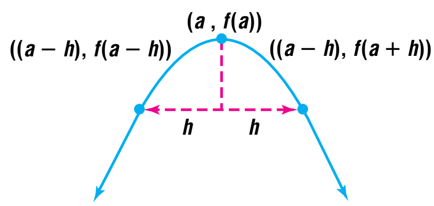
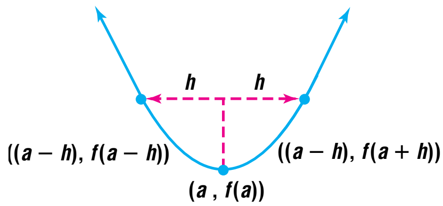
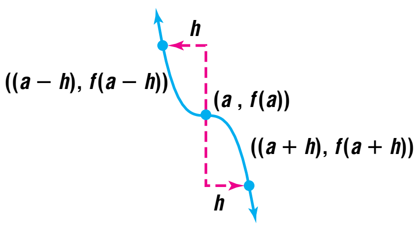
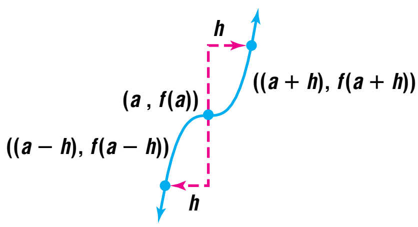
The calculator indicates a relative maximum of about 14.4 at $x \approx -0.67$ and a relative minimum of -33 at $x \approx 2.0$.

All of these approaches give approximations; some more accurate than others. From these three methods, we could estimate that a relative maximum occurs near the point at $(-0.67, 14.4)$ or $\left(-\frac{2}{3}, 14.407\right)$ and a relative minimum near the point at $(2, -33)$.

If you know a critical point of a function, you can determine if it is the location of a relative minimum, a relative maximum, or a point of inflection by testing points on both sides of the critical point. The table below shows how to identify each type of critical point.

Critical Points	
For $f(x)$ with $(a, f(a))$ as a critical point and h as a small value greater than zero	
 $f(a - h) < f(a)$ $f(a + h) < f(a)$ $f(a)$ is a maximum.	 $f(a - h) > f(a)$ $f(a + h) > f(a)$ $f(a)$ is a minimum.
 $f(a - h) > f(a)$ $f(a + h) < f(a)$ $f(a)$ is a point of inflection.	 $f(a - h) < f(a)$ $f(a + h) > f(a)$ $f(a)$ is a point of inflection.

You can also determine whether a critical point is a maximum, minimum, or inflection point by examining the values of a function using a table.

Example 3 **The function $f(x) = 2x^5 - 5x^4 - 10x^3$ has critical points at $x = -1$, $x = 0$, and $x = 3$. Determine whether each of these critical points is the location of a maximum, a minimum, or a point of inflection.**

Evaluate the function at each point. Then check the values of the function around each point. Let $h = 0.1$.

x	$x - 0.1$	$x + 0.1$	$f(x - 0.1)$	$f(x)$	$f(x + 0.1)$	Type of Critical Point
−1	−1.1	−0.9	2.769	3	2.829	maximum
0	−0.1	0.1	0.009	0	−0.010	inflection point
3	2.9	3.1	−187.308	−189	−187.087	minimum

You can verify this solution by graphing f(x) on a graphing calculator.

You can use critical points from the graph of a function to solve real-world problems involving maximization and minimization of values.

Example 4 Real World Application

BUSINESS **A small business owner employing 15 people hires an analyst to help the business maximize profits. The analyst gathers data and develops the mathematical model $P(x) = \frac{1}{3}x^3 - 34x^2 + 1012x$. In this model, P is the owner's monthly profits, in dollars, and x is the number of employees. The model has critical points at $x = 22$ and $x = 46$.**

a. Determine which, if any, of these critical points is a maximum.

b. What does this critical point suggest to the owner about business operations?

c. What are the risks of following the analyst's recommendation?

a. Test values around the points. Let $h = 0.1$.

x	$x - 0.1$	$x + 0.1$	$P(x - 0.1)$	$P(x)$	$P(x + 0.1)$	Type of Critical Point
22	21.9	22.1	9357.21	9357.33	9357.21	maximum
46	45.9	46.1	7053.45	7053.33	7053.45	minimum

The profit will be at a maximum when the owner employs 22 people.

b. The owner should consider expanding the business by increasing the number of employees from 15 to 22.

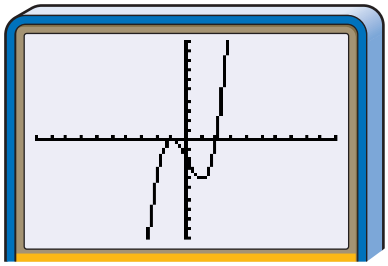
c. It is important that the owner hire qualified employees. Hiring unqualified employees will likely cause profits to decline.

CHECK FOR UNDERSTANDING

Communicating Mathematics

Read and study the lesson to answer each question.

1. **Write** an explanation of how to determine if a critical point is a maximum, minimum, or neither.
2. **Determine** whether the point at $(1, -4)$, a critical point of the graph of $f(x) = x^3 - 3x - 2$ shown at the right, represents a relative maximum, a relative minimum, or a point of inflection. Explain your reasoning.



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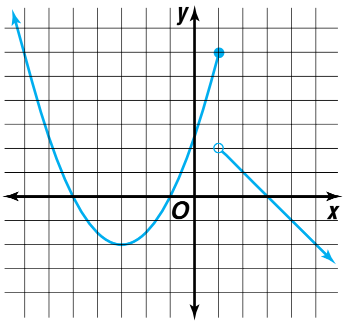
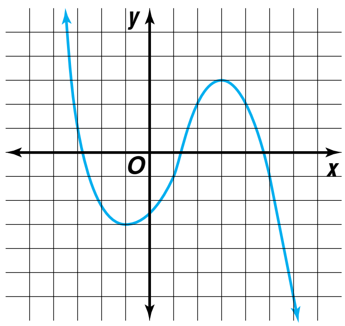
3. **Sketch** the graph of a function that has a relative minimum at $(0, -4)$, a relative maximum at $(-3, 1)$, and an absolute maximum at $(4, 6)$.

Guided Practice

Locate the extrema for the graph of $y = f(x)$. Name and classify the extrema of the function.

4. 

5. 

Use a graphing calculator to graph each function and to determine and classify its extrema.

6. $f(x) = 2x^5 - 5x^4$
7. $g(x) = x^4 + 3x^3 - 2$

Determine whether the given critical point is the location of a *maximum*, a *minimum*, or a *point of inflection*.

8. $y = 3x^3 - 9x - 5$, $x = -1$
9. $y = x^2 + 5x - 6$, $x = -2.5$
10. $y = 2x^3 - x^5$, $x = 0$
11. $y = x^6 - 3x^4 + 3x^2 - 1$, $x = 0$

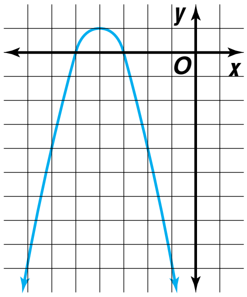
12. **Agriculture** Malik Davis is a soybean farmer. If he harvests his crop now, the yield will average 120 bushels of soybeans per acre and will sell for \$0.48 per bushel. However, he knows that if he waits, his yield will increase by about 10 bushels per week, but the price will decrease by \$0.03 per bushel per week.
    a. If x represents the number of weeks Mr. Davis waits to harvest his crop, write and graph a function $P(x)$ to represent his profit.
    b. How many weeks should Mr. Davis wait in order to maximize his profit?
    c. What is the maximum profit?
    d. What are the risks of waiting?

EXERCISES

Practice

Locate the extrema for the graph of $y = f(x)$. Name and classify the extrema of the function.

13.



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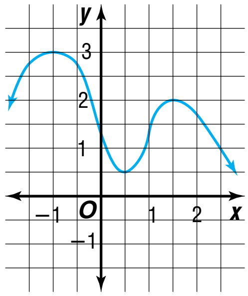

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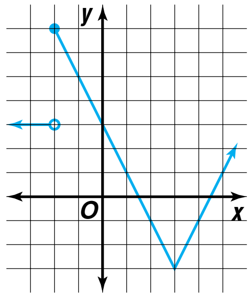

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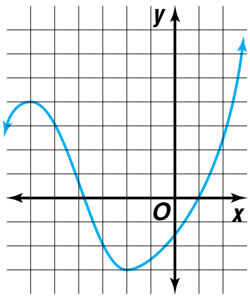

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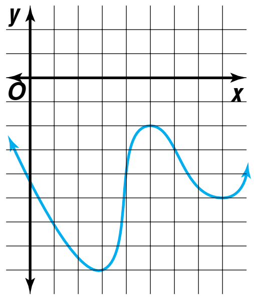

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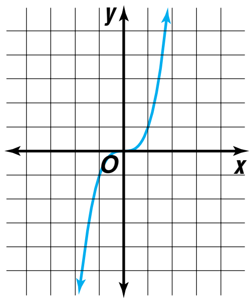


Use a graphing calculator to graph each function and to determine and classify its extrema.

19. $f(x) = -4 + 3x - x^2$

20. $V(w) = w^3 - 7w - 6$

21. $g(x) = 6x^3 + x^2 - 5x - 2$

22. $h(x) = x^4 - 4x^2 - 2$

23. $f(x) = 2x^5 + 4x^2 - 2x - 3$

24. $D(t) = t^3 + t$

25. Determine and classify the extrema of $f(x) = x^4 + 5x^3 + 3x^2 - 4x$.

Determine whether the given critical point is the location of a *maximum*, a *minimum*, or a *point of inflection*.

26. $y = x^3, x = 0$

27. $y = -x^2 + 8x - 10, x = 4$

28. $y = 2x^2 + 10x - 7, x = -2.5$

29. $y = x^4 - 2x^2 + 7, x = 0$

30. $y = \frac{1}{4}x^4 - 2x^2, x = 2$

31. $y = x^3 - 9x^2 + 27x - 27, x = 3$

32. $y = \frac{1}{3}x^3 + \frac{1}{2}x^2 - 2x + 1, x = -2$

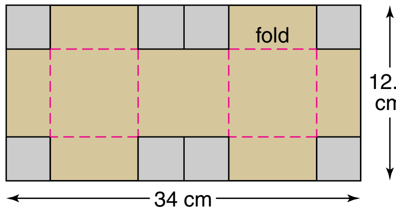
33. $y = x^3 - x^2 + 3, x = \frac{2}{3}$

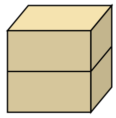
34. A function f has a relative maximum at $x = 2$ and a point of inflection at $x = -1$. Find the critical points of $y = -2f(x + 5) - 1$. Describe what happens at each new critical point.

CONTENTS

Applications and Problem Solving

35. Manufacturing A 12.5 centimeter by 34 centimeter piece of cardboard will have eight congruent squares removed as in the diagram. The box will be folded to create a take-out hamburger box.





a. Find the model for the volume $V(x)$ of the box as a function of the length x of the sides of the eight squares removed.

b. What are the dimensions of each of the eight squares that should be removed to produce a box with maximum volume?

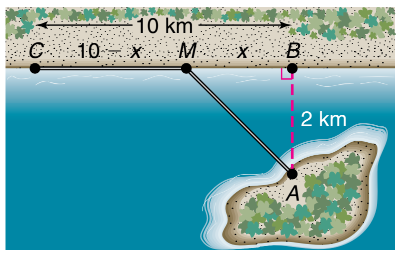
c. Construct a physical model of the box and measure its volume. Compare this result to the result from the mathematical model.

36. Business The Carlisle Innovation Company has created a new product that costs \$25 per item to produce. The company has hired a marketing analyst to help it determine a selling price for the product. After collecting and analyzing data relating selling price s to yearly consumer demand d, the analyst estimates demand for the product using the equation $d = -200s + 15{,}000$.

a. If yearly profit is the difference between total revenue and production costs, determine a selling price s, $s \geq 25$, that will maximize the company's yearly profit, P. (*Hint:* $P = sd - 25d$)

b. What are the risks of determining a selling price using this method?

37. Telecommunications A cable company wants to provide service for residents of an island. The distance from the closest point on the island's beach, point A, directly to the mainland at point B is 2 kilometers. The nearest cable station, point C, is 10 kilometers downshore from point B. It costs \$3500 per kilometer to lay the cable lines underground and \$5000 per kilometer to lay the cable lines under water. The line comes to the mainland at point M. Let x be the distance in kilometers from point B to point M.



a. Write a function to calculate the cost of laying the cable.

b. At what distance x should the cable come to shore to minimize cost?

38. Critical Thinking Which families of graphs have points of inflection but no maximum or minimum points?

39. Physics When the position of a particle as a function of time t is modeled by a polynomial function, then the particle is at rest at each critical point. If a particle has a position given by $s(t) = 2t^3 - 11t^2 + 3t - 9$, find the position of the particle each time it is at rest.

40. Critical Thinking A cubic polynomial can have 1 or 3 critical points. Describe the possible combinations of relative maxima and minima for a cubic polynomial.

Mixed Review

41. Is $y = \dfrac{5x}{x^2 - 3x - 10}$ continuous at $x = 5$? Justify your answer by using the continuity test. *(Lesson 3-5)*

42. Graph the inequality $y \le \frac{1}{5}(x - 2)^3$. *(Lesson 3-3)*

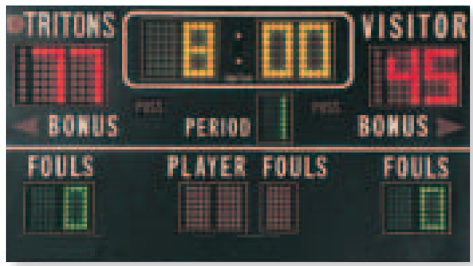
43. Manufacturing The Eastern Minnesota Paper Company can convert wood pulp to either newsprint or notebook paper. The mill can produce up to 200 units of paper a day, and regular customers require 10 units of notebook paper and 80 units of newsprint per day. If the profit on a unit of notebook paper is \$400 and the profit on a unit of newsprint is \$350, how much of each should the plant produce? *(Lesson 2-7)*

44. Geometry Find the system of inequalities that will define a polygonal convex set that includes all points in the interior of a square whose vertices are $A(-3, 4)$, $B(2, 4)$, $C(2, -1)$, and $D(-3, -1)$. *(Lesson 2-6)*

45. Find the determinant for $\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$. Does an inverse exist for this matrix? *(Lesson 2-5)*

46. Find $3A + 2B$ if $A = \begin{bmatrix} 4 & -2 \\ 5 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} -3 & 5 \\ -4 & 3 \end{bmatrix}$. *(Lesson 2-3)*

47. Sports Jon played in two varsity basketball games. He scored 32 points by hitting 17 of his 1-point, 2-point, and 3-point attempts. He made 50% of his 18 2-point field goal attempts. Find the number of 1-point free throws, 2-point field goals, and 3-point field goals Jon scored in these two games. *(Lesson 2-2)*

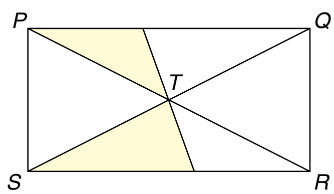


48. Graph $y + 6 \ge 4$. *(Lesson 1-8)*

49. Determine whether the graphs of $2x + 3y = 15$ and $6x = 4y + 16$ are *parallel, coinciding, perpendicular,* or *none of these.* *(Lesson 1-5)*

50. Describe the difference between a relation and a function. How do you test a graph to determine if it is the graph of a function? *(Lesson 1-1)*

51. SAT/ACT Practice Refer to the figure at the right. What percent of the area of rectangle *PQRS* is shaded?



A 20%

B $33\frac{1}{3}\%$

C 30%

D 25%

E 40%

Extra Practice See p. A31.

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