4-1

Polynomial Functions

OBJECTIVES

- Determine roots of polynomial equations.
- Apply the Fundamental Theorem of Algebra.

Real World Application

INVESTMENTS Many grandparents invest in the stock market for their grandchildren's college fund. Eighteen years ago, Della Brooks purchased \$1000 worth of merchandising stocks at the birth of her first grandchild Owen. Ten years ago, she purchased \$500 worth of transportation stocks, and five years ago, she purchased \$250 worth of technology stocks. The stocks will be used to help pay for Owen's college education. If the stocks appreciate at an average annual rate of 12.25%, determine the current value of the college fund. *This problem will be solved in Example 1.*

Appreciation is the increase in value of an item over a period of time. The formula for compound interest can be used to find the value of Owen's college fund after appreciation. The formula is $A = P(1 + r)^t$, where P is the original amount of money invested, r is the interest rate or rate of return (written as a decimal), and t is the time invested (in years).

Example 1 Real World Application

INVESTMENTS **The value of Owen's college fund is the sum of the current values of his grandmother's investments.**

a. Write a function in one variable that models the value of the college fund for any rate of return.

b. Use the function to determine the current value of the college fund for an average annual rate of 12.25%.

a. Let x represent $1 + r$ and $T(x)$ represent the total current value of the three stocks. The times invested, which are the exponents of x, are 18, 10, and 5, respectively.

Total	=	*merchandising*	+	*transportation*	+	*technology*
$T(x)$	=	$1000x^{18}$	+	$500x^{10}$	+	$250x^5$

b. Since $r = 0.1225$, $x = 1 + 0.1225$ or 1.1225. Now evaluate $T(x)$ for $x = 1.1225$.

$$T(x) = 1000x^{18} + 500x^{10} + 250x^5$$
$$T(1.1225) = 1000(1.1225)^{18} + 500(1.1225)^{10} + 250(1.1225)^5$$
$$T(1.1225) \approx 10{,}038.33$$

The present value of Owen's college fund is about \$10,038.33.

The expression $1000x^{18} + 500x^{10} + 250x^5$ is a **polynomial in one variable.**

Polynomial in One Variable	A polynomial in one variable, x, is an expression of the form $a_0x^n + a_1x^{n-1} + \dots + a_{n-2}x^2 + a_{n-1}x + a_n$. The coefficients $a_0, a_1, a_2, \dots, a_n$ represent complex numbers (real or imaginary), a_0 is not zero, and n represents a nonnegative integer.

The **degree** of a polynomial in one variable is the greatest exponent of its variable. The coefficient of the variable with the greatest exponent is called the **leading coefficient.** For the expression $1000x^{18} + 500x^{10} + 250x^5$, 18 is the degree, and 1000 is the leading coefficient.

If a function is defined by a polynomial in one variable with real coefficients, like $T(x) = 1000x^{18} + 500x^{10} + 250x^5$, then it is a **polynomial function.** If $f(x)$ is a polynomial function, the values of x for which $f(x) = 0$ are called the **zeros** of the function. *If the function is graphed, these zeros are also the x-intercepts of the graph.*

Example 2 **Consider the polynomial function $f(x) = x^3 - 6x^2 + 10x - 8$.**

a. State the degree and leading coefficient of the polynomial.

b. Determine whether 4 is a zero of $f(x)$.

a. $x^3 - 6x^2 + 10x - 8$ has a degree of 3 and a leading coefficient of 1.

b. Evaluate $f(x) = x^3 - 6x^2 + 10x - 8$ for $x = 4$. That is, find $f(4)$.

$f(4) = 4^3 - 6(4^2) + 10(4) - 8 \quad x = 4$

$f(4) = 64 - 96 + 40 - 8$

$f(4) = 0$

Since $f(4) = 0$, 4 is a zero of $f(x) = x^3 - 6x^2 + 10x - 8$.

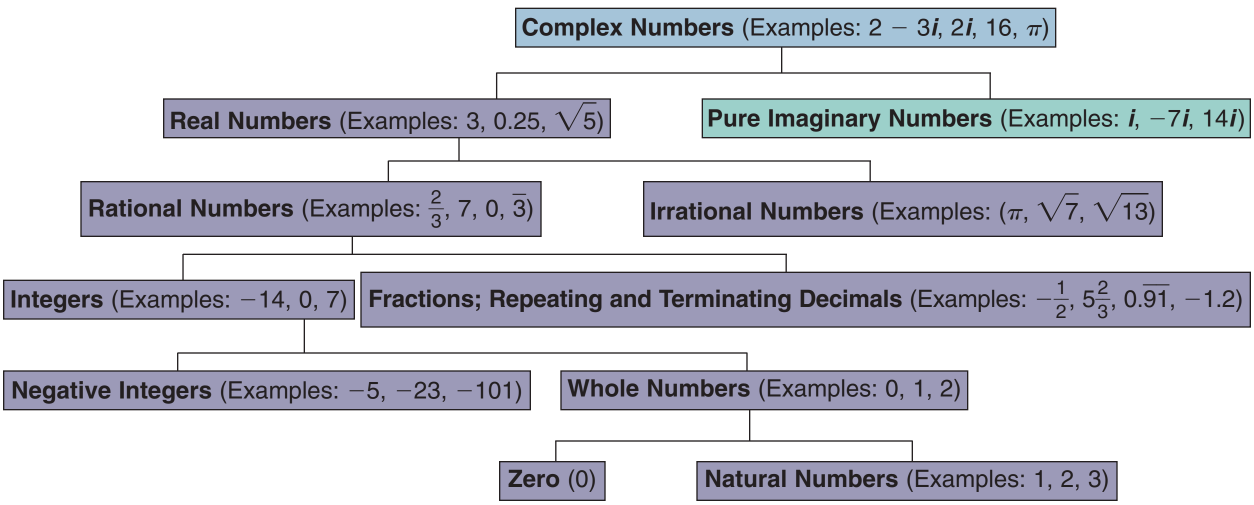
Graphing Calculator Tip

To find a value of a polynomial for a given value of x, enter the polynomial in the **Y=** list. Then use the **1:value** option in the **CALC** menu.

Since 4 is a zero of $f(x) = x^3 - 6x^2 + 10x - 8$, it is also a solution for the **polynomial equation** $x^3 - 6x^2 + 10x - 8 = 0$. The solution for a polynomial equation is called a **root.** The words *zero* and *root* are often used interchangeably, but technically, you find the *zero of a function* and the *root of an equation.*

A root or zero may also be an **imaginary number** such as $3\boldsymbol{i}$. By definition, the imaginary unit $\boldsymbol{i}$ equals $\sqrt{-1}$. Since $\boldsymbol{i} = \sqrt{-1}$, $\boldsymbol{i}^2 = -1$. It also follows that $\boldsymbol{i}^3 = \boldsymbol{i}^2 \times \boldsymbol{i}$ or $-\boldsymbol{i}$ and $\boldsymbol{i}^4 = \boldsymbol{i}^2 \times \boldsymbol{i}^2$ or 1.

The imaginary numbers combined with the real numbers compose the set of **complex numbers.** A complex number is any number of the form $a + b\boldsymbol{i}$ where a and b are real numbers. If $b = 0$, then the complex number is a real number. If $a = 0$ and $b \neq 0$, then the complex number is called a **pure imaginary number.**



One of the most important theorems in mathematics is the **Fundamental Theorem of Algebra.**

Fundamental Theorem of Algebra	Every polynomial equation with degree greater than zero has at least one root in the set of complex numbers.

A corollary to the Fundamental Theorem of Algebra states that the degree of a polynomial indicates the number of possible roots of a polynomial equation.

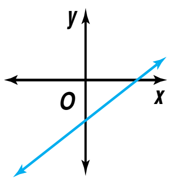
Corollary to the Fundamental Theorem of Algebra

Every polynomial $P(x)$ of degree n $(n > 0)$ can be written as the product of a constant k $(k \neq 0)$ and n linear factors.

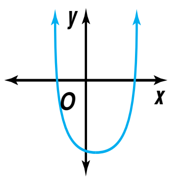
$$P(x) = k(x - r_1)(x - r_2)(x - r_3) \ldots (x - r_n)$$

Thus, a polynomial equation of degree n has exactly n complex roots, namely $r_1, r_2, r_3, \ldots, r_n$.

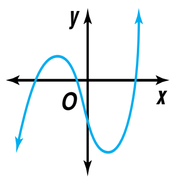
The general shapes of the graphs of polynomial functions with positive leading coefficients and degree greater than 0 are shown below. These graphs also show the *maximum* number of times the graph of each type of polynomial may cross the x-axis.



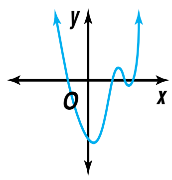
Degree 1



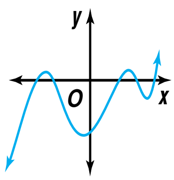
Degree 2



Degree 3

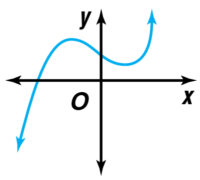


Degree 4

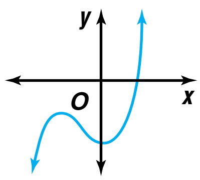


Degree 5

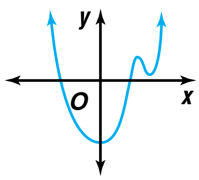
Since the x-axis only represents real numbers, imaginary roots cannot be determined by using a graph. The graphs below have the general shape of a third-degree function and a fourth-degree function. In these graphs, the third-degree function only crosses the x-axis once, and the fourth-degree function crosses the x-axis twice or not at all.



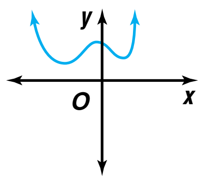
Degree 3
1 x-intercept



Degree 3
1 x-intercept



Degree 4
2 x-intercepts



Degree 4
0 x-intercepts

The graph of a polynomial function with odd degree must cross the x-axis at least once. The graph of a function with even degree may or may not cross the x-axis. If it does, it will cross an even number of times. Each x-intercept represents a real root of the corresponding polynomial equation.

If you know the roots of a polynomial equation, you can use the corollary to the Fundamental Theorem of Algebra to find the polynomial equation. That is, if a and b are roots of the equation, the equation must be $(x - a)(x - b) = 0$.

Examples

3 **a. Write a polynomial equation of least degree with roots 2, $4i$, and $-4i$.**

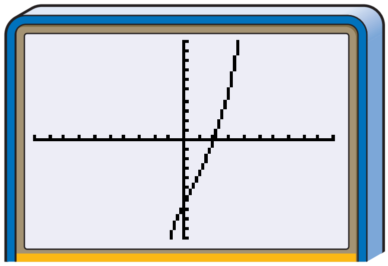
b. Does the equation have an odd or even degree? How many times does the graph of the related function cross the x-axis?

a. If $x = 2$, then $x - 2$ is a factor of the polynomial. Likewise, if $x = 4\boldsymbol{i}$ and $x = -4\boldsymbol{i}$, then $x - 4\boldsymbol{i}$ and $x - (-4\boldsymbol{i})$ are factors of the polynomial. Therefore, the linear factors for the polynomial are $x - 2$, $x - 4\boldsymbol{i}$, and $x + 4\boldsymbol{i}$. Now find the products of these factors.

$$(x - 2)(x - 4\boldsymbol{i})(x + 4\boldsymbol{i}) = 0$$
$$(x - 2)(x^2 - 16\boldsymbol{i}^2) = 0$$
$$(x - 2)(x^2 + 16) = 0 \quad -16\boldsymbol{i}^2 = -16(-1) \text{ or } 16$$
$$x^3 - 2x^2 + 16x - 32 = 0$$

A polynomial equation with roots 2, $4\boldsymbol{i}$, and $-4\boldsymbol{i}$ is $x^3 - 2x^2 + 16x - 32 = 0$.

b. The degree of this equation is 3. Thus, the equation has an odd degree since 3 is an odd number. Since two of the roots are imaginary, the graph will only cross the x-axis once. The graphing calculator image at the right verifies these conclusions.



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4 **State the number of complex roots of the equation $9x^4 - 35x^2 - 4 = 0$. Then find the roots and graph the related function.**

The polynomial has a degree of 4, so there are 4 complex roots.

Factor the equation to find the roots.

$$9x^4 - 35x^2 - 4 = 0$$
$$(9x^2 + 1)(x^2 - 4) = 0$$
$$(9x^2 + 1)(x + 2)(x - 2) = 0$$

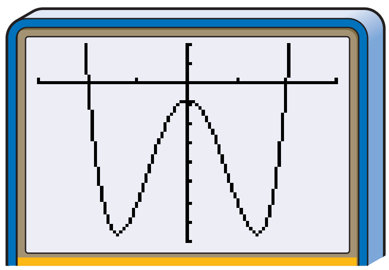
To find each root, set each factor equal to zero.

$$9x^2 + 1 = 0$$
$$x^2 = -\frac{1}{9} \quad \textit{Solve for } x^2.$$
$$x = \pm\sqrt{\frac{1}{9} \cdot (-1)} \quad \textit{Take the square root of each side.}$$
$$x = \pm\frac{1}{3}\sqrt{-1} \text{ or } \pm\frac{1}{3}\boldsymbol{i}$$

$$x + 2 = 0 \qquad x - 2 = 0$$
$$x = -2 \qquad x = 2$$

The roots are $\pm\frac{1}{3}\boldsymbol{i}$, -2, and 2.

Use a table of values or a graphing calculator to graph the function. The x-intercepts are -2 and 2.



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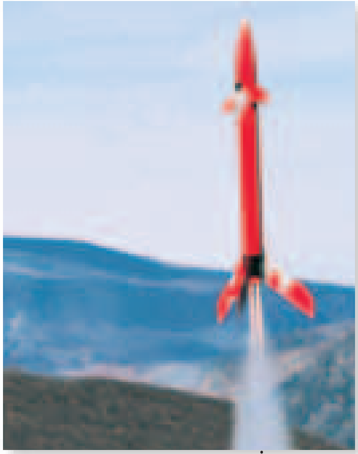
The function has an even degree and has 2 real zeros.

Example 5 Real World Application

METEOROLOGY **A meteorologist sends a temperature probe on a small weather rocket through a cloud layer. The launch pad for the rocket is 2 feet off the ground. The height of the rocket after launching is modeled by the equation $h = -16t^2 + 232t + 2$, where h is the height of the rocket in feet and t is the elapsed time in seconds.**



a. When will the rocket be 114 feet above the ground?

b. Verify your answer using a graph.

a.

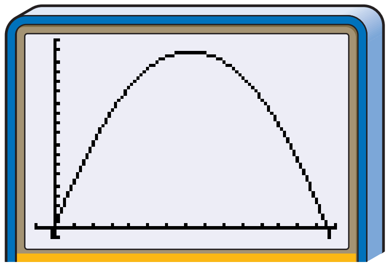
$$\begin{aligned} h &= -16t^2 + 232t + 2 & \\ 114 &= -16t^2 + 232t + 2 & \text{Replace } h \text{ with 114.} \\ 0 &= -16t^2 + 232t - 112 & \text{Subtract 114 from each side.} \\ 0 &= -8(2t^2 - 29t + 14) & \text{Factor.} \\ 0 &= -8(2t - 1)(t - 14) & \text{Factor.} \end{aligned}$$

$$2t - 1 = 0 \quad \text{or} \quad t - 14 = 0$$

$$t = \frac{1}{2} \qquad\qquad 14 = t$$

The weather rocket will be 114 feet above the ground after $\frac{1}{2}$ second and again after 14 seconds.

b. To verify the answer, graph $h(t) = -16t^2 + 232t - 112$. The graph appears to verify this solution.



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CHECK FOR UNDERSTANDING

Communicating Mathematics

Read and study the lesson to answer each question.

1. **Write** several sentences about the relationship between zeros and roots.
2. **Explain** why zeros of a function are also the x-intercepts of its graph.
3. **Define** a complex number and tell under what conditions it will be a pure imaginary number. Write two examples and two **nonexamples** of a pure imaginary numbers.
4. **Sketch** the general graph of a sixth-degree function.

Guided Practice

State the degree and leading coefficient of each polynomial.

5. $a^3 + 6a + 14$
6. $5m^2 + 8m^5 - 2$

Determine whether each number is a root of $x^3 - 5x^2 - 3x - 18 = 0$. Explain.

7. 5
8. 6

Write a polynomial equation of least degree for each set of roots. Does the equation have an odd or even degree? How many times does the graph of the related function cross the *x*-axis?

9. $-5, 7$ **10.** $6, 2\boldsymbol{i}, -2\boldsymbol{i}, \boldsymbol{i}, -\boldsymbol{i}$

State the number of complex roots of each equation. Then find the roots and graph the related functions.

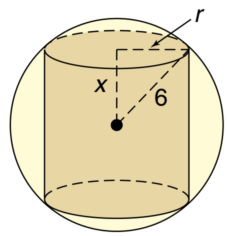
11. $x^2 - 14x + 49 = 0$ **12.** $a^3 + 2a^2 - 8a = 0$ **13.** $t^4 - 1 = 0$

14. Geometry A cylinder is inscribed in a sphere with a radius of 6 units as shown.

a. Write a function that models the volume of the cylinder in terms of x. (*Hint:* The volume of a cylinder equals $\pi r^2 h$.)

b. Write this function as a polynomial function.

c. Find the volume of the cylinder if $x = 4$.



EXERCISES

Practice

State the degree and leading coefficient of each polynomial.

15. $5t^4 + t^3 - 7$ **16.** $3x^7 - 4x^5 + x^3$ **17.** $9a^2 + 5a^3 - 10$

18. $14b - 25b^5$ **19.** $p^5 + 7p^3 - p^6$ **20.** $14y + 30 + y^2$

21. Determine if $x^3 + 3x + \sqrt{5}$ is a polynomial in one variable. Explain.

22. Is $\frac{1}{a} + a^2$ a polynomial in one variable? Explain.

Determine whether each number is a root of $a^4 - 13a^2 + 12a = 0$. Explain.

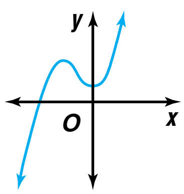
23. 0 **24.** -1 **25.** 1 **26.** -4 **27.** -3 **28.** 3

29. Is -2 a root of $b^4 - 3b^2 - 2b + 4 = 0$?

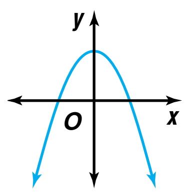
30. Is -1 a root of $x^4 - 4x^3 - x^2 + 4x = 0$?

31. Each graph represents a polynomial function. State the number of complex zeros and the number of real zeros of each function.

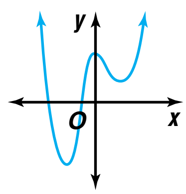
a.



b.



c.



Write a polynomial equation of least degree for each set of roots. Does the equation have an odd or even degree? How many times does the graph of the related function cross the *x*-axis?

32. $-2, 3$ **33.** $-1, 1, 5$ **34.** $-2, -0.5, 4$

35. $-3, -2\boldsymbol{i}, 2\boldsymbol{i}$ **36.** $-5\boldsymbol{i}, -\boldsymbol{i}, \boldsymbol{i}, 5\boldsymbol{i}$ **37.** $-1, 1, 4, -4, 5$

38. Write a polynomial equation of least degree whose roots are -1, 1, 3, and -3.

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State the number of complex roots of each equation. Then find the roots and graph the related function.

39. $x + 8 = 0$ **40.** $a^2 - 81 = 0$ **41.** $b^2 + 36 = 0$

42. $t^3 + 2t^2 - 4t - 8 = 0$ **43.** $n^3 - 9n = 0$ **44.** $6c^3 - 3c^2 - 45c = 0$

45. $a^4 + a^2 - 2 = 0$ **46.** $x^4 - 10x^2 + 9 = 0$ **47.** $4m^4 + 17m^2 + 4 = 0$

48. Solve $(u + 1)(u^2 - 1) = 0$ and graph the related polynomial function.

49. Sketch a fourth-degree equation for each situation.

a. no x-intercept **b.** one x-intercept **c.** two x-intercepts

d. three x-intercepts **e.** four x-intercepts **f.** five x-intercepts

Graphing Calculator

50. Use a graphing calculator to graph $f(x) = x^4 - 2x^2 + 1$.

a. What is the maximum number of x-intercepts possible for this function?

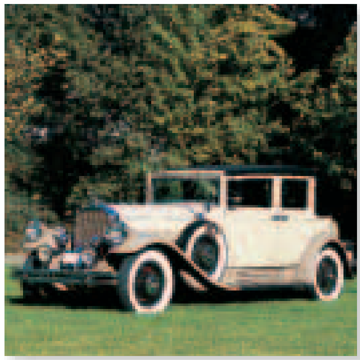
b. How many x-intercepts are there? Name the intercept(s).

c. Why are there fewer x-intercepts than the maximum number? (*Hint:* The factored form of the polynomial is $(x^2 - 1)^2$.)

Applications and Problem Solving

Real World Application

51. Classic Cars Sonia Orta invests in vintage automobiles. Three years ago, she purchased a 1953 Corvette roadster for \$99,000. Two years ago, she purchased a 1929 Pierce-Arrow Model 125 for \$55,000. A year ago she purchased a 1909 Cadillac Model Thirty for \$65,000.



a. Let x represent 1 plus the average rate of appreciation. Write a function in terms of x that models the value of the automobiles.

b. If the automobiles appreciate at an average annual rate of 15%, find the current value of the three automobiles.

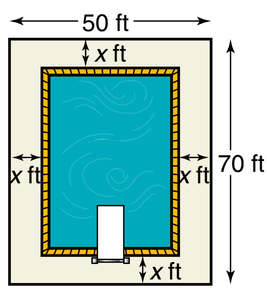
52. Critical Thinking One of the zeros of a polynomial function is 1. After translating the graph of the function left 2 units, 1 is a zero of the new function. What do you know about the original function?

53. Aeronautics At liftoff, the space shuttle *Discovery* has a constant acceleration, a, of 16.4 feet per second squared. The function $d(t) = \frac{1}{2}at^2$ can be used to determine the distance from Earth for each time interval, t, after takeoff.

a. Find its distance from Earth after 30 seconds, 1 minute, and 2 minutes.

b. Study the pattern of answers to part a. If the time the space shuttle is in flight doubles, how does the distance from Earth change? Explain.

54. Construction The Santa Fe Recreation Department has a 50-foot by 70-foot area for construction of a new public swimming pool. The pool will be surrounded by a concrete sidewalk of constant width. Because of water restrictions, the pool can have a maximum area of 2400 square feet. What should be the width of the sidewalk that surrounds the pool?



55. Marketing Each week, Marino's Pizzeria sells an average of 160 large supreme pizzas for \$16 each. Next week, the pizzeria plans to run a sale on these large supreme pizzas. The owner estimates that for each 40¢ decrease in the price, the store will sell approximately 16 more large pizzas. If the owner wants to sell \$4,000 worth of the large supreme pizzas next week, determine the sale price.

56. Critical Thinking If B and C are the real roots of $x^2 + Bx + C = 0$, where $B \neq 0$ and $C \neq 0$, find the values of B and C.

Mixed Review

57. Create a function in the form $y = f(x)$ that has a vertical asymptote at $x = -2$ and $x = 0$, and a hole at $x = 2$. *(Lesson 3-7)*

58. Construction Selena wishes to build a pen for her animals. He has 52 yards of fencing and wants to build a rectangular pen. *(Lesson 3-6)*

a. Find a model for the area of the pen as a function of the length and width of the rectangle.

b. What are the dimensions that would produce the maximum area?

59. Describe how the graphs of $y = 2x^3$ and $y = 2x^3 + 1$ are related. *(Lesson 3-2)*

60. Find the coordinates of P' if $P(4, 9)$ and P' are symmetric with respect to $M(-1, 9)$. *(Lesson 3-1)*

61. Find the determinant for $\begin{bmatrix} -15 & 5 \\ -9 & 3 \end{bmatrix}$. Tell whether an inverse exists for the matrix. *(Lesson 2-5)*

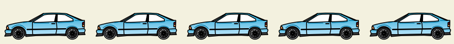
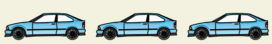
62. If $A = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -9 & 2 \\ 5 & 7 & -6 \end{bmatrix}$, find AB. *(Lesson 2-3)*

63. Graph $x + 4y < 9$. *(Lesson 1-8)*

64. The slope of $\overleftrightarrow{AB}$ is 0.6. The slope of $\overleftrightarrow{CD}$ is $\frac{3}{5}$. State whether the lines are *parallel, perpendicular,* or *neither.* Explain. *(Lesson 1-5)*

65. Find $[f \circ g](x)$ and $[g \circ f](x)$ for the functions $f(x) = x^2 - 4$ and $g(x) = \frac{1}{2}x + 6$. *(Lesson 1-2)*

66. SAT Practice In 2003, Bob's Quality Cars sold 270 more cars than in 2004. How many cars does each  represent?

Year	Cars Sold by Bob's Quality Cars
2003	
2004	

A 135
B 125
C 100
D 90
E 85

Extra Practice See p. A32.

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