4-2

Quadratic Equations

OBJECTIVES
- Solve quadratic equations.
- Use the discriminant to describe the roots of quadratic equations.

Real World Application

BASEBALL On September 8, 1998, Mark McGwire of the St. Louis Cardinals broke the home-run record with his 62nd home run of the year. He went on to hit 70 home runs for the season. Besides hitting home runs, McGwire also occasionally popped out. Suppose the ball was 3.5 feet above the ground when he hit it straight up with an initial velocity of 80 feet per second. The function $d(t) = 80t - 16t^2 + 3.5$ gives the ball's height above the ground in feet as a function of time in seconds. How long did the catcher have to get into position to catch the ball after it was hit? *This problem will be solved in Example 3.*

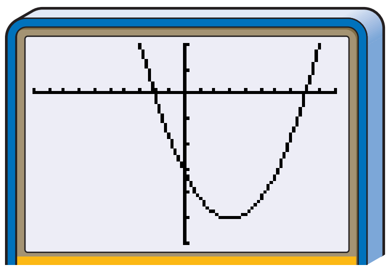
A quadratic equation is a polynomial equation with a degree of two. Solving quadratic equations by graphing usually does not yield exact answers. Also, some quadratic expressions are not factorable over the integers. Therefore, alternative strategies for solving these equations are needed. One such alternative is solving quadratic equations by **completing the square.**

Completing the square is a useful method when the quadratic is not easily factorable. It can be used to solve any quadratic equation. Remember that, for any number b, the square of the binomial $x + b$ has the form $x^2 + 2bx + b^2$. When completing the square, you know the first term and middle term and need to supply the last term. This term equals the square of half the coefficient of the middle term. For example, to complete the square of $x^2 + 8x$, find $\frac{1}{2}(8)$ and square the result. So, the third term is 16, and the expression becomes $x^2 + 8x + 16$. Note that this technique works only if the coefficient of x^2 is 1.

Example 1 **Solve $x^2 - 6x - 16 = 0$.**

This equation can be solved by graphing, factoring, or completing the square.

Method 1
Solve the equation by graphing the related function $f(x) = x^2 - 6x - 16$. The zeros of the function appear to be -2 and 8.



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Method 2
Solve the equation by factoring.

$x^2 - 6x - 16 = 0$

$(x + 2)(x - 8) = 0$ *Factor.*

$x + 2 = 0$ or $x - 8 = 0$

$x = -2$ $\quad$ $x = 8$

The roots of the equation are -2 and 8.

Method 3
Solve the equation by completing the square.

$$\begin{aligned} x^2 - 6x - 16 &= 0 \\ x^2 - 6x &= 16 && \textit{Add 16 to each side.} \\ x^2 - 6x + 9 &= 16 + 9 && \textit{Complete the square by adding } \left(\tfrac{-6}{2}\right)^2 \textit{ or 9 to each side.} \\ (x-3)^2 &= 25 && \textit{Factor the perfect square trinomial.} \\ x - 3 &= \pm 5 && \textit{Take the square root of each side.} \end{aligned}$$

$x - 3 = 5$ or $x - 3 = -5$
$x = 8$ $\quad x = -2$

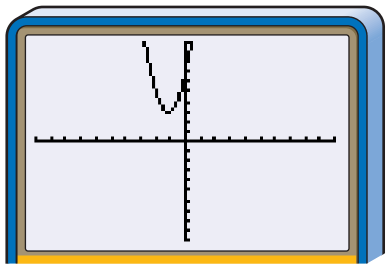
The roots of the equation are 8 and -2.

Although factoring may be an easier method to solve this particular equation, completing the square can always be used to solve any quadratic equation.

When solving a quadratic equation by completing the square, the leading coefficient must be 1. When the leading coefficient of a quadratic equation is not 1, you must first divide each side of the equation by that coefficient before completing the square.

Example 2 Solve $3n^2 + 7n + 7 = 0$ by completing the square.

Notice that the graph of the related function, $y = 3x^2 + 7x + 7$, does not cross the x-axis. Therefore, the roots of the equation are imaginary numbers. Completing the square can be used to find the roots of any equation, including one with no real roots.



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$$\begin{aligned} 3n^2 + 7n + 7 &= 0 \\ n^2 + \frac{7}{3}n + \frac{7}{3} &= 0 && \textit{Divide each side by 3.} \\ n^2 + \frac{7}{3}n &= -\frac{7}{3} && \textit{Subtract } \tfrac{7}{3} \textit{ from each side.} \\ n^2 + \frac{7}{3}n + \frac{49}{36} &= -\frac{7}{3} + \frac{49}{36} && \textit{Complete the square by adding } \left(\tfrac{7}{6}\right)^2 \textit{ or } \tfrac{49}{36} \textit{ to each side.} \\ \left(n + \frac{7}{6}\right)^2 &= -\frac{35}{36} && \textit{Factor the perfect square trinomial.} \\ n + \frac{7}{6} &= \pm \boldsymbol{i}\frac{\sqrt{35}}{6} && \textit{Take the square root of each side.} \\ n &= -\frac{7}{6} \pm \boldsymbol{i}\frac{\sqrt{35}}{6} && \textit{Subtract } \tfrac{7}{6} \textit{ from each side.} \end{aligned}$$

The roots of the equation are $-\frac{7}{6} \pm \boldsymbol{i}\frac{\sqrt{35}}{6}$ or $\frac{-7 \pm \boldsymbol{i}\sqrt{35}}{6}$.

Completing the square can be used to develop a general formula for solving any quadratic equation of the form $ax^2 + bx + c = 0$. This formula is called the **Quadratic Formula.**

Quadratic Formula

The roots of a quadratic equation of the form $ax^2 + bx + c = 0$ with $a \neq 0$ are given by the following formula.

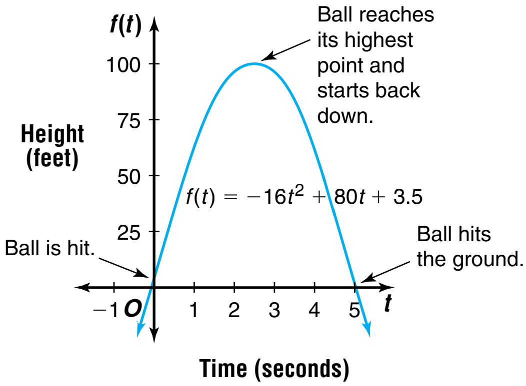
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The quadratic formula can be used to solve any quadratic equation. It is usually easier than completing the square.

Example 3 Real World Application

BASEBALL **Refer to the application at the beginning of the lesson. How long did the catcher have to get into position to catch the ball after if was hit?**

The catcher must get into position to catch the ball before $80t - 16t^2 + 3.5 = 0$. This equation can be written as $-16t^2 + 80t + 3.5 = 0$. Use the Quadratic Formula to solve this equation.



$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$t = \frac{-80 \pm \sqrt{80^2 - 4(-16)(3.5)}}{2(-16)} \qquad a = -16,\ b = 80, \text{ and } c = 3.5$$

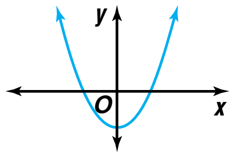
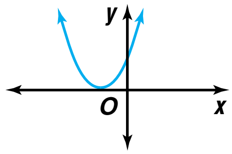
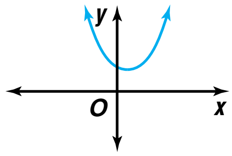
$$t = \frac{-80 \pm \sqrt{6624}}{-32}$$

$$t = \frac{-80 + \sqrt{6624}}{-32} \quad \text{or} \quad t = \frac{-80 - \sqrt{6624}}{-32}$$

$$t \approx -0.04 \qquad\qquad t \approx 5.04$$

The roots of the equation are about -0.04 and 5.04. Since the catcher has a positive amount of time to catch the ball, he will have about 5 seconds to get into position to catch the ball.

In the quadratic formula, the radicand $b^2 - 4ac$ is called the **discriminant** of the equation. The discriminant tells the nature of the roots of a quadratic equation or the zeros of the related quadratic function.

Discriminant	Nature of Roots/Zeros	Graph
$b^2 - 4ac > 0$	two distinct real roots/zeros	
$b^2 - 4ac = 0$	exactly one real root/zero (The one real root is actually a double root.)	
$b^2 - 4ac < 0$	no real roots/zero (two distinct imaginary roots/zeros)	

Example 4 **Find the discriminant of $x^2 - 4x + 15 = 0$ and describe the nature of the roots of the equation. Then solve the equation by using the Quadratic Formula.**

The value of the discriminant, $b^2 - 4ac$, is $(-4)^2 - 4(1)(15)$ or -44. Since the value of the discriminant is less than zero, there are no real roots.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

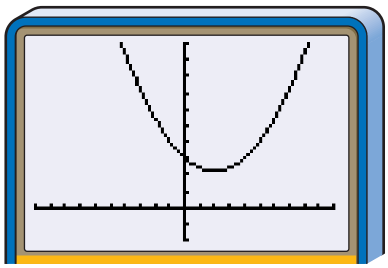
$$x = \frac{-(-4) \pm \sqrt{-44}}{2(1)}$$

$$x = \frac{4 \pm 2\boldsymbol{i}\sqrt{11}}{2}$$

$$x = 2 \pm \boldsymbol{i}\sqrt{11}$$

The roots are $2 + \boldsymbol{i}\sqrt{11}$ and $2 - \boldsymbol{i}\sqrt{11}$.

The graph of $y = x^2 - 4x + 15$ verifies that there are no real roots.



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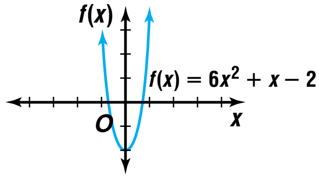
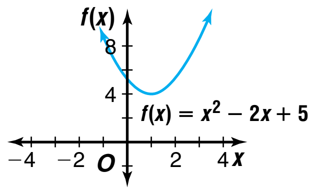
The roots of the equation in Example 4 are the complex numbers $2 + \boldsymbol{i}\sqrt{11}$ and $2 - \boldsymbol{i}\sqrt{11}$. A pair of complex numbers in the form $a + b\boldsymbol{i}$ and $a - b\boldsymbol{i}$ are called **conjugates.** Imaginary roots of polynomial equations with real coefficients always occur in conjugate pairs. Some other examples of complex conjugates are listed below.

$\boldsymbol{i}$ and $-\boldsymbol{i}$ $\qquad$ $-1 + \boldsymbol{i}$ and $-1 - \boldsymbol{i}$ $\qquad$ $-\boldsymbol{i}\sqrt{2}$ and $\boldsymbol{i}\sqrt{2}$

Complex Conjugates Theorem

Suppose a and b are real numbers with $b \neq 0$. If $a + b\boldsymbol{i}$ is a root of a polynomial equation with real coefficients, then $a - b\boldsymbol{i}$ is also a root of the equation. $a + b\boldsymbol{i}$ and $a - b\boldsymbol{i}$ are conjugate pairs.

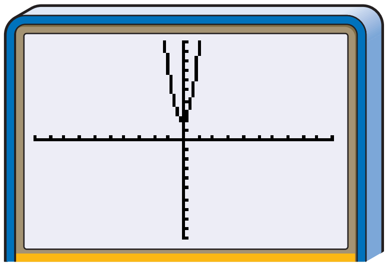
There are four methods used to solve quadratic equations. Two methods work for any quadratic equation. One method approximates any real roots, and one method only works for equations that can be factored over the integers.

Solution Method	Situation	Examples
Graphing	Usually, only approximate solutions are shown. If roots are imaginary (discriminant is less than zero), the graph has no *x*-intercepts, and the solutions must be found by another method. Graphing is a good method to verify solutions.	$6x^2 + x - 2 = 0$  $x = -\frac{2}{3}$ or $x = \frac{1}{2}$ $x^2 - 2x + 5 = 0$ discriminant: $(-2)^2 - 4(1)(5) = -16$  The equation has no real roots.
Factoring	When *a*, *b*, and *c* are integers and the discriminant is a perfect square or zero, this method is useful. It cannot be used if the discriminant is less than zero.	$g^2 + 2g - 8 = 0$ discriminant: $2^2 - 4(1)(-8) = 36$ $g^2 + 2g - 8 = 0$ $(g + 4)(g - 2) = 0$ $g + 4 = 0$ or $g - 2 = 0$ $g = -4$ $\quad$ $g = 2$
Completing the Square	This method works for any quadratic equation. There is more room for an arithmetic error than when using the Quadratic Formula.	$r^2 + 4r - 6 = 0$ $r^2 + 4r = 6$ $r^2 + 4r + 4 = 6 + 4$ $(r + 2)^2 = 10$ $r + 2 = \pm\sqrt{10}$ $r = -2 \pm \sqrt{10}$
Quadratic Formula	This method works for any quadratic equation.	$2s^2 + 5s + 4 = 0$ $s = \frac{-5 \pm \sqrt{5^2 - 4(2)(4)}}{2(2)}$ $s = \frac{-5 \pm \sqrt{-7}}{4}$ $s = \frac{-5 \pm i\sqrt{7}}{4}$

Example 5 **Solve $6x^2 + x + 2 = 0$.**

Method 1: Graphing

Graph $y = 6x^2 + x + 2$.



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The graph does not touch the x-axis, so there are no real roots for this equation. You cannot determine the roots from the graph.

Method 2: Factoring

Find the discriminant.

$b^2 - 4ac = 1^2 - 4(6)(2)$ or -47

The discriminant is less than zero, so factoring cannot be used to solve the equation.

Method 3: Completing the Square

$$6x^2 + x + 2 = 0$$

$$x^2 + \frac{1}{6}x + \frac{1}{3} = 0$$

$$x^2 + \frac{1}{6}x = -\frac{1}{3}$$

$$x^2 + \frac{1}{6}x + \frac{1}{144} = -\frac{1}{3} + \frac{1}{144}$$

$$\left(x + \frac{1}{12}\right)^2 = -\frac{47}{144}$$

$$x + \frac{1}{12} = \pm i\frac{\sqrt{47}}{12}$$

$$x = -\frac{1}{12} \pm i\frac{\sqrt{47}}{12}$$

$$x = \frac{-1 \pm i\sqrt{47}}{12}$$

Completing the square works, but this method requires a lot of steps.

Method 4: Quadratic Formula

For this equation, $a = 6$, $b = 1$, $c = 2$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-1 \pm \sqrt{1^2 - 4(6)(2)}}{2(6)}$$

$$x = \frac{-1 \pm \sqrt{-47}}{12}$$

$$x = \frac{-1 \pm i\sqrt{47}}{12}$$

The Quadratic Formula works and requires fewer steps than completing the square.

The roots of the equation are $\frac{-1 \pm i\sqrt{47}}{12}$.

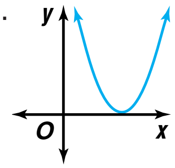
CHECK FOR UNDERSTANDING

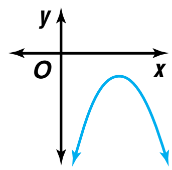
Communicating Mathematics

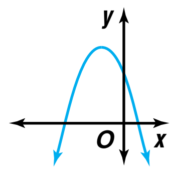
Read and study the lesson to answer each question.

1. **Write** a short paragraph explaining how to solve $t^2 - 6t - 4 = 0$ by completing the square.
2. **Discuss** which method of solving $5p^2 - 13p + 7 = 0$ would be most appropriate. Explain. Then solve.

3. **Describe** the discriminant of the equation represented by each graph.

a. 

b. 

c. 

4. *Math Journal* **Solve** $x^2 + 4x - 5 = 0$ using each of the four methods discussed in this lesson. Which method do you prefer? Explain.

Guided Practice

Solve each equation by completing the square.

5. $x^2 + 8x - 20 = 0$

6. $2a^2 + 11a - 21 = 0$

Find the discriminant of each equation and describe the nature of the roots of the equation. Then solve the equation by using the Quadratic Formula.

7. $m^2 + 12m + 36 = 0$

8. $t^2 - 6t + 13 = 0$

Solve each equation.

9. $p^2 - 6p + 5 = 0$

10. $r^2 - 4r + 10 = 0$

11. **Electricity** On a cold day, a 12-volt car battery has a resistance of 0.02 ohms. The power available to start the motor is modeled by the equation $P = 12\,I - 0.02\,I^2$, where I is the current in amperes. What current is needed to produce 1600 watts of power to start the motor?

EXERCISES

Practice

Solve each equation by completing the square.

12. $z^2 - 2z - 24 = 0$

13. $p^2 - 3p - 88 = 0$

14. $x^2 - 10x + 21 = 0$

15. $d^2 - \frac{3}{4}d + \frac{1}{8} = 0$

16. $3g^2 - 12g = -4$

17. $t^2 - 3t - 7 = 0$

18. What value of c makes $x^2 - x + c$ a perfect square?

19. Describe the nature of the roots of the equation $4n^2 + 6n + 25$. Explain.

Find the discriminant of each equation and describe the nature of the roots of the equation. Then solve the equation by using the Quadratic Formula.

20. $6m^2 + 7m - 3 = 0$

21. $s^2 - 5s + 9 = 0$

22. $36d^2 - 84d + 49 = 0$

23. $4x^2 - 2x + 9 = 0$

24. $3p^2 + 4p = 8$

25. $2k^2 + 5k = 9$

26. What is the conjugate of $-7 - \boldsymbol{i}\sqrt{5}$?

27. Name the conjugate of $5 - 2\boldsymbol{i}$.

Solve each equation.

28. $3s^2 - 5s + 9 = 0$ **29.** $x^2 - 3x - 28 = 0$ **30.** $4w^2 + 19w - 5 = 0$

31. $4r^2 - r = 5$ **32.** $p^2 + 2p + 8 = 0$ **33.** $x^2 - 2x\sqrt{6} - 2 = 0$

Applications and Problem Solving

34. Health Normal systolic blood pressure is a function of age. For a woman, the normal systolic pressure P in millimeters of mercury (mm Hg) is modeled by $P = 0.01A^2 + 0.05A + 107$, where A is age in years.

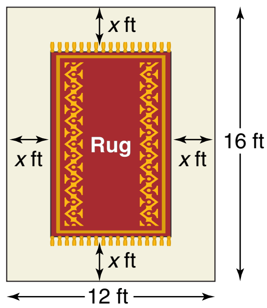
a. Use this model to determine the normal systolic pressure of a 25-year-old woman.

b. Use this model to determine the age of a woman whose normal systolic pressure is 125 mm Hg.

c. Sketch the graph of the function. Describe what happens to the normal systolic pressure as a woman gets older.

35. Critical Thinking Consider the equation $x^2 + 8x + c = 0$. What can you say about the value of c if the equation has two imaginary roots?

36. Interior Design Abey Numkena is an interior designer. She has been asked to locate an oriental rug for a new corporate office. As a rule, the rug should cover $\frac{1}{2}$ of the total floor area with a uniform width surrounding the rug.



a. If the dimensions of the room are 12 feet by 16 feet, write an equation to model the situation.

b. Graph the related function.

c. What are the dimensions of the rug?

37. Entertainment In an action movie, a stuntwoman jumps off a building that is 50 feet tall with an upward initial velocity of 5 feet per second. The distance $d(t)$ traveled by a free falling object can be modeled by the formula $d(t) = v_0t - \frac{1}{2}gt^2$, where v_0 is the initial velocity and g represents the acceleration due to gravity. The acceleration due to gravity is 32 feet per second squared.

a. Draw a graph that relates the woman's distance traveled with the time since the jump.

b. Name the x-intercepts of the graph.

c. What is the meaning of the x-intercepts of the graph?

d. Write an equation that could be used to determine when the stuntwoman will reach the safety pad on the ground. (*Hint:* The ground is -50 feet from the starting point.)

e. How long will it take the stuntwoman to reach the safety pad on the ground?

38. Critical Thinking Derive the quadratic formula by completing the square if $ax^2 + bx + c = 0$, $a \neq 0$.

Extra Practice See p. A32.

CONTENTS

Mixed Review

39. State the number of complex roots of the equation $18a^2 + 3a - 1 = 0$. Then find the roots and graph the related function. *(Lesson 4-1)*

40. Graph $y < |x| - 2$. *(Lesson 3-5)*

41. Find the inverse of $f(x) = (x - 9)^2$. *(Lesson 3-4)*

42. Solve the system of equations, $3x + 4y = 375$ and $5x + 2y = 345$. *(Lesson 2-1)*

43. **Sales** The Computer Factory is selling a 300 MHz computer system for \$595 and a 350 MHz computer system for \$619. At this rate, what would be the cost of a 400 MHz computer system? *(Lesson 1-4)*

44. Find the slope of the line whose equation is $3y + 8x = 12$. *(Lesson 1-3)*

45. **SAT/ACT Practice** The trinomial $x^2 + x - 20$ is exactly divisible by which binomial?

A $x - 4$ **B** $x + 4$ **C** $x + 6$ **D** $x - 10$ **E** $x - 5$

CAREER CHOICES

Environmental Engineering

Would you like a career where you will constantly be learning and have the opportunity to work both outdoors and indoors? Environmental engineering has become an important profession in the past twenty-five years.

As an environmental engineer, you might design, build, or maintain systems for controlling wastes produced by cities or industry. These wastes can include solid waste, waste water, hazardous waste, or air pollutants. You could work for a private company, a consulting firm, or the Environmental Protection Agency. Opportunities for advancement in this field include becoming a supervisor or consultant. You might even have the opportunity to identify a new specialty area in the field of environmental engineering!

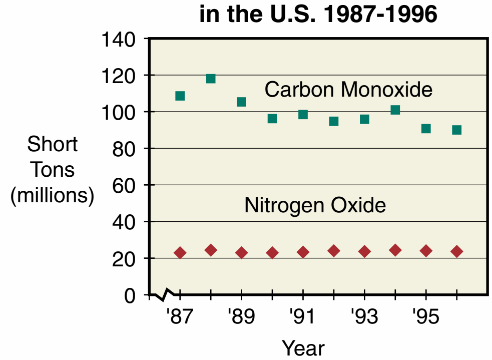
Career Overview

Degree Required:
Bachelor's degree in environmental engineering

Related Courses:
biology, chemistry, mathematics

Outlook:
number of jobs expected to increase though the year 2006

Emissions of Two Air Pollutants in the U.S. 1987-1996



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