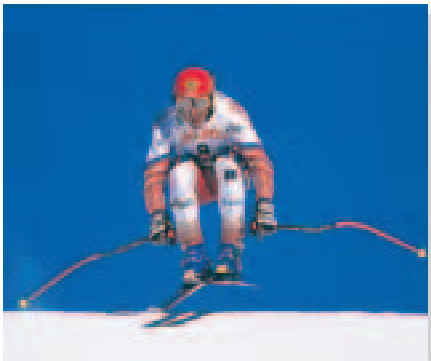
4-3 The Remainder and Factor Theorems

OBJECTIVE

- Find the factors of polynomials using the Remainder and Factor Theorems.

Real World Application

SKIING On December 13, 1998, Olympic champion Hermann (The Herminator) Maier won the super-G at Val d'Isere, France. His average speed was 73 meters per second. The average recreational skier skis at a speed of about 5 meters per second. Suppose you were skiing at a speed of 5 meters per second and heading downhill, accelerating at a rate of 0.8 meter per second squared. How far will you travel in 30 seconds? *This problem will be solved in Example 1.*



Hermann Maier

Consider the polynomial function $f(a) = 2a^2 + 3a - 8$. Since 2 is a factor of 8, it may be possible that $a - 2$ is a factor of $2a^2 + 3a - 8$. Suppose you use long division to divide the polynomial by $a - 2$.

$$\begin{array}{r l}
 & 2a + 7 \quad \leftarrow \textit{quotient} \\
\textit{divisor} \rightarrow a - 2 \,\big)\!\! & \overline{2a^2 + 3a - 8} \quad \leftarrow \textit{dividend} \\
 & \underline{2a^2 - 4a} \\
 & 7a - 8 \\
 & \underline{7a - 14} \\
 & 6 \quad \leftarrow \textit{remainder}
\end{array}$$

From arithmetic, you may remember that the dividend equals the product of the divisor and the quotient plus the remainder. For example, $44 \div 7 = 6$ R2, so $44 = 7(6) + 2$. This relationship can be applied to polynomials.

You may want to verify that $(a - 2)(2a + 7) + 6 = 2a^2 + 3a - 8$.

$$f(a) = (a - 2)(2a + 7) + 6 \quad \Longleftrightarrow \quad f(a) = 2a^2 + 3a - 8$$

Let $a = 2$.

$$f(2) = (2 - 2)[2(2) + 7] + 6 \qquad f(2) = 2(2^2) + 3(2) - 8$$

$$= 0 + 6 \text{ or } 6 \qquad\qquad = 8 + 6 - 8 \text{ or } 6$$

Notice that the value of $f(2)$ is the same as the remainder when the polynomial is divided by $a - 2$. This example illustrates the **Remainder Theorem.**

Remainder Theorem

If a polynomial $P(x)$ is divided by $x - r$, the remainder is a constant $P(r)$, and

$$P(x) = (x - r) \cdot Q(x) + P(r),$$

where $Q(x)$ is a polynomial with degree one less than the degree of $P(x)$.

The Remainder Theorem provides another way to find the value of the polynomial function $P(x)$ for a given value of r. The value will be the remainder when $P(x)$ is divided by $x - r$.

Example 1 Real World Application

SKIING **Refer to the application at the beginning of the lesson. The formula for distance traveled is $d(t) = v_0t + \frac{1}{2}at^2$, where $d(t)$ is the distance traveled, v_0 is the initial velocity, t is the time, and a is the acceleration. Find the distance traveled after 30 seconds.**

The distance formula becomes $d(t) = 5t + \frac{1}{2}(0.8)t^2$ or $d(t) = 0.4t^2 + 5t$. You can use one of two methods to find the distance after 30 seconds.

Method 1

Divide $0.4t^2 + 5t$ by $t - 30$.

$$\begin{array}{r|l} & 0.4t + 17 \\ \hline t - 30 & 0.4t^2 + 5t \\ & \underline{0.4t^2 - 12t} \\ & \quad 17t + 0 \\ & \quad \underline{17t - 510} \\ & \qquad 510 \rightarrow D(30) = 510 \end{array}$$

Method 2

Evaluate $d(t)$ for $t = 30$.

$$\begin{aligned} d(t) &= 0.4t^2 + 5t \\ d(30) &= 0.4(30^2) + 5(30) \\ &= 0.4(900) + 5(30) \\ &= 510 \end{aligned}$$

By either method, the result is the same. You will travel 510 meters in 30 seconds.

Long division can be very time consuming. **Synthetic division** is a shortcut for dividing a polynomial by a binomial of the form $x - r$. The steps for dividing $x^3 + 4x^2 - 3x - 5$ by $x + 3$ using synthetic division are shown below.

Step 1 Arrange the terms of the polynomial in descending powers of x. Insert zeros for any missing powers of x. Then, write the coefficients as shown.

$$\begin{array}{cccc} x^3 & + 4x^2 & - 3x & - 5 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 1 & 4 & -3 & -5 \end{array}$$

For $x + 3$, the value of r is -3.

Step 2 Write the constant r of the divisor $x - r$. In this case, write -3.

$$\begin{array}{r|rrrr} -3 & 1 & 4 & -3 & -5 \end{array}$$

Step 3 Bring down the first coefficient.

$$\begin{array}{r|rrrr} -3 & 1 & 4 & -3 & -5 \\ \hline & 1 & & & \end{array}$$

Step 4 Multiply the first coefficient by r. Then write the product under the next coefficient. Add.

$$\begin{array}{r|rrrr} -3 & 1 & 4 & -3 & -5 \\ \times & & -3 & & \\ \hline & 1 & 1 & & \end{array}$$

Step 5 Multiply the sum by r. Then write the product under the next coefficient. Add.

$$\begin{array}{r|rrrr} -3 & 1 & 4 & -3 & -5 \\ \times & & -3 & -3 & \\ \hline & 1 & 1 & -6 & \end{array}$$

Notice that a vertical bar separates the quotient from the remainder.

Step 6 Repeat Step 5 for all coefficients in the dividend.

$$\begin{array}{r|rrr|r} -3 & 1 & 4 & -3 & -5 \\ \times & & -3 & -3 & 18 \\ \hline & 1 & 1 & -6 & 13 \end{array}$$

Step 7 The final sum represents the remainder, which in this case is 13. The other numbers are the coefficients of the quotient polynomial, which has a degree one less than the dividend. Write the quotient $x^2 + x - 6$ with remainder 13. *Check the results using long division.*

Example 2 **Divide $x^3 - x^2 + 2$ by $x + 1$ using synthetic division.**

$$\begin{array}{r|rrr|r} -1 & 1 & -1 & 0 & 2 \\ & & -1 & 2 & -2 \\ \hline & 1 & -2 & 2 & 0 \end{array}$$

Notice there is no x term. A zero is placed in this position as a placeholder.

The quotient is $x^2 - 2x + 2$ with a remainder of 0.

In Example 2, the remainder is 0. Therefore, $x + 1$ is a factor of $x^3 - x^2 + 2$. If $f(x) = x^3 - x^2 + 2$, then $f(-1) = (-1)^3 - (-1)^2 + 2$ or 0. This illustrates the **Factor Theorem,** which is a special case of the Remainder Theorem.

Factor Theorem The binomial $x - r$ is a factor of the polynomial $P(x)$ if and only if $P(r) = 0$.

Example 3 **Use the Remainder Theorem to find the remainder when $2x^3 - 3x^2 + x$ is divided by $x - 1$. State whether the binomial is a factor of the polynomial. Explain.**

Find $f(1)$ to see if $x - 1$ is a factor.

$f(x) = 2x^3 - 3x^2 + x$

$f(1) = 2(1^3) - 3(1^2) + 1$ *Replace x with 1.*

$= 2(1) - 3(1) + 1$ or 0

Since $f(1) = 0$, the remainder is 0. So the binomial $(x - 1)$ is a factor of the polynomial by the Factor Theorem.

When a polynomial is divided by one of its binomial factors $x - r$, the quotient is called a **depressed polynomial.** A depressed polynomial has a degree less than the original polynomial. In Example 3, $x - 1$ is a factor of $2x^3 - 3x^2 + x$. Use synthetic division to find the depressed polynomial.

$$\begin{array}{r|rrr|r} 1 & 2 & -3 & 1 & 0 \\ & & 2 & -1 & 0 \\ \hline & 2 & -1 & 0 & 0 \end{array}$$

$$\downarrow \quad \downarrow \quad \downarrow$$

$$2x^2 - 1x + 0$$

Thus, $(2x^3 - 3x^2 + x) \div (x - 1) = 2x^2 - x$.

The depressed polynomial is $2x^2 - x$.

A depressed polynomial may also be the product of two polynomial factors, which would give you other zeros of the polynomial function. In this case, $2x^2 - x$ equals $x(2x - 1)$. So, the zeros of the polynomial function $f(x) = 2x^3 - 3x^2 + x$ are 0, $\frac{1}{2}$, and 1.

Note that the values of r where no remainder occurs are also factors of the constant term of the polynomial.

You can also find factors of a polynomial such as $x^3 + 2x^2 - 16x - 32$ by using a shortened form of synthetic division to test several values of r. In the table, the first column contains various values of r. The next three columns show the coefficients of the depressed polynomial. The fifth column shows the remainder. Any value of r that results in a remainder of zero indicates that $x - r$ is a factor of the polynomial. The factors of the original polynomial are $x + 4$, $x + 2$, and $x - 4$.

r	1	2	−16	−32
−4	1	−2	−8	0
−3	1	−1	−13	7
−2	1	0	−16	0
−1	1	1	−17	−15
0	1	2	−16	−32
1	1	3	−13	−45
2	1	4	−8	−48
3	1	5	−1	−35
4	1	6	8	0

Look at the pattern of values in the last column. Notice that when r = 1, 2, and 3, the values of f(x) decrease and then increase. This indicates that there is an x-coordinate of a relative minimum between 1 and 3.

Example 4 **Determine the binomial factors of $x^3 - 7x + 6$.**

Graphing Calculator Tip

The **TABLE** feature can help locate integral zeros. Enter the polynomial function as **Y1** in the **Y=** menu and press **TABLE**. Search the **Y1** column to find 0 and then look at the corresponding x-value.

Method 1 Use synthetic division.

r	1	0	−7	6	
−4	1	−4	9	−30	
−3	1	−3	2	0	*x + 3 is a factor.*
−2	1	−2	−3	12	
−1	1	−1	−6	12	
0	1	0	−7	6	
1	1	1	−6	0	*x − 1 is a factor.*
2	1	2	−3	0	*x − 2 is a factor.*

Method 2 Test some values using the Factor Theorem.

$f(x) = x^3 - 7x + 6$
$f(-1) = (-1)^3 - 7(-1) + 6$ or 12
$f(1) = 1^3 - 7(1) + 6$ or 0

Because $f(1) = 0$, $x - 1$ is a factor. Find the depressed polynomial.

$$\begin{array}{r|rrrr} 1 & 1 & 0 & -7 & 6 \\ & & 1 & 1 & -6 \\ \hline & 1 & 1 & -6 & 0 \end{array}$$

The depressed polynomial is $x^2 + x - 6$. Factor the depressed polynomial.

$x^2 + x - 6 = (x + 3)(x - 2)$

The factors of $x^3 + x - 6$ are $x + 3$, $x - 1$, and $x - 2$. *Verify the results.*

The Remainder Theorem can be used to determine missing coefficients.

Example 5 **Find the value of k so that the remainder of $(x^3 + 3x^2 - kx - 24) \div (x + 3)$ is 0.**

If the remainder is to be 0, $x + 3$ must be a factor of $x^3 + 3x^2 - kx - 24$. So, $f(-3)$ must equal 0.

$$\begin{aligned} f(x) &= x^3 + 3x^2 - kx - 24 \\ f(-3) &= (-3)^3 + 3(-3)^2 - k(-3) - 24 \\ 0 &= -27 + 27 + 3k - 24 \quad \textit{Replace } f(-3) \textit{ with } 0. \\ 0 &= 3k - 24 \\ 8 &= k \end{aligned}$$

The value of k is 8. Check using synthetic division.

$$\begin{array}{r|rrrr} -3 & 1 & 3 & -8 & -24 \\ & & -3 & 0 & 24 \\ \hline & 1 & 0 & -8 & 0 \checkmark \end{array}$$

CHECK FOR UNDERSTANDING

Communicating Mathematics

Read and study the lesson to answer each question.

1. **Explain** how the Remainder Theorem and the Factor Theorem are related.

2. **Write** the division problem illustrated by the synthetic division. What is the quotient? What is the remainder?

$$\begin{array}{r|rrrr} 5 & 1 & -4 & -7 & 8 \\ & & 5 & 5 & -10 \\ \hline & 1 & 1 & -2 & -2 \end{array}$$

3. **Compare** the degree of a polynomial and its depressed polynomial.

4. **You Decide** Brittany tells Isabel that if $x + 3$ is a factor of the polynomial function $f(x)$, then $f(3) = 0$. Isabel argues that if $x + 3$ is a factor of $f(x)$, then $f(-3) = 0$. Who is correct? Explain.

Guided Practice

Divide using synthetic division.

5. $(x^2 - x + 4) \div (x - 2)$
6. $(x^3 + x^2 - 17x + 15) \div (x + 5)$

Use the Remainder Theorem to find the remainder for each division. State whether the binomial is a factor of the polynomial.

7. $(x^2 + 2x - 15) \div (x - 3)$
8. $(x^4 + x^2 + 2) \div (x - 3)$

Determine the binomial factors of each polynomial.

9. $x^3 - 5x^2 - x + 5$
10. $x^3 - 6x^2 + 11x - 6$

11. Find the value of k so that the remainder of $(x^3 - 7x + k) \div (x + 1)$ is 2.

12. Let $f(x) = x^7 + x^9 + x^{12} - 2x^2$.
   a. State the degree of $f(x)$.
   b. State the number of complex zeros that $f(x)$ has.
   c. State the degree of the depressed polynomial that would result from dividing $f(x)$ by $x - a$.
   d. Find one factor of $f(x)$.

13. **Geometry** A cylinder has a height 4 inches greater than the radius of its base. Find the radius and the height to the nearest inch if the volume of the cylinder is 5π cubic inches.

interNET CONNECTION

Graphing Calculator Program
For a graphing calculator program that computes the value of a function visit **www.amc.glencoe.com**

EXERCISES

Practice

Divide using synthetic division.

14. $(x^2 + 20x + 91) \div (x + 7)$
15. $(x^3 - 9x^2 + 27x - 28) \div (x - 3)$
16. $(x^4 + x^3 - 1) \div (x - 2)$
17. $(x^4 - 8x^2 + 16) \div (x + 2)$
18. $(3x^4 - 2x^3 + 5x^2 - 4x - 2) \div (x + 1)$
19. $(2x^3 - 2x - 3) \div (x - 1)$

Use the Remainder Theorem to find the remainder for each division. State whether the binomial is a factor of the polynomial.

20. $(x^2 - 2) \div (x - 1)$
21. $(x^5 + 32) \div (x + 2)$
22. $(x^4 - 6x^2 + 8) \div (x - \sqrt{2})$
23. $(x^3 - x + 6) \div (x - 2)$
24. $(4x^3 + 4x^2 + 2x + 3) \div (x - 1)$
25. $(2x^3 - 3x^2 + x) \div (x - 1)$

26. Which binomial is a factor of the polynomial $x^3 + 3x^2 - 2x - 8$?
a. $x - 1$ **b.** $x + 1$ **c.** $x - 2$ **d.** $x + 2$

27. Verify that $x - \sqrt{6}$ is a factor of $x^4 - 36$.

28. Use synthetic division to find all the factors of $x^3 + 7x^2 - x - 7$ if one of the factors is $x + 1$.

Determine the binomial factors of each polynomial.

29. $x^3 + x^2 - 4x - 4$ **30.** $x^3 - x^2 - 49x + 49$ **31.** $x^3 - 5x^2 + 2x + 8$

32. $x^3 - 2x^2 - 4x + 8$ **33.** $x^3 + 4x^2 - x - 4$ **34.** $x^3 + 3x^2 + 3x + 1$

35. How many times is 2 a root of $x^6 - 9x^4 + 24x^2 - 16 = 0$?

36. Determine how many times -1 is a root of $x^3 + 2x^2 - x - 2 = 0$. Then find the other roots.

Find the value of *k* so that each remainder is zero.

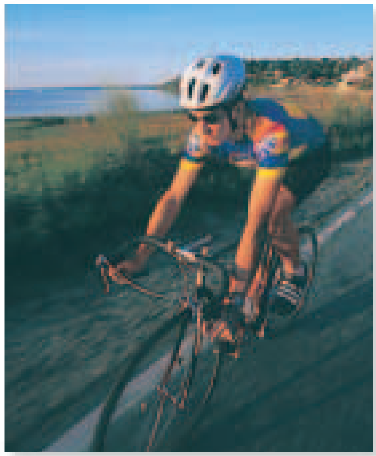
37. $(2x^3 - x^2 + x + k) \div (x - 1)$ **38.** $(x^3 - kx^2 + 2x - 4) \div (x - 2)$

39. $(x^3 + 18x^2 + kx + 4) \div (x + 2)$ **40.** $(x^3 + 4x^2 - kx + 1) \div (x + 1)$

Applications and Problem Solving

Real World Application

41. Bicycling Matthew is cycling at a speed of 4 meters per second. When he starts down a hill, the bike accelerates at a rate of 0.4 meter per second squared. The vertical distance from the top of the hill to the bottom of the hill is 25 meters. Use the equation $d(t) = v_0 t + \frac{1}{2}at^2$ to find how long it will take Matthew to ride down the hill.



42. Critical Thinking Determine a and b so that when $x^4 + x^3 - 7x^2 + ax + b$ is divided by $(x - 1)(x + 2)$, the remainder is 0.

43. Sculpting Esteban is preparing to start an ice sculpture. He has a block of ice that is 3 feet by 4 feet by 5 feet. Before he starts, he wants to reduce the volume of the ice by shaving off the same amount from the length, the width, and the height.

a. Write a polynomial function to model the situation.

b. Graph the function.

c. He wants to reduce the volume of the ice to $\frac{3}{5}$ of the original volume. Write an equation to model the situation.

d. How much should he take from each dimension?

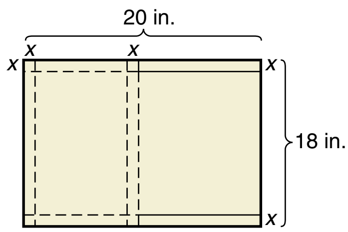
44. Manufacturing An 18-inch by 20-inch sheet of cardboard is cut and folded to make a box for the Great Pecan Company.

a. Write an polynomial function to model the volume of the box.

b. Graph the function.

c. The company wants the box to have a volume of 224 cubic inches. Write an equation to model this situation.

d. Find a positive integer for x.



45. **Critical Thinking** Find a, b, and c for $P(x) = ax^2 + bx + c$ if $P(3 + 4\boldsymbol{i}) = 0$ and $P(3 - 4\boldsymbol{i}) = 0$.

Mixed Review

46. Solve $r^2 + 5r - 8 = 0$ by completing the square. *(Lesson 4-2)*

47. Determine whether each number is a root of $x^4 - 4x^3 - x^2 + 4x = 0$. *(Lesson 4-1)*
   a. 2 **b.** 0 **c.** -2 **d.** 4

48. Find the critical points of the graph of $f(x) = x^5 - 32$. Determine whether each represents a *maximum*, a *minimum*, or a *point of inflection*. *(Lesson 3-6)*

49. Describe the transformation(s) that have taken place between the parent graph of $y = x^2$ and the graph of $y = 0.5(x + 1)^2$. *(Lesson 3-2)*

50. **Business** Pristine Pipes Inc. produces plastic pipe for use in newly-built homes. Two of the basic types of pipe have different diameters, wall thickness, and strengths. The strength of a pipe is increased by mixing a special additive into the plastic before it is molded. The table below shows the resources needed to produce 100 feet of each type of pipe and the amount of the resource available each week.

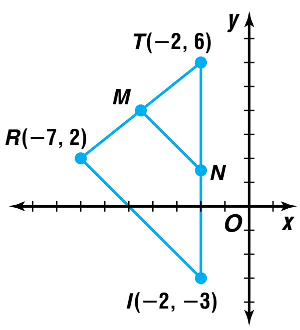
Resource	Pipe A	Pipe B	Resource Availability
Extrusion Dept.	4 hours	6 hours	48 hours
Packaging Dept.	2 hours	2 hours	18 hours
Strengthening Additive	2 pounds	1 pound	16 pounds

If the profit on 100 feet of type A pipe is \$34 and of type B pipe is \$40, how much of each should be produced to maximize the profit? *(Lesson 2-7)*

51. Solve the system of equations. *(Lesson 2-2)*

$$4x + 2y + 3z = 6$$
$$2x + 7y = 3z$$
$$-3x - 9y + 13 = -2z$$

52. **Geometry** Show that the line segment connecting the midpoints of sides $\overline{TR}$ and $\overline{TI}$ is parallel to $\overline{RI}$. *(Lesson 1-5)*



53. **SAT/ACT Practice** If $a > b$ and $c < 0$, which of the following are true?
I. $ac < bc$
II. $a + c > b + c$
III. $a - c < b - c$

A I only **B** II only **C** III only
D I and II only **E** I, II, and III

Extra Practice See p. A32.