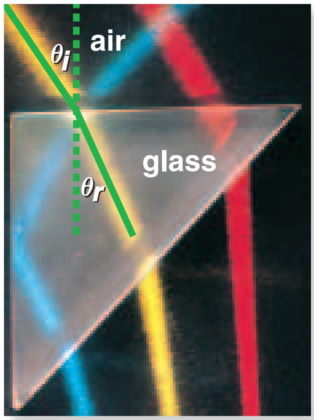
5-2 Trigonometric Ratios in Right Triangles

OBJECTIVE

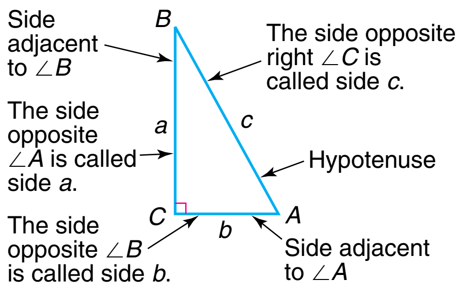
- Find the values of trigonometric ratios for acute angles of right triangles.

Real World Application

PHYSICS As light passes from one substance such as air to another substance such as glass, the light is bent. The relationship between the angle of incidence θ_i and the angle of refraction θ_r is given by Snell's Law, $\frac{\sin \theta_i}{\sin \theta_r} = n$, where $\sin \theta$ represents a trigonometric ratio and n is a constant called the *index of refraction.* Suppose a ray of light passes from air with an angle of incidence of 50° to glass with an angle of refraction of 32° 16′. Find the index of refraction of the glass. *This problem will be solved in Example 2.*

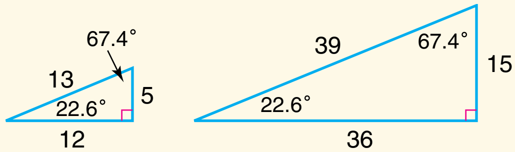


In a right triangle, one of the angles measures 90°, and the remaining two angles are *acute* and *complementary.* The longest side of a right triangle is known as the **hypotenuse** and is opposite the right angle. The other two sides are called **legs.** The leg that is a side of an acute angle is called the **side adjacent** to the angle. The other leg is the **side opposite** the angle.



GRAPHING CALCULATOR EXPLORATION

Use a graphing calculator to find each ratio for the 22.6° angle in each triangle. Record each ratio as a decimal. Make sure your calculator is in degree mode.



$$R_1 = \frac{\text{side opposite}}{\text{hypotenuse}} \qquad R_2 = \frac{\text{side adjacent}}{\text{hypotenuse}}$$

$$R_3 = \frac{\text{side opposite}}{\text{side adjacent}}$$

Find the same ratios for the 67.4° angle in each triangle.

TRY THESE

1. Draw two other triangles that are similar to the given triangles.

2. Find each ratio for the 22.6° angle in each triangle.

3. Find each ratio for the 67.4° angle in each triangle.

WHAT DO YOU THINK?

4. Make a conjecture about R_1, R_2, and R_3 for any right triangle with a 22.6° angle.

5. Is your conjecture true for any 67.4° angle in a right triangle?

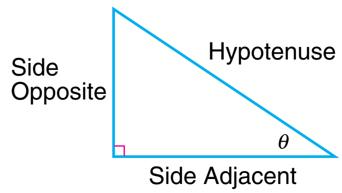
6. Do you think your conjecture is true for any acute angle of a right triangle? Why?

If two angles of a triangle are congruent to two angles of another triangle, the triangles are similar. If an acute angle of one right triangle is congruent to an acute angle of another right triangle, the triangles are similar, and the ratios of the corresponding sides are equal. Therefore, any two congruent angles of different right triangles will have equal ratios associated with them.

In right triangles, the Greek letter θ (theta) is often used to denote a particular angle.

The ratios of the sides of the right triangles can be used to define the **trigonometric ratios.** The ratio of the side opposite θ and the hypotenuse is known as the **sine.** The ratio of the side adjacent θ and the hypotenuse is known as the **cosine.** The ratio of the side opposite θ and the side adjacent θ is known as the **tangent.**

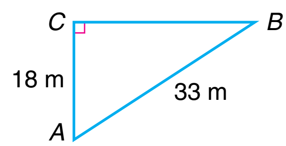
	Words	Symbol	Definition
Trigonometric Ratios	sine θ	sin θ	$\sin\theta = \frac{\text{side opposite}}{\text{hypotenuse}}$
	cosine θ	cos θ	$\cos\theta = \frac{\text{side adjacent}}{\text{hypotenuse}}$
	tangent θ	tan θ	$\tan\theta = \frac{\text{side opposite}}{\text{side adjacent}}$



SOH-CAH-TOA is a mnemonic device commonly used for remembering these ratios.

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

Example 1 **Find the values of the sine, cosine, and tangent for ∠B.**



First, find the length of $\overline{BC}$

$(AC)^2 + (BC)^2 = (AB)^2$ *Pythagorean Theorem*

$18^2 + (BC)^2 = 33^2$ *Substitute 18 for AC and 33 for AB.*

$(BC)^2 = 765$

$BC = \sqrt{765}$ or $3\sqrt{85}$ *Take the square root of each side. Disregard the negative root.*

Then write each trigonometric ratio.

$\sin B = \frac{\text{side opposite}}{\text{hypotenuse}}$ $\qquad \cos B = \frac{\text{side adjacent}}{\text{hypotenuse}}$ $\qquad \tan B = \frac{\text{side opposite}}{\text{side adjacent}}$

$\sin B = \frac{18}{33}$ or $\frac{6}{11}$ $\qquad \cos B = \frac{3\sqrt{85}}{33}$ or $\frac{\sqrt{85}}{11}$ $\qquad \tan B = \frac{18}{3\sqrt{85}}$ or $\frac{6\sqrt{85}}{85}$

Trigonometric ratios are often simplified, but never written as mixed numbers.

In Example 1, you found the exact values of the sine, cosine, and tangent ratios. You can use a calculator to find the approximate decimal value of any of the trigonometric ratios for a given angle.

Example **2** **PHYSICS** **Refer to the application at the beginning of the lesson. Find the index of refraction of the glass.**

Real World Application

$\frac{\sin \theta_i}{\sin \theta_r} = n$ *Snell's Law*

$\frac{\sin 50°}{\sin 32° 16'} = n$ *Substitute 50° for θ_i and 32° 16′ for θ_r.*

$\frac{0.7660444431}{0.5338605056} \approx n$ *Use a calculator to find each sine ratio.*

$1.434914992 \approx n$ *Use a calculator to find the quotient.*

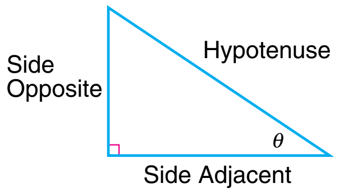
The index of refraction of the glass is about 1.4349.

Graphing Calculator Tip

If using your graphing calculator to do the calculation, make sure you are in degree mode.

In addition to the trigonometric ratios sine, cosine, and tangent, there are three other trigonometric ratios called **cosecant**, **secant**, and **cotangent.** These ratios are the reciprocals of sine, cosine, and tangent, respectively.

	Words	Symbol	Definition
Reciprocal Trigonometric Ratios	cosecant θ	csc θ	$\csc \theta = \frac{1}{\sin \theta}$ or $\frac{\text{hypotenuse}}{\text{side opposite}}$
	secant θ	sec θ	$\sec \theta = \frac{1}{\cos \theta}$ or $\frac{\text{hypotenuse}}{\text{side adjacent}}$
	cotangent θ	cot θ	$\cot \theta = \frac{1}{\tan \theta}$ or $\frac{\text{side adjacent}}{\text{side opposite}}$



These definitions are called the reciprocal identities.

Examples **3** **a. If $\cos \theta = \frac{3}{4}$, find $\sec \theta$.**

$\sec \theta = \frac{1}{\cos \theta}$

$\sec \theta = \frac{1}{\frac{3}{4}}$ or $\frac{4}{3}$

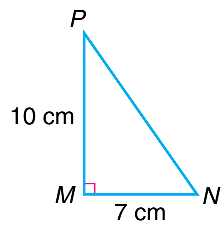
b. If $\csc \theta = 1.345$, find $\sin \theta$.

$\sin \theta = \frac{1}{\csc \theta}$

$\sin \theta = \frac{1}{1.345}$ or about 0.7435

4 **Find the values of the six trigonometric ratios for $\angle P$.**

First determine the length of the hypotenuse.



$(MP)^2 + (MN)^2 = (NP)^2$ *Pythagorean Theorem*

$10^2 + 7^2 = (NP)^2$ *Substitute 10 for MP and 7 for MN.*

$149 = (NP)^2$

$\sqrt{149} = NP$ *Disregard the negative root.*

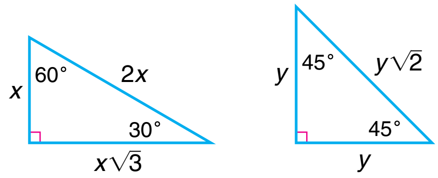
$\sin P = \frac{\text{side opposite}}{\text{hypotenuse}}$ $\cos P = \frac{\text{side adjacent}}{\text{hypotenuse}}$ $\tan P = \frac{\text{side opposite}}{\text{side adjacent}}$

$\sin P = \frac{7}{\sqrt{149}}$ or $\frac{7\sqrt{149}}{149}$ $\cos P = \frac{10}{\sqrt{149}}$ or $\frac{10\sqrt{149}}{149}$ $\tan P = \frac{7}{10}$

$\csc P = \frac{\text{hypotenuse}}{\text{side opposite}}$ $\sec P = \frac{\text{hypotenuse}}{\text{side adjacent}}$ $\cot P = \frac{\text{side adjacent}}{\text{side opposite}}$

$\csc P = \frac{\sqrt{149}}{7}$ $\sec P = \frac{\sqrt{149}}{10}$ $\cot P = \frac{10}{7}$

Consider the special relationships among the sides of $30°-60°-90°$ and $45°-45°-90°$ triangles.



These special relationships can be used to determine the trigonometric ratios for 30°, 45°, and 60°. You should memorize the sine, cosine, and tangent values for these angles.

θ	sin θ	cos θ	tan θ	csc θ	sec θ	cot θ
30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2\sqrt{3}}{3}$	$\sqrt{3}$
45°	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	$\sqrt{2}$	$\sqrt{2}$	1
60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2	$\frac{\sqrt{3}}{3}$

Note that $\sin 30° = \cos 60°$ and $\cos 30° = \sin 60°$. This is an example showing that the sine and cosine are **cofunctions.** That is, if θ is an acute angle, $\sin \theta = \cos (90° - \theta)$. Similar relationships hold true for the other trigonometric ratios.

Cofunctions

$\sin \theta = \cos (90° - \theta)$ $\quad$ $\cos \theta = \sin (90° - \theta)$

$\tan \theta = \cot (90° - \theta)$ $\quad$ $\cot \theta = \tan (90° - \theta)$

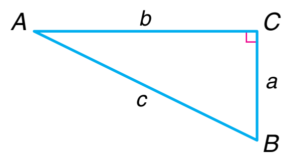
$\sec \theta = \csc (90° - \theta)$ $\quad$ $\csc \theta = \sec (90° - \theta)$

CHECK FOR UNDERSTANDING

Communicating Mathematics

Read and study the lesson to answer each question.

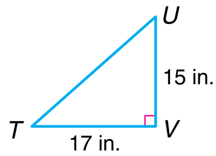
1. **Explain** in your own words how to decide which side is opposite the given acute angle of a right triangle and which side is adjacent to the given angle.
2. **State** the reciprocal ratios of sine, cosine, and tangent.
3. **Write** each trigonometric ratio for $\angle A$ in triangle *ABC.*



4. **Compare** sin *A* and cos *B,* csc *A* and sec *B,* and tan *A* and cot *B.*

Guided Practice

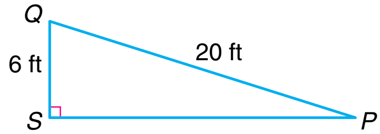
5. Find the values of the sine, cosine, and tangent for $\angle T$.



6. If $\sin\theta = \frac{2}{5}$, find $\csc\theta$.

7. If $\cot\theta = 1.5$, find $\tan\theta$.

8. Find the values of the six trigonometric ratios for $\angle P$.



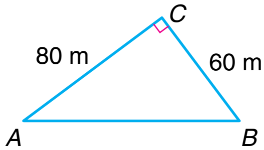
9. **Physics** You may have polarized sunglasses that eliminate glare by polarizing the light. When light is polarized, all of the waves are traveling in parallel planes. Suppose vertically polarized light with intensity I_o strikes a polarized filter with its axis at an angle of θ with the vertical. The intensity of the transmitted light I_t and θ are related by the equation $\cos\theta = \sqrt{\frac{I_t}{I_o}}$. If θ is 45°, write I_t as a function of I_o.

EXERCISES

Practice

Find the values of the sine, cosine, and tangent for each $\angle A$.

10.



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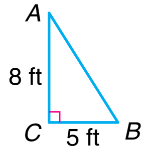

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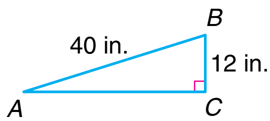
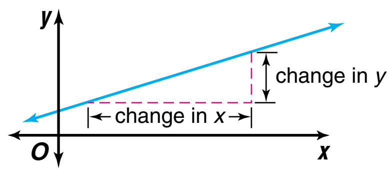


13. The slope of a line is the ratio of the change of y to the change of x. Name the trigonometric ratio of θ that equals the slope of line m.



14. If $\tan\theta = \frac{1}{3}$, find $\cot\theta$.

15. If $\sin\theta = \frac{3}{7}$, find $\csc\theta$.

16. If $\sec\theta = \frac{5}{9}$, find $\cos\theta$.

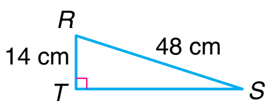
17. If $\csc\theta = 2.5$, find $\sin\theta$.

18. If $\cot\theta = 0.75$, find $\tan\theta$.

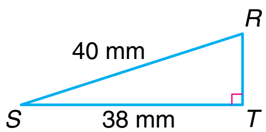
19. If $\cos\theta = 0.125$, find $\sec\theta$.

Find the values of the six trigonometric ratios for each $\angle R$.

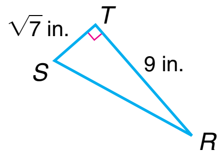
20.



21.



22.



23. If $\tan\theta = 1.3$, what is the value of $\cot(90° - \theta)$?

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24. Use a calculator to determine the value of each trigonometric ratio.

a. sin 52° 47′ b. cos 79° 15′ c. tan 88° 22′ 45″

d. cot 36° (*Hint*: Tangent and cotangent have a reciprocal relationship.)

Graphing Calculator

25. Use the table function on a graphing calculator to complete the table. Round values to three decimal places.

θ	**72°**	**74°**	**76°**	**78°**	**80°**	**82°**	**84°**	**86°**	**88°**
sin	0.951	0.961							
cos	0.309								

a. What value does sin θ approach as θ approaches 90°?

b. What value does cos θ approach as θ approaches 90°?

26. Use the table function on a graphing calculator to complete the table. Round values to three decimal places.

θ	**18°**	**16°**	**14°**	**12°**	**10°**	**8°**	**6°**	**4°**	**2°**
sin	0.309	0.276							
cos	0.951								
tan									

a. What value does sin θ approach as θ approaches 0°?

b. What value does cos θ approach as θ approaches 0°?

c. What value does tan θ approach as θ approaches 0°?

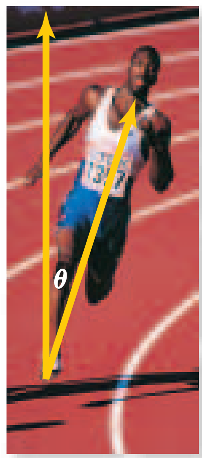
Applications and Problem Solving

Real World Application

27. **Physics** Suppose a ray of light passes from air to Lucite. The measure of the angle of incidence is 45°, and the measure of an angle of refraction is 27° 55′. Use Snell's Law, which is stated in the application at the beginning of the lesson, to find the index of refraction for Lucite.

28. **Critical Thinking** The sine of an acute $\angle R$ of a right triangle is $\frac{3}{7}$. Find the values of the other trigonometric ratios for this angle.

29. **Track** When rounding a curve, the acute angle θ that a runner's body makes with the vertical is called the angle of incline. It is described by the equation $\tan \theta = \frac{v^2}{gr}$, where v is the velocity of the runner, g is the acceleration due to gravity, and r is the radius of the track. The acceleration due to gravity is a constant 9.8 meters per second squared. Suppose the radius of the track is 15.5 meters.



a. What is the runner's velocity if the angle of incline is 11°?

b. Find the runner's velocity if the angle of incline is 13°.

c. What is the runner's velocity if the angle of incline is 15°?

d. Should a runner increase or decrease her velocity to increase his or her angle of incline?

30. **Critical Thinking** Use the fact that $\sin \theta = \frac{\text{side opposite}}{\text{hypotenuse}}$ and $\cos \theta = \frac{\text{side adjacent}}{\text{hypotenuse}}$ to write an expression for tan θ in terms of sin θ and cos θ.

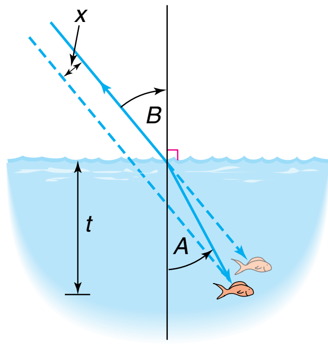
31. **Architecture** The angle of inclination of the sun affects the heating and cooling of buildings. The angle is greater in the summer than the winter. The sun's angle of inclination also varies according to the latitude. The sun's angle of inclination at noon equals $90° - L - 23.5° \times \cos\left[\frac{(N + 10)360}{365}\right]$. In this expression, L is the latitude of the building site, and N is the number of days elapsed in the year.

   a. The latitude of Brownsville, Texas, is 26°. Find the angle of inclination for Brownsville on the first day of summer (day 172) and on the first day of winter (day 355).

   b. The latitude of Nome, Alaska, is 64°. Find the angle of inclination for Nome on the first day of summer and on the first day of winter.

   c. Which city has the greater change in the angle of inclination?

32. **Biology** An object under water is not exactly where it appears to be. The displacement x depends on the angle A at which the light strikes the surface of the water from below, the depth t of the object, and the angle B at which the light leaves the surface of the water. The measure of displacement is modeled by the equation $x = t\left(\frac{\sin (B - A)}{\cos A}\right)$. Suppose a biologist is trying to net a fish under water. Find the measure of displacement if t measures 10 centimeters, the measure of angle A is 41°, and the measure of angle B is 60°.



Mixed Review

33. Change 88.37° to degrees, minutes, and seconds. *(Lesson 5-1)*

34. Find the number of possible positive real zeros and the number of possible negative real zeros for $f(x) = x^4 + 2x^3 - 6x - 1$. *(Lesson 4-4)*

35. **Business** Luisa Diaz is planning to build a new factory for her business. She hires an analyst to gather data and develop a mathematical model. In the model $P(x) = 18 + 92x - 2x^2$, P is Ms. Diaz's monthly profit, and x is the number of employees needed to staff the new facility. *(Lesson 3-6)*

   a. How many employees should she hire to maximize profits?

   b. What is her maximum profit?

36. Find the value of $\begin{vmatrix} 7 & -3 & 5 \\ 4 & 0 & -1 \\ 8 & 2 & 0 \end{vmatrix}$. *(Lesson 2-5)*

37. Write the slope-intercept form of the equation of the line that passes through points at (2, 5) and (6, 3). *(Lesson 1-4)*

38. **SAT/ACT Practice** The area of a right triangle is 12 square inches. The ratio of the lengths of its legs is 2:3. Find the length of the hypotenuse.

   A $\sqrt{13}$ in. **B** 26 in. **C** $2\sqrt{13}$ in. **D** 52 in. **E** $4\sqrt{13}$ in.

Extra Practice See p. A34.