5-3

Trigonometric Functions on the Unit Circle

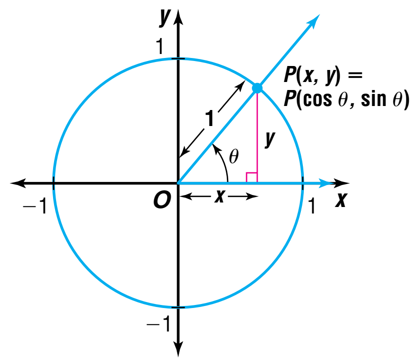
OBJECTIVES
- Find the values of the six trigonometric functions using the unit circle.
- Find the values of the six trigonometric functions of an angle in standard position given a point on its terminal side.

Real World Application

FOOTBALL The longest punt in NFL history was 98 yards. The punt was made by Steve O'Neal of the New York Jets in 1969. When a football is punted, the angle made by the initial path of the ball and the ground affects both the height and the distance the ball will travel. If a football is punted from ground level, the maximum height it will reach is given by the formula $h = \frac{v_0^2 \sin^2 \theta}{2g}$, where v_0 is the initial velocity, θ is the measure of the angle between the ground and the initial path of the ball, and g is the acceleration due to gravity. The value of g is 9.8 meters per second squared. Suppose the initial velocity of the ball is 28 meters per second. Describe the possible maximum height of the ball if the angle is between 0° and 90°. *This problem will be solved in Example 2.*

A **unit circle** is a circle of radius 1. Consider a unit circle whose center is at the origin of a rectangular coordinate system. The unit circle is symmetric with respect to the x-axis, the y-axis, and the origin.

Consider an angle θ between 0° and 90° in standard position. Let $P(x, y)$ be the point of intersection of the angle's terminal side with the unit circle. If a perpendicular segment is drawn from point P to the x-axis, a right triangle is created. In the triangle, the side adjacent to angle θ is along the x-axis and has length x. The side opposite angle θ is the perpendicular segment and has length y. According to the Pythagorean Theorem, $x^2 + y^2 = 1$. We can find values for $\sin \theta$ and $\cos \theta$ using the definitions used in Lesson 5-2.



$$\sin \theta = \frac{\text{side opposite}}{\text{hypotenuse}} \qquad \cos \theta = \frac{\text{side adjacent}}{\text{hypotenuse}}$$

$$\sin \theta = \frac{y}{1} \text{ or } y \qquad \cos \theta = \frac{x}{1} \text{ or } x$$

Right triangles can also be formed for angles greater than 90°. In these cases, the reference angle is one of the acute angles. Similar results will occur. Thus, **sine** θ can be redefined as the y-coordinate and **cosine** θ can be redefined as the x-coordinate.

Sine and Cosine If the terminal side of an angle θ in standard position intersects the unit circle at $P(x, y)$, then $\cos \theta = x$ and $\sin \theta = y$.

Since there is exactly one point $P(x, y)$ for any angle θ, the relations $\cos\theta = x$ and $\sin\theta = y$ are functions of θ. Because they are both defined using the unit circle, they are often called **circular functions.**

The domain of the sine and cosine functions is the set of real numbers, since $\sin\theta$ and $\cos\theta$ are defined for any angle θ. The range of the sine and the cosine functions is the set of real numbers between -1 and 1 inclusive, since $(\cos\theta, \sin\theta)$ are the coordinates of points on the unit circle.

In addition to the sine and cosine functions, the four other **trigonometric functions** can also be defined using the unit circle.

$$\tan\theta = \frac{\text{side opposite}}{\text{side adjacent}} = \frac{y}{x} \qquad \csc\theta = \frac{\text{hypotenuse}}{\text{side opposite}} = \frac{1}{y}$$

$$\sec\theta = \frac{\text{hypotenuse}}{\text{side adjacent}} = \frac{1}{x} \qquad \cot\theta = \frac{\text{side adjacent}}{\text{side opposite}} = \frac{x}{y}$$

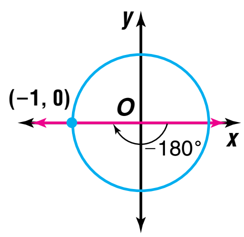
Since division by zero is undefined, there are several angle measures that are excluded from the domain of the tangent, cotangent, secant, and cosecant functions.

Examples

1 Use the unit circle to find each value.

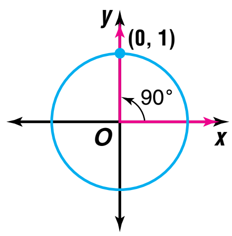
a. cos (−180°)

The terminal side of a $-180°$ angle in standard position is the negative x-axis, which intersects the unit circle at $(-1, 0)$. The x-coordinate of this ordered pair is $\cos(-180°)$. Therefore, $\cos(-180°) = -1$.



b. sec 90°

The terminal side of a $90°$ angle in standard position is the positive y-axis, which intersects the unit circle at $(0, 1)$. According to the definition of secant, $\sec 90° = \frac{1}{x}$ or $\frac{1}{0}$, which is undefined. Therefore, $\sec 90°$ is undefined.



2 FOOTBALL Refer to the application at the beginning of the lesson. Describe the possible maximum height of the ball if the angle is between 0° and 90°.

Real World Application

The expression $\sin^2\theta$ means the square of the sine of θ or $(\sin\theta)^2$.

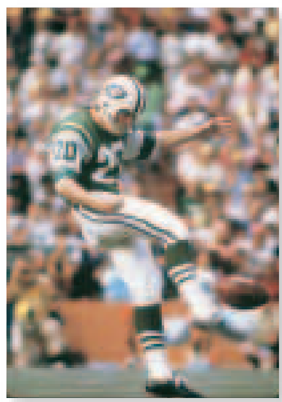
Find the value of h when $\theta = 0°$.

$$h = \frac{v_0^2 \sin^2\theta}{2g}$$

$$h = \frac{28^2 \sin^2 0°}{2(9.8)} \qquad v_0 = 28,\ \theta = 0°,\ g = 9.8$$

$$h = \frac{28^2(0^2)}{2(9.8)} \qquad \sin 0° = 0$$

$$h = 0$$



Find the value of h when $\theta = 90°$.

$$h = \frac{v_0^{\,2} \sin^2 \theta}{2g}$$

$$h = \frac{28^2 \sin^2 90°}{2(9.8)} \qquad v_0 = 28,\ \theta = 90°,\ g = 9.8$$

$$h = \frac{28^2(1^2)}{2(9.8)} \qquad \sin 90° = 1$$

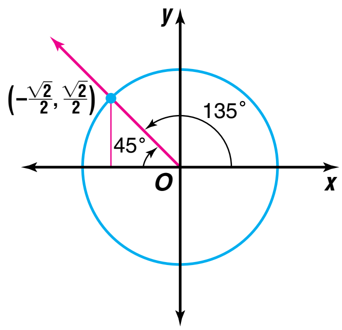
$$h = 40$$

The maximum height of the ball is between 0 meters and 40 meters.

The radius of a circle is defined as a positive value. Therefore, the signs of the six trigonometric functions are determined by the signs of the coordinates of x and y in each quadrant.

Example 3 **Use the unit circle to find the values of the six trigonometric functions for a 135° angle.**

Since 135° is between 90° and 180°, the terminal side is in the second quadrant. Therefore, the reference angle is $180° - 135°$ or 45°. The terminal side of a 45° angle intersects the unit circle at a point with coordinates $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$. Because the terminal side of a 135° angle is in the second quadrant, the x-coordinate is negative, and the y-coordinate is positive. The point of intersection has coordinates $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$.



$$\sin 135° = y \qquad \cos 135° = x \qquad \tan 135° = \frac{y}{x}$$

$$\sin 135° = \frac{\sqrt{2}}{2} \qquad \cos 135° = -\frac{\sqrt{2}}{2} \qquad \tan 135° = \frac{\frac{\sqrt{2}}{2}}{-\frac{\sqrt{2}}{2}}$$

$$\tan 135° = -1$$

$$\csc 135° = \frac{1}{y} \qquad \sec 135° = \frac{1}{x} \qquad \cot 135° = \frac{x}{y}$$

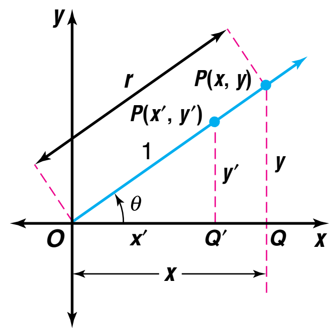
$$\csc 135° = \frac{1}{\frac{\sqrt{2}}{2}} \qquad \sec 135° = \frac{1}{-\frac{\sqrt{2}}{2}} \qquad \cot 135° = \frac{\frac{-\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}}$$

$$\csc 135° = \frac{2}{\sqrt{2}} \qquad \sec 135° = -\frac{2}{\sqrt{2}} \qquad \cot 135° = -1$$

$$\csc 135° = \sqrt{2} \qquad \sec 135° = -\sqrt{2}$$

The sine and cosine functions of an angle in standard position may also be determined using the ordered pair of any point on its terminal side and the distance between that point and the origin.

Suppose $P(x, y)$ and $P'(x', y')$ are two points on the terminal side of an angle with measure θ, where P' is on the unit circle. Let $OP = r$. By the Pythagorean Theorem, $r = \sqrt{x^2 + y^2}$. Since P' is on the unit circle, $OP' = 1$. Triangles $OP'Q'$ and OPQ are similar. Thus, the lengths of corresponding sides are proportional.



$$\frac{x'}{1} = \frac{x}{r} \qquad \frac{y'}{1} = \frac{y}{r}$$

Therefore, $\cos\theta = x'$ or $\frac{x}{r}$ and $\sin\theta = y'$ or $\frac{y}{r}$.

All six trigonometric functions can be determined using x, y, and r. The ratios do not depend on the choice of P. They depend only on the measure of θ.

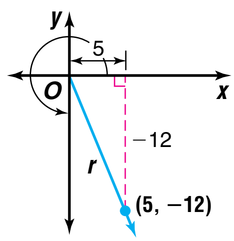
Trigonometric Functions of an Angle in Standard Position

For any angle in standard position with measure θ, a point $P(x, y)$ on its terminal side, and $r = \sqrt{x^2 + y^2}$, the trigonometric functions of θ are as follows.

$$\sin\theta = \frac{y}{r} \qquad \cos\theta = \frac{x}{r} \qquad \tan\theta = \frac{y}{x}$$

$$\csc\theta = \frac{r}{y} \qquad \sec\theta = \frac{r}{x} \qquad \cot\theta = \frac{x}{y}$$

Example 4 **Find the values of the six trigonometric functions for angle θ in standard position if a point with coordinates $(5, -12)$ lies on its terminal side.**



You know that $x = 5$ and $y = -12$. You need to find r.

$r = \sqrt{x^2 + y^2}$ *Pythagorean Theorem*

$r = \sqrt{5^2 + (-12)^2}$ *Substitute 5 for x and −12 for y.*

$r = \sqrt{169}$ or 13 *Disregard the negative root.*

Now write the ratios.

$\sin\theta = \frac{y}{r}$ $\qquad$ $\cos\theta = \frac{x}{r}$ $\qquad$ $\tan\theta = \frac{y}{x}$

$\sin\theta = \frac{-12}{13}$ or $-\frac{12}{13}$ $\qquad$ $\cos\theta = \frac{5}{13}$ $\qquad$ $\tan\theta = \frac{-12}{5}$ or $-\frac{12}{5}$

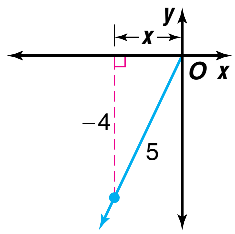
$\csc\theta = \frac{r}{y}$ $\qquad$ $\sec\theta = \frac{r}{x}$ $\qquad$ $\cot\theta = \frac{x}{y}$

$\csc\theta = \frac{13}{-12}$ or $-\frac{13}{12}$ $\qquad$ $\sec\theta = \frac{13}{5}$ $\qquad$ $\cot\theta = \frac{5}{-12}$ or $-\frac{5}{12}$

If you know the value of one of the trigonometric functions and the quadrant in which the terminal side of θ lies, you can find the values of the remaining five functions.

Example 5 **Suppose θ is an angle in standard position whose terminal side lies in Quadrant III. If $\sin\theta = -\frac{4}{5}$, find the values of the remaining five trigonometric functions of θ.**

To find the other function values, you must find the coordinates of a point on the terminal side of θ. Since $\sin\theta = -\frac{4}{5}$ and r is always positive, $r = 5$ and $y = -4$.



Find x.

$r^2 = x^2 + y^2$ *Pythagorean Theorem*

$5^2 = x^2 + (-4)^2$ *Substitute 5 for r and −4 for y.*

$9 = x^2$

$\pm 3 = x$ *Take the square root of each side.*

Since the terminal side of θ lies in Quadrant III, x must be negative.

Thus, $x = -3$.

Now use the values of x, y, and r to find the remaining five trigonometric functions of θ.

$\cos\theta = \frac{x}{r}$

$\cos\theta = \frac{-3}{5}$ or $-\frac{3}{5}$

$\tan\theta = \frac{y}{x}$

$\tan\theta = \frac{-4}{-3}$ or $\frac{4}{3}$

$\csc\theta = \frac{r}{y}$

$\csc\theta = \frac{5}{-4}$ or $-\frac{5}{4}$

$\sec\theta = \frac{r}{x}$

$\sec\theta = \frac{5}{-3}$ or $-\frac{5}{3}$

$\cot\theta = \frac{x}{y}$

$\cot\theta = \frac{-3}{-4}$ or $\frac{3}{4}$

Notice that the cosine and secant have the same sign. This will always be true since $\sec\theta = \frac{1}{\cos\theta}$. Similar relationships exist for the other reciprocal identities. *You will complete a chart for this in Exercise 4.*

CHECK FOR UNDERSTANDING

Communicating Mathematics

Read and study the lesson to answer each question.

1. **Explain** why csc 180° is undefined.
2. **Show** that the value of sin θ increases as θ goes from 0° to 90° and then decreases as θ goes from 90° to 180°.
3. **Confirm** that $\cot \theta = \frac{\cos \theta}{\sin \theta}$.
4. *Math Journal* **Draw** a unit circle. Use the drawing to complete the chart below that indicates the sign of the trigonometric functions in each quadrant.

Function	Quadrant			
	I	II	III	IV
sin α or csc α	+			
cos α or sec α	+			
tan α or cot α				

Guided Practice

Use the unit circle to find each value.

5. tan 180°
6. sec (−90°)

Use the unit circle to find the values of the six trigonometric functions for each angle.

7. 30°
8. 225°

Find the values of the six trigonometric functions for angle θ in standard position if a point with the given coordinates lies on its terminal side.

9. (3, 4)
10. (−6, 6)

Suppose θ is an angle in standard position whose terminal side lies in the given quadrant. For each function, find the values of the remaining five trigonometric functions for θ.

11. $\tan \theta = -1$; Quadrant IV
12. $\cos \theta = -\frac{1}{2}$; Quadrant II
13. **Map Skills** The distance around Earth along a given latitude can be found using the formula $C = 2\pi r \cos L$, where r is the radius of Earth and L is the latitude. The radius of Earth is approximately 3960 miles. Describe the distances along the latitudes as you go from 0° at the equator to 90° at the poles.

EXERCISES

Practice

Use the unit circle to find each value.

14. sin 90°
15. tan 360°
16. cot (−180°)
17. csc 270°
18. cos (−270°)
19. sec 180°
20. Find two values of θ for which $\sin \theta = 0$.
21. If $\cos \theta = 0$, what is $\sec \theta$?

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CONTENTS

Use the unit circle to find the values of the six trigonometric functions for each angle.

22. $45°$ **23.** $150°$ **24.** $315°$ **25.** $210°$ **26.** $330°$ **27.** $420°$

28. Find $\cot\,(-45°)$.

29. Find $\csc 390°$.

Find the values of the six trigonometric functions for angle θ in standard position if a point with the given coordinates lies on its terminal side.

30. $(-4, -3)$ **31.** $(-6, 6)$ **32.** $(2, 0)$ **33.** $(1, -8)$ **34.** $(5, -3)$ **35.** $(-8, 15)$

36. The terminal side of one angle in standard position contains the point with coordinates $(5, -6)$. The terminal side of another angle in standard position contains the point with coordinates $(-5, 6)$. Compare the sines of these angles.

37. If $\sin \theta < 0$, where would the terminal side of the angle be located?

Suppose θ is an angle in standard position whose terminal side lies in the given quadrant. For each function, find the values of the remaining five trigonometric functions for θ.

38. $\cos \theta = -\frac{12}{13}$; Quadrant III

39. $\csc \theta = 2$; Quadrant II

40. $\sin \theta = -\frac{1}{5}$; Quadrant IV

41. $\tan \theta = 2$; Quadrant I

42. $\sec \theta = \sqrt{3}$; Quadrant IV

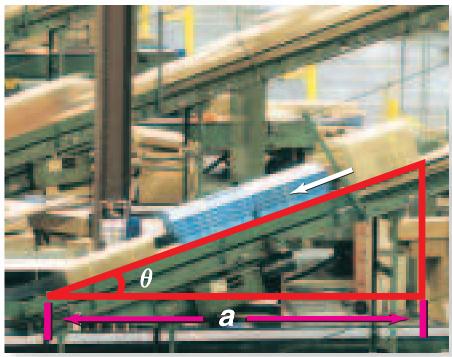
43. $\cot \theta = 1$; Quadrant III

44. If $\csc \theta = -2$ and θ lies in Quadrant III, find $\tan \theta$.

Applications and Problem Solving

Real World Application

45. Physics If you ignore friction, the amount of time required for a box to slide down an inclined plane is $\sqrt{\frac{2a}{g \sin \theta \cos \theta}}$, where a is the horizontal distance defined by the inclined plane, g is the acceleration due to gravity, and θ is the angle of the inclined plane. For what values of θ is the expression undefined?



46. Critical Thinking For each statement, describe k.

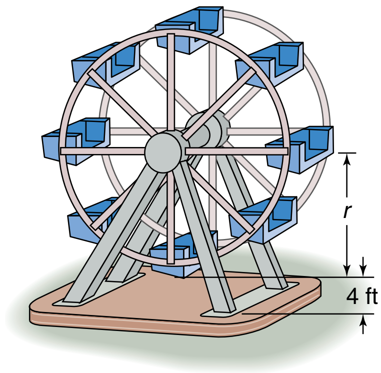
a. $\tan\,(k \cdot 90°) = 0$

b. $\sec\,(k \cdot 90°)$ is undefined.

47. Physics For polarized light, $\cos \theta = \sqrt{\frac{I_t}{I_o}}$, where θ is the angle of the axis of the polarized filter with the vertical, I_t is the intensity of the transmitted light, and I_O is the intensity of the vertically-polarized light striking the filter. Under what conditions would $I_t = I_O$?

48. Critical Thinking The terminal side of an angle θ in standard position coincides with the line $y = -3x$ and lies in Quadrant II. Find the six trigonometric functions of θ.

49. Entertainment Domingo decides to ride the Ferris wheel at the local carnival. When he gets into the seat that is at the bottom of the Ferris wheel, he is 4 feet above the ground.



a. If the radius of the Ferris wheel is 36 feet, how far above the ground will Domingo be when his seat reaches the top?

b. The Ferris wheel rotates 300° counterclockwise and stops to let another passenger on the ride. How far above the ground is Domingo when the Ferris wheel stops?

c. Suppose the radius of the Ferris wheel is only 30 feet. How far above the ground is Domingo after the Ferris wheel rotates 300°?

d. Suppose the radius of the Ferris wheel is r. Write an expression for the distance from the ground to Domingo after the Ferris wheel rotates 300°.

Mixed Review

50. If $\csc \theta = \frac{7}{5}$, find $\sin \theta$. *(Lesson 5-2)*

51. If a $-840°$ angle is in standard position, determine a coterminal angle that is between 0° and 360° and state the quadrant in which the terminal side lies. *(Lesson 5-1)*

52. Solve $5 - \sqrt{b + 2} = 0$. *(Lesson 4-7)*

53. Solve $4x^2 - 9x + 5 = 0$ by using the quadratic formula. *(Lesson 4-2)*

54. If y varies directly as x and $y = 9$ when x is -15, find y when $x = 21$. *(Lesson 3-8)*

55. Graph the inverse of $f(x) = x^2 - 16$. *(Lesson 3-4)*

56. Find the multiplicative inverse of $\begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix}$. *(Lesson 2-5)*

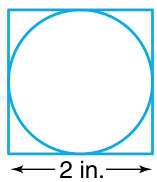
57. Solve the system of equations. *(Lesson 2-2)*

$8m - 3n - 4p = 6$
$4m + 9n - 2p = -4$
$6m + 12n + 5p = -1$

58. State whether each of the points at $(9, 3)$, $(-1, 2)$, and $(2, -2)$ satisfy the inequality $2x - 4y \leq 7$. *(Lesson 1-8)*

59. Manufacturing The length of a nail is $2\frac{1}{2}$ inches. The manufacturer randomly measures the nails to test if their equipment is working properly. If the discrepancy is more than $\frac{1}{8}$ inch, adjustments must be made. Identify the type of function that models this situation. Then write a function for the situation. *(Lesson 1-7)*

60. SAT/ACT Practice In the figure at the right, the largest possible circle is cut out of a square piece of tin. What is the approximate total area of the remaining pieces of tin?



A 0.13 in^2 **B** 0.75 in^2 **C** 0.86 in^2

D 1.0 in^2 **E** 3.14 in^2

Extra Practice See p. A34.