5-5

Solving Right Triangles

OBJECTIVES
- Evaluate inverse trigonometric functions.
- Find missing angle measurements.
- Solve right triangles.

Real World Application

SECURITY A security light is being installed outside a loading dock. The light is mounted 20 feet above the ground. The light must be placed at an angle so that it will illuminate the end of the parking lot. If the end of the parking lot is 100 feet from the loading dock, what should be the angle of depression of the light? *This problem will be solved in Example 4.*

In Lesson 5-3, you learned to use the unit circle to determine the value of trigonometric functions. Some of the frequently-used values are listed below.

θ	0°	30°	45°	60°	90°	120°	135°	150°	180°
sin θ	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
cos θ	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1
tan θ	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	undefined	$-\sqrt{3}$	-1	$-\frac{\sqrt{3}}{3}$	0

θ	210°	225°	240°	270°	300°	315°	330°	360°
sin θ	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0
cos θ	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
tan θ	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	undefined	$-\sqrt{3}$	-1	$-\frac{\sqrt{3}}{3}$	0

Sometimes you know a trigonometric value of an angle, but not the angle. In this case, you need to use an **inverse** of the trigonometric function. The inverse of the sine function is the **arcsine relation.**

An equation such as $\sin x = \frac{\sqrt{3}}{2}$ can be written as $x = \arcsin \frac{\sqrt{3}}{2}$, which is read "$x$ is an angle whose sine is $\frac{\sqrt{3}}{2}$," or "x equals the arcsine of $\frac{\sqrt{3}}{2}$." The solution, x, consists of all angles that have $\frac{\sqrt{3}}{2}$ as the value of sine x.

Similarly, the inverse of the cosine function is the **arccosine relation,** and the inverse of the tangent function is the **arctangent relation.**

The equations in each row of the table below are equivalent. You can use these equations to rewrite trigonometric expressions.

Inverses of the Trigonometric Functions	Trigonometric Function	Inverse Trigonometric Relation
	$y = \sin x$	$x = \sin^{-1} y$ or $x = \arcsin y$
	$y = \cos x$	$x = \cos^{-1} y$ or $x = \arccos y$
	$y = \tan x$	$x = \tan^{-1} y$ or $x = \arctan y$

Examples

1 Solve each equation.

a. $\sin x = \frac{\sqrt{3}}{2}$

If $\sin x = \frac{\sqrt{3}}{2}$, then x is an angle whose sine is $\frac{\sqrt{3}}{2}$.

$x = \arcsin \frac{\sqrt{3}}{2}$

From the table on page 305, you can determine that x equals 60°, 120°, or any angle coterminal with these angles.

b. $\cos x = -\frac{\sqrt{2}}{2}$

If $\cos x = -\frac{\sqrt{2}}{2}$, then x is an angle whose cosine is $-\frac{\sqrt{2}}{2}$.

$x = \arccos -\frac{\sqrt{2}}{2}$

From the table, you can determine that x equals 135°, 225°, or any angle coterminal with these angles.

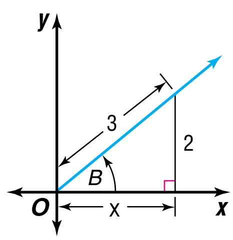
2 Evaluate each expression. Assume that all angles are in Quadrant I.

a. $\tan\left(\tan^{-1} \frac{6}{11}\right)$

Let $A = \tan^{-1} \frac{6}{11}$. Then $\tan A = \frac{6}{11}$ by the definition of inverse. Therefore, by substitution, $\tan\left(\tan^{-1} \frac{6}{11}\right) = \frac{6}{11}$.

b. $\cos\left(\arcsin \frac{2}{3}\right)$

Let $B = \arcsin \frac{2}{3}$. Then $\sin B = \frac{2}{3}$ by the definition of inverse. Draw a diagram of the $\angle B$ in Quadrant I.



$r^2 = x^2 + y^2$ *Pythagorean Theorem*

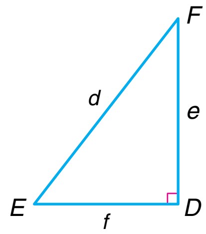
$3^2 = x^2 + 2^2$ *Substitute 3 for r and 2 for y.*

$\sqrt{5} = x$ *Take the square root of each side. Disregard the negative root.*

Since $\cos = \frac{\text{side adjacent}}{\text{hypotenuse}}$, $\cos B = \frac{\sqrt{5}}{3}$ and $\cos\left(\arcsin \frac{2}{3}\right) = \frac{\sqrt{5}}{3}$.

Inverse trigonometric relations can be used to find the measure of angles of right triangles. Calculators can be used to find values of the inverse trigonometric relations.

Example 3 **If $f = 17$ and $d = 32$, find E.**



In this problem, you want to know the measure of an acute angle in a right triangle. You know the side adjacent to the angle and the hypotenuse. The cosine function relates the side adjacent to the angle and the hypotenuse.

Remember that in trigonometry the measure of an angle is symbolized by the angle vertex letter.

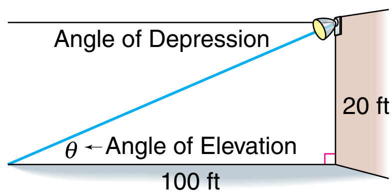
$\cos E = \frac{f}{d}$	*$\cos = \frac{\text{side adjacent}}{\text{hypotenuse}}$*
$\cos E = \frac{17}{32}$	*Substitute 17 for f and 32 for d.*
$E = \cos^{-1}\frac{17}{32}$	*Definition of inverse*
$E \approx 57.91004874$	*Use a calculator.*

Therefore, E measures about 57.9°.

Trigonometry can be used to find the angle of elevation or the angle of depression.

Example 4 Real World Application **SECURITY** **Refer to the application at the beginning of the lesson. What should be the angle of depression of the light?**

The angle of depression from the light and the angle of elevation to the light are equal in measure. To find the angle of elevation, use the tangent function.

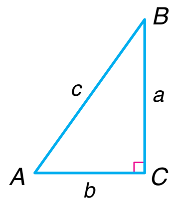


$\tan \theta = \frac{20}{100}$	*$\tan = \frac{\text{side opposite}}{\text{side adjacent}}$*
$\theta = \tan^{-1}\frac{20}{100}$	*Definition of inverse*
$\theta \approx 11.30993247$	*Use a calculator.*

The angle of depression should be about 11.3°.

You can use trigonometric functions and inverse relations to solve right triangles. To **solve a triangle** means to find all of the measures of its sides and angles. Usually, two measures are given. Then you can find the remaining measures.

Example 5 **Solve each triangle described, given the triangle at the right.**



a. $A = 33°$, $b = 5.8$

Find B.

$$33° + B = 90° \quad \textit{Angles A and B are complementary.}$$
$$B = 57°$$

Whenever possible, use measures given in the problem to find the unknown measures.

Find a.

$$\tan A = \frac{a}{b}$$
$$\tan 33° = \frac{a}{5.8}$$
$$5.8 \tan 33° = a$$
$$3.766564041 \approx a$$

Find c.

$$\cos A = \frac{b}{c}$$
$$\cos 33° = \frac{5.8}{c}$$
$$c \cos 33° = 5.8$$
$$c = \frac{5.8}{\cos 33°}$$
$$c \approx 6.915707098$$

Therefore, $B = 57°$, $a \approx 3.8$, and $c \approx 6.9$.

b. $a = 23$, $c = 45$

Find b.

$$a^2 + b^2 = c^2$$
$$23^2 + b^2 = 45^2$$
$$b = \sqrt{1496}$$
$$b \approx 38.67815921$$

Find A.

$$\sin A = \frac{a}{c}$$
$$\sin A = \frac{23}{45}$$
$$A = \sin^{-1}\frac{23}{45}$$
$$A \approx 30.73786867$$

Find B.

$$30.73786867 + B \approx 90$$
$$B \approx 59.26213133$$

Therefore, $b \approx 38.7$, $A \approx 30.7°$, and $B \approx 59.3°$

CHECK FOR UNDERSTANDING

Communicating Mathematics

Read and study the lesson to answer each question.

1. **Tell** whether the solution to each equation is an angle measure or a linear measurement.
   a. $\tan 34° 15' = \frac{x}{12}$ b. $\tan x = 3.284$
2. **Describe** the relationship of the two acute angles of a right triangle.
3. **Counterexample** You can usually solve a right triangle if you know two measures besides the right angle. Draw a right triangle and label two measures other than the right angle such that you cannot solve the triangle.
4. **You Decide** Marta and Rebecca want to determine the degree measure of angle ϑ if $\cos \vartheta = 0.9876$. Marta tells Rebecca to press [2nd] [COS^{-1}] .9876 on the calculator. Rebecca disagrees. She says to press [COS] .9876 [)] [x^{-1}]. Who is correct? Explain.

Guided Practice

Solve each equation if $0° \leq x \leq 360°$.

5. $\cos x = \frac{1}{2}$

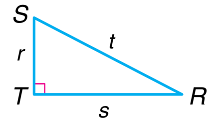
6. $\tan x = \frac{-\sqrt{3}}{3}$

Evaluate each expression. Assume that all angles are in Quadrant I.

7. $\sin\left(\sin^{-1}\frac{\sqrt{3}}{2}\right)$

8. $\tan\left(\cos^{-1}\frac{3}{5}\right)$

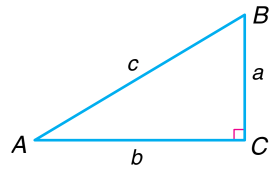
Solve each problem. Round to the nearest tenth.



9. If $r = 7$ and $s = 10$, find R.

10. If $r = 12$ and $t = 20$, find S.

Solve each triangle described, given the triangle at the right. Round to the nearest tenth if necessary.



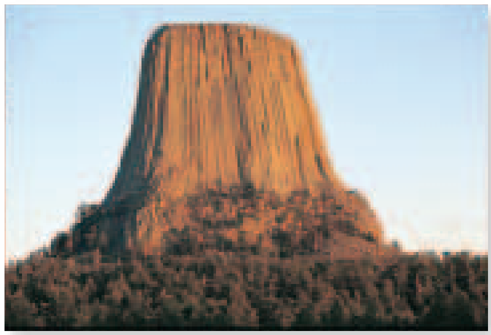
11. $B = 78°$, $a = 41$

12. $a = 11$, $b = 21$

13. $A = 32°$, $c = 13$

14. National Monuments In 1906, Teddy Roosevelt designated Devils Tower National Monument in northeast Wyoming as the first national monument in the United States. The tower rises 1280 feet above the valley of the Bell Fourche River.

a. If the shadow of the tower is 2100 feet long at a certain time, find the angle of elevation of the sun.



Devils Tower National Monument

b. How long is the shadow when the angle of elevation of the sun is 38°?

c. If a person at the top of Devils Tower sees a hiker at an angle of depression of 65°, how far is the hiker from the base of Devils Tower?

EXERCISES

Practice

Solve each equation if $0° \leq x \leq 360°$.

15. $\sin x = 1$

16. $\tan x = -\sqrt{3}$

17. $\cos x = \frac{\sqrt{3}}{2}$

18. $\cos x = 0$

19. $\sin x = -\frac{\sqrt{2}}{2}$

20. $\tan x = -1$

21. Name four angles whose sine equals $\frac{1}{2}$.

Evaluate each expression. Assume that all angles are in Quadrant I.

22. $\cos\left(\arccos\frac{4}{5}\right)$

23. $\tan\left(\tan^{-1}\frac{2}{3}\right)$

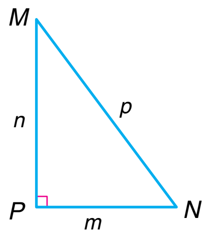
24. $\sec\left(\cos^{-1}\frac{2}{5}\right)$

25. $\csc(\arcsin 1)$

26. $\tan\left(\cos^{-1}\frac{5}{13}\right)$

27. $\cos\left(\sin^{-1}\frac{2}{5}\right)$

Solve each problem. Round to the nearest tenth.



28. If $n = 15$ and $m = 9$, find N.

29. If $m = 8$ and $p = 14$, find M.

30. If $n = 22$ and $p = 30$, find M.

31. If $m = 14.3$ and $n = 18.8$, find N.

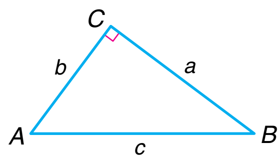
32. If $p = 17.1$ and $m = 7.2$, find N.

33. If $m = 32.5$ and $p = 54.7$, find M.

34. **Geometry** If the legs of a right triangle are 24 centimeters and 18 centimeters long, find the measures of the acute angles.

35. **Geometry** The base of an isosceles triangle is 14 inches long. Its height is 8 inches. Find the measure of each angle of the triangle.

Solve each triangle described, given the triangle at the right. Round to the nearest tenth, if necessary.



36. $a = 21, c = 30$

37. $A = 35°, b = 8$

38. $B = 47°, b = 12.5$

39. $a = 3.8, b = 4.2$

40. $c = 9.5, b = 3.7$

41. $a = 13.3, A = 51.5°$

42. $B = 33°, c = 15.2$

43. $c = 9.8, A = 14°$

Applications and Problem Solving

Real World Application

44. **Railways** The steepest railway in the world is the Katoomba Scenic Railway in Australia. The passenger car is pulled up the mountain by twin steel cables. It travels along the track 1020 feet to obtain a change in altitude of 647 feet.

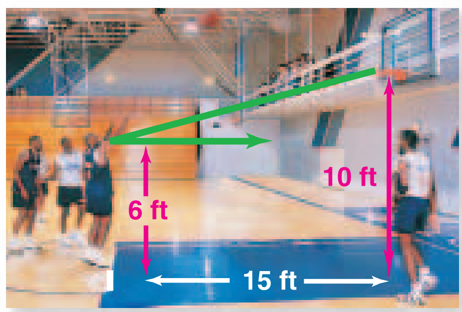
a. Find the angle of elevation of the railway.

b. How far does the car travel in a horizontal direction?

45. **Critical Thinking** Explain why each expression is impossible.

a. $\sin^{-1} 2.4567$ **b.** $\sec^{-1} 0.5239$ **c.** $\cos^{-1} (-3.4728)$

46. **Basketball** The rim of a basketball hoop is 10 feet above the ground. The free-throw line is 15 feet from the basket rim. If the eyes of a basketball player are 6 feet above the ground, what is the angle of elevation of the player's line of sight when shooting a free throw to the rim of the basket?



47. **Road Safety** Several years ago, a section on I-75 near Cincinnati, Ohio, had a rise of 8 meters per 100 meters of horizontal distance. However, there were numerous accidents involving large trucks on this section of highway. Civil engineers decided to reconstruct the highway so that there is only a rise of 5 meters per 100 meters of horizontal distance.

a. Find the original angle of elevation.

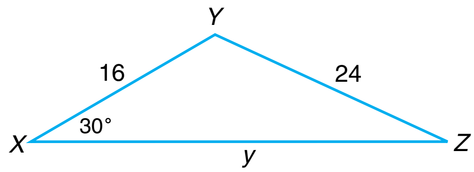
b. Find the new angle of elevation.

48. **Air Travel** At a local airport, a light that produces a powerful white-green beam is placed on the top of a 45-foot tower. If the tower is at one end of the runway, find the angle of depression needed so that the light extends to the end of the 2200-foot runway.

49. **Civil Engineering** Highway curves are usually banked or tilted inward so that cars can negotiate the curve more safely. The proper banking angle θ for a car making a turn of radius r feet at a velocity of v feet per second is given by the equation is $\tan \theta = \frac{v^2}{gr}$. In this equation, g is the acceleration due to gravity or 32 feet per second squared. An engineer is designing a curve with a radius of 1200 feet. If the speed limit on the curve will be 65 miles per hour, at what angle should the curve be banked? (*Hint*: Change 65 miles per hour to feet per second.)

50. **Physics** According to Snell's Law, $\frac{\sin \theta_i}{\sin \theta_r} = n$, where θ_i is the angle of incidence, θ_r is the angle of refraction, and n is the index of refraction. The index of refraction for a diamond is 2.42. If a beam of light strikes a diamond at an angle of incidence of 60°, find the angle of refraction.

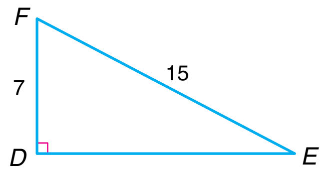
51. **Critical Thinking** Solve the triangle. (*Hint*: Draw the altitude from Y.)



Mixed Review

52. **Aviation** A traffic helicopter is flying 1000 feet above the downtown area. To the right, the pilot sees the baseball stadium at an angle of depression of 63°. To the left, the pilot sees the football stadium at an angle of depression of 18°. Find the distance between the two stadiums. *(Lesson 5-4)*

53. Find the six trigonometric ratios for $\angle F$. *(Lesson 5-2)*



54. Approximate the real zeros of the function $f(x) = 3x^3 - 16x^2 + 12x + 6$ to the nearest tenth. *(Lesson 4-5)*

55. Determine whether the graph of $y^3 - x^2 = 2$ is symmetric with respect to the x-axis, y-axis, the graph of $y = x$, the graph of $y = -x$, or none of these. *(Lesson 3-1)*

56. Use a reflection matrix to find the coordinates of the vertices of a pentagon reflected over the y-axis if the coordinates of the vertices of the pentagon are $(-5, -3)$, $(-5, 4)$, $(-3, 6)$, $(-1, 3)$, and $(-2, -2)$. *(Lesson 2-4)*

57. Find the sum of the matrices $\begin{bmatrix} 4 & -3 & 2 \\ 8 & -2 & 0 \\ 9 & 6 & -3 \end{bmatrix}$ and $\begin{bmatrix} -2 & 2 & -2 \\ -5 & 1 & 1 \\ -7 & 2 & -2 \end{bmatrix}$. *(Lesson 2-3)*

Extra Practice See p. A35.

CONTENTS

58. Write a linear equation of best fit for a set of data using the ordered pairs (1880, 42.5) and (1950, 22.2). *(Lesson 1-6)*

59. Write the equation $2x + 5y - 10 = 0$ in slope-intercept form. Then name the slope and y-intercept. *(Lesson 1-5)*

60. **SAT/ACT Practice** The Natural Snack Company mixes a pounds of peanuts that cost b cents per pound with c pounds of rice crackers that cost d cents per pound to make Oriental Peanut Mix. What should the price in cents for a pound of Oriental Peanut Mix be if the company wants to make a profit of 10¢ per pound?

A $\frac{ab + cd}{a + c} + 10$ **B** $\frac{b + d}{a + c} + 10$ **C** $\frac{ab + cd}{a + c} + 0.10$

D $\frac{b + d}{a + c} + 0.10$ **E** $\frac{b + d + 10}{a + c}$

CAREER CHOICES

Architecture

Are you creative and concerned with accuracy and detail? Do you like to draw and design new things? You may want to consider a career in architecture.

An architect plans, designs, and oversees the construction of all types of buildings, a job that is very complex. An architect needs to stay current with new construction methods and design. An architect must also be knowledgeable about engineering principles.

As an architect, you would work closely with others such as consulting engineers and building contractors. Your projects could include large structures such as shopping malls or small structures such as single-family houses. There are also specialty fields in architecture such as interior design, landscape architecture, and products and material design.

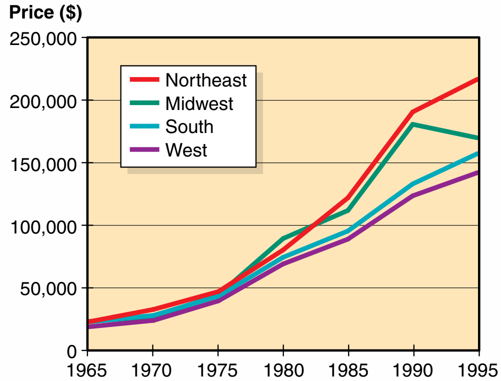
Career Overview

Degree Preferred:
bachelor's degree in architecture

Related Courses:
mathematics, physics, art, computer science

Outlook:
number of jobs expected to increase as fast as the average through the year 2006

Average Sale Prices of New Single-Family Houses by Region



Source: *The New York Times Almanac*

For more information on careers in architecture, visit: **www.amc.glencoe.com**