5-6

The Law of Sines

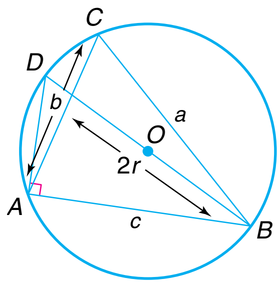
OBJECTIVES

- Solve triangles by using the Law of Sines if the measures of two angles and a side are given.
- Find the area of a triangle if the measures of two sides and the included angle or the measures of two angles and a side are given.

Real World Application

BASEBALL A baseball fan is sitting directly behind home plate in the last row of the upper deck of Comiskey Park in Chicago. The angle of depression to home plate is 29° 54′, and the angle of depression to the pitcher's mound is 24° 12′. In major league baseball, the distance between home plate and the pitcher's mound is 60.5 feet. How far is the fan from home plate? *This problem will be solved in Example 2.*

The **Law of Sines** can be used to solve triangles that are not right triangles. Consider $\triangle ABC$ inscribed in circle O with diameter $\overline{DB}$. Let $2r$ be the measure of the diameter. Draw $\overline{AD}$. Then $\angle D \cong \angle C$ since they intercept the same arc. So, $\sin D = \sin C$. $\angle DAB$ is inscribed in a semicircle, so it is a right angle. $\sin D = \frac{c}{2r}$. Thus, since $\sin D = \sin C$, it follows that $\sin C = \frac{c}{2r}$ or $\frac{c}{\sin C} = 2r$.



Similarly, by drawing diameters through A and C, $\frac{b}{\sin B} = 2r$ and $\frac{a}{\sin A} = 2r$. Since each rational expression equals 2r, the following is true.

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

These equations state that the ratio of the length of any side of a triangle to the sine of the angle opposite that side is a constant for a given triangle. These equations are collectively called the Law of Sines.

Law of Sines

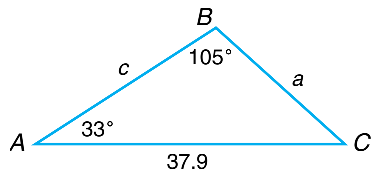
Let $\triangle ABC$ be any triangle with a, b, and c representing the measures of the sides opposite the angles with measures A, B, and C, respectively. Then, the following is true.

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

From geometry, you know that a unique triangle can be formed if you know the measures of two angles and the included side (ASA) or the measures of two angles and the non-included side (AAS). Therefore, there is one unique solution when you use the Law of Sines to solve a triangle given the measures of two angles and one side. *In Lesson 5-7, you will learn to use the Law of Sines when the measures of two sides and a nonincluded angle are given.*

Examples

1 **Solve $\triangle ABC$ if $A = 33°$, $B = 105°$, and $b = 37.9$.**



First, find the measure of $\angle C$.
$C = 180° - (33° + 105°)$ or $42°$

Use the Law of Sines to find a and c.

$$\frac{a}{\sin A} = \frac{b}{\sin B} \qquad\qquad \frac{c}{\sin C} = \frac{b}{\sin B}$$

$$\frac{a}{\sin 33°} = \frac{37.9}{\sin 105°} \qquad\qquad \frac{c}{\sin 42°} = \frac{37.9}{\sin 105°}$$

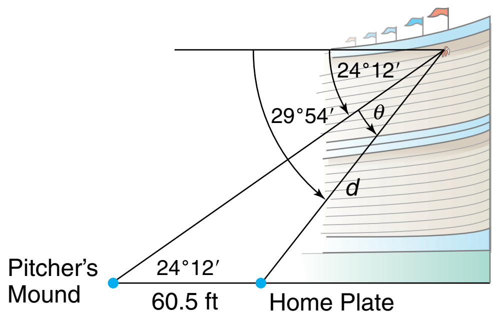
$$a = \frac{37.9 \sin 33°}{\sin 105°} \qquad\qquad c = \frac{37.9 \sin 42°}{\sin 105°}$$

$a \approx 21.36998397$ *Use a calculator.* $\qquad c \approx 26.25465568$ *Use a calculator.*

Therefore, $C = 42°$, $a \approx 21.4$, and $c \approx 26.3$.

2 **BASEBALL** **Refer to the application at the beginning of the lesson. How far is the fan from home plate?**

Make a diagram for the problem. Remember that the angle of elevation is congruent to the angle of depression, because they are alternate interior angles.



First, find θ.

$\theta = 29° \, 54' - 24° \, 12'$ or $5° \, 42'$

Use the Law of Sines to find d.

$$\frac{d}{\sin 24° \, 12'} = \frac{60.5}{\sin 5° \, 42'}$$

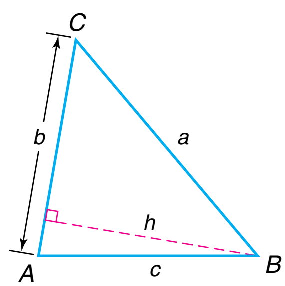
$$d = \frac{60.5 \sin 24° \, 12'}{\sin 5° \, 42'}$$

$d \approx 249.7020342$ *Use a calculator.*

The fan is about 249.7 feet from home plate.

The area of any triangle can be expressed in terms of two sides of a triangle and the measure of the included angle. Suppose you know the measures of $\overline{AC}$ and $\overline{AB}$ and the measure of the included $\angle A$ in $\triangle ABC$. Let K represent the measure of the area of $\triangle ABC$, and let h represent the measure of the altitude from B. Then $K = \frac{1}{2}bh$. But, $\sin A = \frac{h}{c}$ or $h = c \sin A$. If you substitute $c \sin A$ for h, the result is the following formula.

Use K for area instead of A to avoid confusion with angle A.



$$K = \frac{1}{2}bc \sin A$$

If you drew altitudes from A and C, you could also develop two similar formulas.

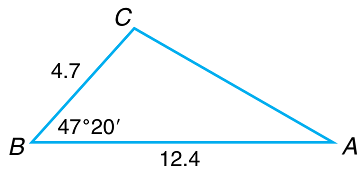
Area of Triangles

Let $\triangle ABC$ be any triangle with a, b, and c representing the measures of the sides opposite the angles with measurements A, B, and C, respectively. Then the area K can be determined using one of the following formulas.

$$K = \frac{1}{2}bc \sin A \qquad K = \frac{1}{2}ac \sin B$$

$$K = \frac{1}{2}ab \sin C$$

Example 3 **Find the area of $\triangle ABC$ if $a = 4.7$, $c = 12.4$, and $B = 47° 20'$.**



$K = \frac{1}{2}ac \sin B$

$K = \frac{1}{2}(4.7)(12.4) \sin 47° 20'$

$K \approx 21.42690449$ *Use a calculator.*

The area of $\triangle ABC$ is about 21.4 square units.

You can also find the area of a triangle if you know the measures of one side and two angles of the triangle. By the Law of Sines, $\frac{b}{\sin B} = \frac{c}{\sin C}$ or $b = \frac{c \sin B}{\sin C}$. If you substitute $\frac{c \sin B}{\sin C}$ for b in $K = \frac{1}{2}bc \sin A$, the result is $K = \frac{1}{2}c^2 \frac{\sin A \sin B}{\sin C}$. Two similar formulas can be developed.

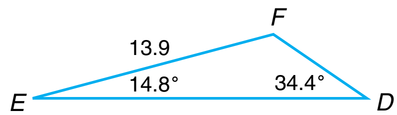
Area of Triangles

Let $\triangle ABC$ be any triangle with a, b, and c representing the measures of the sides opposite the angles with measurements A, B, and C respectively. Then the area K can be determined using one of the following formulas.

$$K = \frac{1}{2}a^2 \frac{\sin B \sin C}{\sin A} \qquad K = \frac{1}{2}b^2 \frac{\sin A \sin C}{\sin B}$$

$$K = \frac{1}{2}c^2 \frac{\sin A \sin B}{\sin C}$$

Example 4 **Find the area of $\triangle DEF$ if $d = 13.9$, $D = 34.4°$, and $E = 14.8°$.**



First find the measure of $\angle F$.
$F = 180° - (34.4° + 14.8°)$ or $130.8°$

Then, find the area of the triangle.

$K = \frac{1}{2}d^2 \frac{\sin E \sin F}{\sin D}$

$K = \frac{1}{2}(13.9)^2 \frac{\sin 14.8° \sin 130.8°}{\sin 34.4°}$

$K \approx 33.06497958$ *Use a calculator.*

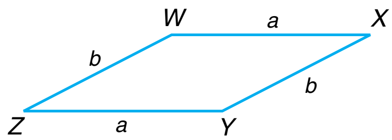
The area of $\triangle DEF$ is about 33.1 square units.

CHECK FOR UNDERSTANDING

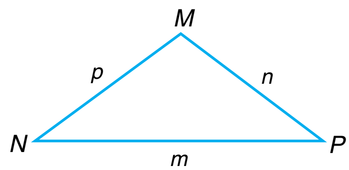
Communicating Mathematics

Read and study the lesson to answer each question.

1. **Show** that the Law of Sines is true for a 30°-60° right triangle.
2. **Draw and label** a triangle that has a unique solution and can be solved using the Law of Sines.
3. **Write** a formula for the area of parallelogram *WXYZ* in terms of *a, b,* and *X*.



4. **You Decide** Roderick says that triangle *MNP* has a unique solution if *M, N,* and *m* are known. Jane disagrees. She says that a triangle has a unique solution if *M, N,* and *p* are known. Who is correct? Explain.



Guided Practice

Solve each triangle. Round to the nearest tenth.

5. $A = 40°$, $B = 59°$, $c = 14$
6. $a = 8.6$, $A = 27.3°$, $B = 55.9°$
7. If $B = 17°\ 55'$, $C = 98°\ 15'$, and $a = 17$, find c.

Find the area of each triangle. Round to the nearest tenth.

8. $A = 78°$, $b = 14$, $c = 12$
9. $A = 22°$, $B = 105°$, $b = 14$
10. **Baseball** Refer to the application at the beginning of the lesson. How far is the baseball fan from the pitcher's mound?

EXERCISES

Practice

Solve each triangle. Round to the nearest tenth.

11. $A = 40°$, $C = 70°$, $a = 20$
12. $B = 100°$, $C = 50°$, $c = 30$
13. $b = 12$, $A = 25°$, $B = 35°$
14. $A = 65°$, $B = 50°$, $c = 12$
15. $a = 8.2$, $B = 24.8°$, $C = 61.3°$
16. $c = 19.3$, $A = 39°\ 15'$, $C = 64°\ 45'$
17. If $A = 37°\ 20'$, $B = 51°\ 30'$, and $c = 125$, find b.
18. What is a if $b = 11$, $B = 29°\ 34'$, and $C = 23°\ 48'$?

Find the area of each triangle. Round to the nearest tenth.

19. $A = 28°$, $b = 14$, $c = 9$
20. $a = 5$, $B = 37°$, $C = 84°$
21. $A = 15°$, $B = 113°$, $b = 7$
22. $b = 146.2$, $c = 209.3$, $A = 62.2°$
23. $B = 42.8°$, $a = 12.7$, $c = 5.8$
24. $a = 19.2$, $A = 53.8°$, $C = 65.4°$
25. **Geometry** The adjacent sides of a parallelogram measure 14 centimeters and 20 centimeters, and one angle measures 57°. Find the area of the parallelogram.
26. **Geometry** A regular pentagon is inscribed in a circle whose radius measures 9 inches. Find the area of the pentagon.
27. **Geometry** A regular octagon is inscribed in a circle with radius of 5 feet. Find the area of the octagon.

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CONTENTS

Applications and Problem Solving

Real World Application

28. **Landscaping** A landscaper wants to plant begonias along the edges of a triangular plot of land in Winton Woods Park. Two of the angles of the triangle measure 95° and 40°. The side between these two angles is 80 feet long.

a. Find the measure of the third angle.

b. Find the length of the other two sides of the triangle.

c. What is the perimeter of this triangular plot of land?

29. **Critical Thinking** For $\triangle MNP$ and $\triangle RST$, $\angle M \cong \angle R$, $\angle N \cong \angle S$, and $\angle P \cong \angle T$. Use the Law of Sines to show $\triangle MNP \sim \triangle RST$.

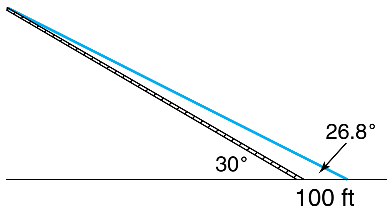
30. **Architecture** The center of the Pentagon in Arlington, Virginia, is a courtyard in the shape of a regular pentagon. The pentagon could be inscribed in a circle with radius of 300 feet. Find the area of the courtyard.

31. **Ballooning** A hot air balloon is flying above Groveburg. To the left side of the balloon, the balloonist measures the angle of depression to the Groveburg soccer fields to be 20° 15′. To the right side of the balloon, the balloonist measures the angle of depression to the high school football field to be 62° 30′. The distance between the two athletic complexes is 4 miles.

a. Find the distance from the balloon to the soccer fields.

b. What is the distance from the balloon to the football field?

32. **Cable Cars** The Duquesne Incline is a cable car in Pittsburgh, Pennsylvania, which transports passengers up and down a mountain. The track used by the cable car has an angle of elevation of 30°. The angle of elevation to the top of the track from a point that is horizontally 100 feet from the base of the track is about 26.8°. Find the length of the track.

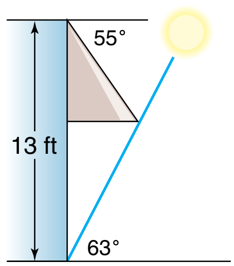


33. **Air Travel** In order to avoid a storm, a pilot starts the flight 13° off course. After flying 80 miles in this direction, the pilot turns the plane to head toward the destination. The angle formed by the course of the plane during the first part of the flight and the course during the second part of the flight is 160°.

a. What is the distance of the flight?

b. Find the distance of a direct flight to the destination.

34. **Architecture** An architect is designing an overhang above a sliding glass door. During the heat of the summer, the architect wants the overhang to prevent the rays of the sun from striking the glass at noon. The overhang has an angle of depression of 55° and starts 13 feet above the ground. If the angle of elevation of the sun during this time is 63°, how long should the architect make the overhang?



35. **Critical Thinking** Use the Law of Sines to show that each statement is true for any $\triangle ABC$.

a. $\frac{a}{b} = \frac{\sin A}{\sin B}$

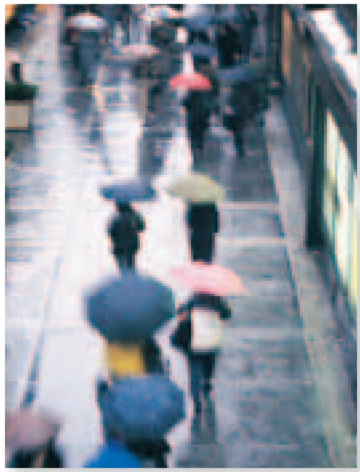
b. $\frac{a - c}{c} = \frac{\sin A - \sin C}{\sin C}$

c. $\frac{a + c}{a - c} = \frac{\sin A + \sin C}{\sin A - \sin C}$

d. $\frac{b}{a + b} = \frac{\sin B}{\sin A + \sin B}$

Mixed Review

36. **Meteorology** If raindrops are falling toward Earth at a speed of 45 miles per hour and a horizontal wind is blowing at a speed of 20 miles per hour, at what angle do the drops hit the ground? *(Lesson 5-5)*



37. Suppose θ is an angle in standard position whose terminal side lies in Quadrant IV. If $\sin \theta = -\frac{1}{6}$, find the values of the remaining five trigonometric functions for θ. *(Lesson 5-3)*

38. Identify all angles that are coterminal with an $83°$ angle. *(Lesson 5-1)*

39. **Business** A company is planning to buy new carts to store merchandise. The owner believes they need at least 2 standard carts and at least 4 deluxe carts. The company can afford to purchase a maximum of 15 carts at this time; however, the supplier has only 8 standard carts and 11 deluxe carts in stock. Standard carts can hold up to 100 pounds of merchandise, and deluxe carts can hold up to 250 pounds of merchandise. How many standard carts and deluxe carts should be purchased to maximize the amount of merchandise that can be stored? *(Lesson 2-7)*

40. Solve the system of equations algebraically. *(Lesson 2-2)*

$4x + y + 2z = 0$
$3x + 4y - 2z = 20$
$-2x + 5y + 3z = 14$

41. Graph $-6 \leq 3x - y \leq 12$. *(Lesson 1-8)*

42. **SAT Practice** Eight cubes, each with an edge of length one inch, are positioned together to create a large cube. What is the difference in the surface area of the large cube and the sum of the surface areas of the small cubes?

A 24 in^2
B 16 in^2
C 12 in^2
D 8 in^2
E 0 in^2

Extra Practice See p. A35.

HISTORY of MATHEMATICS

ANGLES

When someone uses the word "angle", what images does that conjure up in your mind? An angle seems like a simple figure, but historically mathematicians, and even philosophers, have engaged in trying to describe or define an angle. This textbook says, "an angle may be generated by the rotation of two rays that share a fixed endpoint known as the vertex." Let's look at various ideas about angles throughout history.


Autumn Borts

Early Evidence Babylonians (4000–3000 B.C.) were some of the first peoples to leave samples of their use of geometry in the form of inscribed clay tablets.

The first written mathematical work containing definitions for angles was **Euclid's** *The Elements.* Little is known about the life of **Euclid** (about 300 B.C.), but his thirteen-volume work, *The Elements,* has strongly influenced the teaching of geometry for over 2000 years. The first copy of *The Elements* was printed by modern methods in 1482 and has since been edited and translated into over 1000 editions. In *Book I* of *The Elements,* Euclid presents the definitions for various types of angles.

Euclid's definition of a plane angle differed from an earlier Greek idea that an angle was a deflection or a breaking of lines.

Greek mathematicians were not the only scholars interested in angles. **Aristotle** (384–322 B.C.) had devised three categories in which to place mathematical concepts—a quantity, a quality, or a relation. Greek philosophers argued as to which category an angle belonged. **Proclus** (410–485) felt that an angle was a combination of the three, saying "it needs the quantity involved in magnitude, thereby becoming susceptible of equality, inequality, and the like; it needs the quality given it by its form; and lastly, the relation subsisting between the lines or the planes bounding it."

The Renaissance In 1634, Pierre Herigone first used "$<$" as a symbol for an angle in his book *Cursus Mathematicus.* This symbol was already being used for "less than," so, in 1657, **William Oughtred** used the symbol "$\angle$" in his book *Trigonometria.*

Modern Era Various symbols for angle, including $\wedge$, $\urcorner$, $\widehat{ab}$, and $\widehat{ABC}$, were used during the 1700s and 1800s. In 1923, the National Committee on Mathematical Requirements recommended that "$\angle$" be used as a standard symbol in the U.S.

Today, artists like **Autumn Borts** use angles in their creation of Native American pottery. Ms. Borts adorns water jars with carved motifs of both traditional and contemporary designs. She is carrying on the Santa Clara style of pottery and has been influenced by her mother, grandmother, and great grandmother.

ACTIVITIES

1. In a previous course, you have probably drawn triangles in a plane and measured the interior angles to find the angle sum of the triangles. Triangles can also be constructed on a sphere. Get a globe. Use tape and string to form at least three different triangles. Measure the interior angles of the triangles. What appears to be true about the sum of the angles?

2. Research Euclid's famous work, *The Elements.* Find and list any postulates he wrote about angles.

3. interNET CONNECTION Find out more about the personalities referenced in this article and others who contributed to the history of angles. Visit **www.amc.glencoe.com**