5-7

The Ambiguous Case for the Law of Sines

OBJECTIVES

- Determine whether a triangle has zero, one, or two solutions.
- Solve triangles using the Law of Sines.

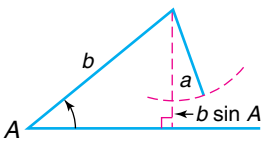
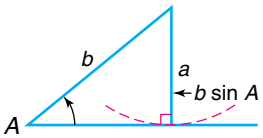
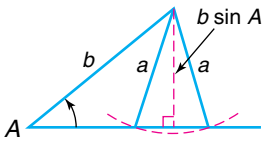
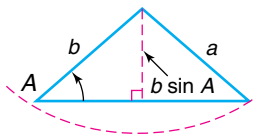
Real World Application

TOURISM Visitors near a certain national park can tune to a local radio station to find out about the activities that are happening in the park. The transmission tower for the radio station is along Park Road about 30 miles from the intersection of this road and the interstate. The interstate and the road form a 47° angle. If the transmitter has a range of 25 miles, how far along the interstate can the passengers in a car hear the broadcast? *This problem will be solved in Example 3.*

From geometry, you know that the measures of two sides and a nonincluded angle do not necessarily define a unique triangle. However, one of the following will be true.

1. No triangle exists.
2. Exactly one triangle exists.
3. Two triangles exist.

In other words, there may be no solution, one solution, or two solutions. A situation with two solutions is called the **ambiguous case.** Suppose you know the measures of a, b, and A. Consider the following cases.

Case 1: $A < 90°$			
	$a < b \sin A$	$a = b \sin A$	$a > b \sin A$
$a < b$	 no solution	 one solution	 two solutions
$a \geq b$	 one solution		

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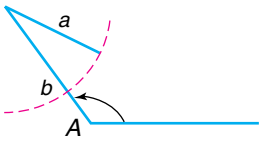
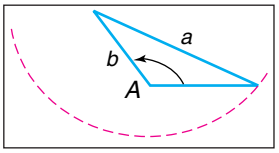
Case 2: $A \geq 90°$

$a \leq b$



no solution

$a > b$



one solution

Example 1 **Determine the number of possible solutions for each triangle.**

a. $A = 30°$, $a = 8$, $b = 10$

Since $30° < 90°$, consider Case I.

$b \sin A = 10 \sin 30°$
$b \sin A = 10(0.5)$
$b \sin A = 5$

Since $5 < 8 < 10$, there are two solutions for the triangle.

b. $b = 8$, $c = 10$, $B = 118°$

Since $118° \geq 90°$, consider Case II.

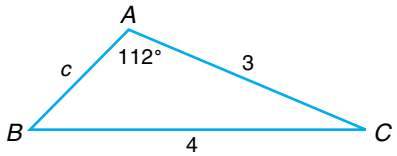
In this triangle, $8 \leq 10$, so there are no solutions.

Once you have determined that there are one or two solutions for a triangle given the measures of two sides and a nonincluded angle, you can use the Law of Sines to solve the triangle.

Example 2 **Find all solutions for each triangle. If no solutions exist, write *none*.**

a. $a = 4$, $b = 3$, $A = 112°$

Since $112° \geq 90°$, consider Case II. In this triangle, $4 > 3$, so there is one solution. First, use the Law of Sines to find B.



$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

$$\frac{4}{\sin 112°} = \frac{3}{\sin B}$$

$$\sin B = \frac{3 \sin 112°}{4}$$

$$B = \sin^{-1}\left(\frac{3 \sin 112°}{4}\right)$$

$$B \approx 44.05813517 \quad \textit{Use a calculator.}$$

So, $B \approx 44.1°$.

Use the value of B to find C and c.

$C \approx 180° - (112° + 44.1°)$ or about $23.9°$

$$\frac{a}{\sin A} = \frac{c}{\sin C}$$

$$\frac{4}{\sin 112°} \approx \frac{c}{\sin 23.9°}$$

$$c \approx \frac{4 \sin 23.9°}{\sin 112°}$$

$$c \approx 1.747837108 \quad \textit{Use a calculator.}$$

The solution of this triangle is $B \approx 44.1°$, $C \approx 23.9°$, and $c \approx 1.7$.

b. $A = 51°$, $a = 40$, $c = 50$

Since $51° < 90°$, consider Case I.

$c \sin A = 50 \sin 51°$

≈ 38.85729807 *Use a calculator.*

Since $38.9 < 40 < 50$, there are two solutions for the triangle.

Use the Law of Sines to find C.

$$\frac{a}{\sin A} = \frac{c}{\sin C}$$

$$\frac{40}{\sin 51°} = \frac{50}{\sin C}$$

$$\sin C = \frac{50 \sin 51°}{40}$$

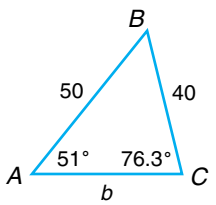
$$C = \sin^{-1}\left(\frac{50 \sin 51°}{40}\right)$$

$C \approx 76.27180414$ *Use a calculator.*

Notice that the sum of the two measures for C is 180°.

So, $C \approx 76.3°$. Since we know there are two solutions, there must be another possible measurement for C. In the second case, C must be less than 180° and have the same sine value. Since we know that if $\alpha < 90$, $\sin \alpha = \sin (180 - \alpha)$, $180° - 76.3°$ or $103.7°$ is another possible measure for C. Now solve the triangle for each possible measure of C.

Solution I



$B \approx 180° - (51° + 76.3°)$
$B \approx 52.7°$

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

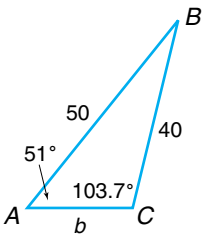
$$\frac{40}{\sin 51°} \approx \frac{b}{\sin 52.7°}$$

$$b \approx \frac{40 \sin 52.7°}{\sin 51°}$$

$b \approx 40.94332444$

One solution is $B \approx 52.7°$, $C \approx 76.3°$, and $b \approx 40.9$.

Solution II



$B \approx 180° - (51° + 103.7°)$
$B \approx 25.3°$

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

$$\frac{40}{\sin 51°} \approx \frac{b}{\sin 25.3°}$$

$$b \approx \frac{40 \sin 25.3°}{\sin 51°}$$

$b \approx 21.99627275$

Another solution is $B \approx 25.3°$, $C \approx 103.7°$, and $b \approx 22.0$.

GRAPHING CALCULATOR EXPLORATION

You can store values in your calculator and use these values in other computations. In solving triangles, you can store a value for a missing part of the triangle and then use this value when solving for the other missing parts.

TRY THESE

1. Rework Example 2a. Use a calculator to solve for B and store this value. Use the stored value to solve for C and c. Round the answers to the nearest tenth after you have completed all computations.
2. Rework Example 2b. Use a calculator to solve for each possible value for C. Store these values. Use the stored values to solve for B and b in each possible triangle. Round the answers to the nearest tenth after you have completed all computations.

WHAT DO YOU THINK?

3. Compare your answers with those in the examples.
4. Why do you think your answers may vary slightly from those of classmates or the textbook?

Example 3 **TOURISM** **Refer to the application at the beginning of the lesson. How far along the interstate can the passengers in a car hear the broadcast?**

Real World Application

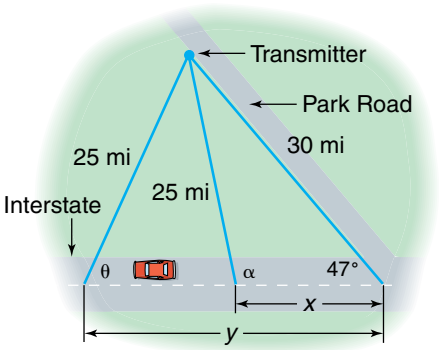
Consider Case 1 because $47° < 90°$. Since $30 \sin 47° \approx 21.9$ and $21.9 < 25 < 30$, there are two triangles with sides 25 miles and 30 miles long and a nonincluded angle of 47°. θ and α represent the two possible angle measures.

$$\frac{25}{\sin 47°} = \frac{30}{\sin \theta}$$

$$\sin \theta = \frac{30 \sin 47°}{25}$$

$$\theta = \sin^{-1}\left(\frac{30 \sin 47°}{25}\right)$$

$$\theta \approx 61.3571157 \quad \textit{Use a calculator.}$$



So, $\theta \approx 61.4°$ and $\alpha \approx 180° - 61.4°$ or $118.6°$.

$$\frac{25}{\sin 47°} \approx \frac{y}{\sin (180° - (47° + 61.4°))}$$

$$\frac{25}{\sin 47°} \approx \frac{y}{\sin 71.6°}$$

$$y \approx \frac{25 \sin 71.6°}{\sin 47°}$$

$$y \approx 32.4356057$$

$$\frac{25}{\sin 47°} \approx \frac{x}{\sin (180° - (47° + 118.6°))}$$

$$\frac{25}{\sin 47°} \approx \frac{x}{\sin 14.4°}$$

$$x \approx \frac{25 \sin 14.4°}{\sin 47°}$$

$$x \approx 8.5010128$$

So, $y \approx 32.4$ and $x \approx 8.5$. The passengers can hear the broadcast when the distance to the transmitter is 25 miles or less. So they could hear for about $32.4 - 8.5$ or 23.9 miles along the interstate.

Check for Understanding

Communicating Mathematics

Read and study the lesson to answer each question.

1. **Describe** the conditions where the Law of Sines indicates that a triangle cannot exist.
2. **Draw** two triangles where $A = 30°$, $a = 6$, and $b = 10$. Calculate and label the degree measure of each angle rounded to the nearest tenth.
3. **Write** the steps needed to solve a triangle if $A = 120°$, $a = 28$, and $b = 17$.

Guided Practice

Determine the number of possible solutions for each triangle.

4. $A = 113°$, $a = 15$, $b = 8$
5. $B = 44°$, $a = 23$, $b = 12$

Find all solutions for each triangle. If no solutions exist, write *none*. Round to the nearest tenth.

6. $C = 17°$, $a = 10$, $c = 11$
7. $A = 140°$, $b = 10$, $a = 3$
8. $A = 38°$, $b = 10$, $a = 8$
9. $C = 130°$, $c = 17$, $b = 5$
10. **Communications** A vertical radio tower is located on the top of a hill that has an angle of elevation of 10°. A 70-foot guy wire is attached to the tower 45 feet above the hill.
    a. Make a drawing to illustrate the situation.
    b. What angle does the guy wire make with the side of the hill?
    c. How far from the base of the tower is the guy wire anchored to the hill?

Exercises

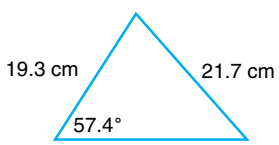
Practice

Determine the number of possible solutions for each triangle.

11. $A = 57°$, $a = 11$, $b = 19$
12. $A = 30°$, $a = 13$, $c = 26$
13. $B = 61°$, $a = 12$, $b = 8$
14. $A = 58°$, $C = 94°$, $b = 17$
15. $C = 100°$, $a = 18$, $c = 15$
16. $B = 37°$, $a = 32$, $b = 27$
17. If $A = 65°$, $a = 55$, and $b = 57$, how many possible values are there for B?

Find all solutions for each triangle. If no solutions exist, write *none*. Round to the nearest tenth.

18. $a = 6$, $b = 8$, $A = 150°$
19. $a = 26$, $b = 29$, $A = 58°$
20. $A = 30°$, $a = 4$, $b = 8$
21. $C = 70°$, $c = 24$, $a = 25$
22. $A = 40°$, $B = 60°$, $c = 20$
23. $a = 14$, $b = 12$, $B = 90°$
24. $B = 36°$, $b = 19$, $c = 30$
25. $A = 107.2°$, $a = 17.2$, $c = 12.2$
26. $A = 76°$, $a = 5$, $b = 20$
27. $C = 47°$, $a = 10$, $c = 16$
28. $B = 40°$, $b = 42$, $c = 60$
29. $b = 40$, $a = 32$, $A = 125.3°$
30. Copy the triangle at the right and label all measurements of the triangle.



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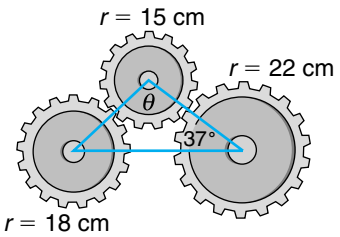
31. Find the perimeter of each of the two noncongruent triangles where $a = 15$, $b = 20$ and $A = 29°$.

32. There are two noncongruent triangles where $B = 55°$, $a = 15$, and $b = 13$. Find the measures of the angles of the triangle with the greater perimeter.

Applications and Problem Solving

33. Gears An engineer designed three gears as shown at the right. What is the measure of θ?



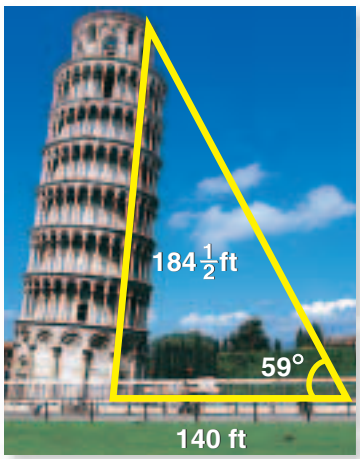
34. Critical Thinking If $b = 14$ and $A = 30°$, determine the possible values of a for each situation.

a. The triangle has no solutions.

b. The triangle has one solution.

c. The triangle has two solutions.

35. Architecture The original height of the Leaning Tower of Pisa was $184\frac{1}{2}$ feet. At a distance of 140 feet from the base of the tower, the angle of elevation from the ground to the top of the tower is 59°. How far is the tower leaning from the original vertical position?



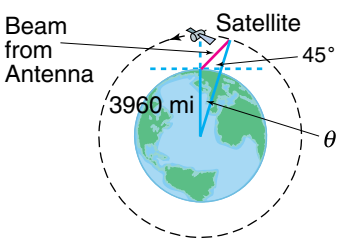
36. Navigation The captain of the Coast Guard Cutter Pendant plans to sail to a port that is 450 miles away and 12° east of north. The captain first sails the ship due north to check a buoy. He then turns the ship and sails 316 miles to the port.

a. In what direction should the captain turn the ship to arrive at the port?

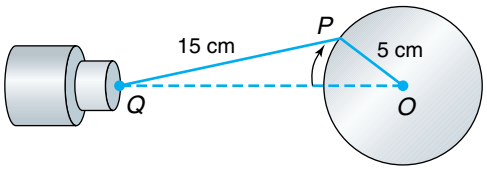
b. How many hours will it take to arrive at the turning point if the captain chooses a speed of 23 miles per hour?

c. Instead of the plan above, the captain decides to sail 200 miles north, turn through an angle of 20° east of north, and then sail along a straight course. Will the ship reach the port by following this plan?

37. Communications A satellite is orbiting Earth every 2 hours. The satellite is directly over a tracking station which has its antenna aimed 45° above the horizon. The satellite is orbiting 1240 miles above Earth, and the radius of Earth is about 3960 miles. How long ago did the satellite pass through the beam of the antenna? (*Hint*: First calculate θ.)



38. Mechanics The blades of a power lawn mower are rotated by a two-stroke engine with a piston sliding back and forth in the engine cylinder. As the piston moves back and forth, the connecting rod rotates the circular crankshaft. Suppose the crankshaft is 5 centimeters long and the connecting rod is 15 centimeters. If the crankshaft rotates 20 revolutions per second and P is at the horizontal position when it begins to rotate, how far is the piston from the rim of the crankshaft after 0.01 second?



39. Critical Thinking If $b = 12$ and $c = 17$, find the values of B for each situation.

a. The triangle has no solutions.

b. The triangle has one solution.

c. The triangle has two solutions.

Mixed Review

40. Geometry Determine the area of a rhombus if the length of a side is 24 inches and one of its angles is 32°. *(Lesson 5-6)*

41. Fire Fighting A fire is sighted from a fire tower in Wayne National Forest in Ohio. The ranger found that the angle of depression to the fire is 22°. If the tower is 75 meters tall, how far is the fire from the base of the tower? *(Lesson 5-4)*

42. State the number of roots of the equation $4x^3 - 4x^2 + 13x - 6 = 0$. Then solve the equation. *(Lesson 4-2)*

43. Determine whether the functions $y = \frac{3x}{x - 1}$ and $y = \frac{x + 1}{3x}$ are inverses of one another. Explain. *(Lesson 3-4)*

44. Solve the system of equations algebraically. *(Lesson 2-1)*

$5x - 2y = 9$

$y = 3x - 1$

45. Write the standard form of the equation whose graph is perpendicular to the graph of $-2x + 5y = 7$ and passes through the point at $(-6, 4)$. *(Lesson 1-5)*

46. SAT Practice **Grid-In** For the triangles shown below, the perimeter of $\triangle ABC$ equals the perimeter of $\triangle XYZ$. If $\triangle ABC$ is equilateral, what is the length of $\overline{AB}$?