5-8

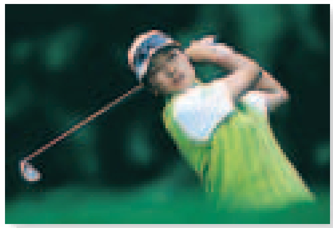
The Law of Cosines

OBJECTIVES

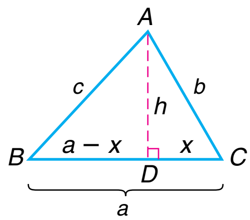
- Solve triangles by using the Law of Cosines.
- Find the area of triangles if the measures of the three sides are given.

Real World Application

GOLF For a right-handed golfer, a *slice* is a shot that curves to the right of its intended path, and a *hook* curves off to the left. Suppose Se Ri Pak hits the ball from the seventh tee at the U.S. Women's Open and the shot is a 160-yard slice 4° from the path straight to the cup. If the tee is 177 yards from the cup, how far does the ball lie from the cup? *This problem will be solved in Example 1.*



From geometry, you know that a unique triangle can be formed if the measures of three sides of a triangle are known and the sum of any two measures is greater than the remaining measure. You also know that a unique triangle can be formed if the measures of two sides and an included angle are known. However, the Law of Sines cannot be used to solve these triangles. Another formula is needed. Consider $\triangle ABC$ with a height of h units and sides measuring a units, b units, and c units. Suppose $\overline{DC}$ is x units long. Then $\overline{BD}$ is $(a - x)$ units long.



The Pythagorean Theorem and the definition of the cosine ratio can be used to show how $\angle C$, a, b, and c are related.

$c^2 = (a - x)^2 + h^2$	*Apply the Pythagorean Theorem for $\triangle ADB$.*
$c^2 = a^2 - 2ax + x^2 + h^2$	*Expand $(a - x)^2$.*
$c^2 = a^2 - 2ax + b^2$	*$b^2 = x^2 + h^2$ in $\triangle ADC$.*
$c^2 = a^2 - 2a(b \cos C) + b^2$	*$\cos C = \frac{x}{b}$, so $b \cos C = x$.*
$c^2 = a^2 + b^2 - 2ab \cos C$	*Simplify.*

By drawing altitudes from B and C, you can derive similar formulas for a^2 and b^2. All three formulas, which make up the **Law of Cosines,** can be summarized as follows.

Law of Cosines

Let $\triangle ABC$ be any triangle with a, b, and c representing the measures of sides opposite angles with measurements A, B, and C, respectively. Then, the following are true.

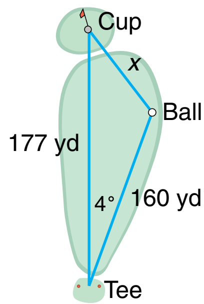
$$a^2 = b^2 + c^2 - 2bc \cos A$$
$$b^2 = a^2 + c^2 - 2ac \cos B$$
$$c^2 = a^2 + b^2 - 2ab \cos C$$

You can use the Law of Cosines to solve the application at the beginning of the lesson.

Example **1** **GOLF** **Refer to the application at the beginning of the lesson. How far does the ball lie from the cup?**

Real World Application

In this problem, you know the measurements of two sides of a triangle and the included angle. Use the Law of Cosines to find the measure of the third side of the triangle.



$x^2 = 177^2 + 160^2 - 2(177)(160)\cos 4°$

$x^2 \approx 426.9721933$ *Use a calculator.*

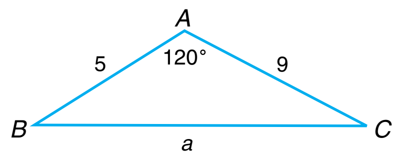
$x \approx 20.66330548$

The ball is about 20.7 yards from the cup.

Many times, you will have to use both the Law of Cosines and the Law of Sines to solve triangles.

Example **2** **Solve each triangle.**

a. $\boldsymbol{A = 120°, b = 9, c = 5}$



Graphing Calculator Tip

If you store the calculated value of *a* in your calculator, your solution will differ slightly from the one using the rounded value of *a*.

$a^2 = b^2 + c^2 - 2bc\cos A$ *Law of Cosines*

$a^2 = 9^2 + 5^2 - 2(9)(5)\cos 120°$ *Substitute 9 for b, 5 for c, and 120° for A.*

$a^2 = 151$ *Use a calculator.*

$a \approx 12.28820573$

So, $a \approx 12.3$.

$\frac{a}{\sin A} = \frac{b}{\sin B}$ *Law of Sines*

$\frac{12.3}{\sin 120°} \approx \frac{9}{\sin B}$ *Substitute 12.3 for a, 9 for b, and 120° for A.*

$\sin B \approx \frac{9 \sin 120°}{12.3}$

$B \approx \sin^{-1}\left(\frac{9 \sin 120°}{12.3}\right)$

$B \approx 39.32193819$ *Use a calculator.*

So, $B \approx 39.3°$.

$C \approx 180° - (120° + 39.3°)$

$C \approx 20.7°$

The solution of this triangle is $a \approx 12.3$, $B \approx 39.3°$, and $C \approx 20.7°$.

b. $\boldsymbol{a = 24, b = 40, c = 18}$

Recall that θ and $180° - \theta$ have the same sine function value, but different cosine function values. Therefore, a good strategy to use when all three sides are given is to use the Law of Cosines to determine the measure of the possible obtuse angle first. Since b is the longest side, B is the angle with the greatest measure, and therefore a possible obtuse angle.

$$
\begin{aligned}
b^2 &= a^2 + c^2 - 2ac \cos B && \textit{Law of Cosines}\\
40^2 &= 24^2 + 18^2 - 2(24)(18) \cos B\\
\frac{40^2 - 24^2 - 18^2}{-2(24)(18)} &= \cos B\\
\cos^{-1}\left(\frac{40^2 - 24^2 - 18^2}{-2(24)(18)}\right) &= B\\
144.1140285 &\approx B && \textit{Use a calculator.}
\end{aligned}
$$

So, $B \approx 144.1°$.

$$
\begin{aligned}
\frac{a}{\sin A} &= \frac{b}{\sin B} && \textit{Law of Sines}\\
\frac{24}{\sin A} &\approx \frac{40}{\sin 144.1°}\\
\sin A &\approx \frac{24 \sin 144.1°}{40}\\
A &\approx \sin^{-1}\left(\frac{24 \sin 144.1°}{40}\right)\\
A &\approx 20.59888389 && \textit{Use a calculator.}
\end{aligned}
$$

So, $A \approx 20.6°$.

$C \approx 180 - (20.6 + 144.1)$
$C \approx 15.3$

The solution of this triangle is $A \approx 20.6°$, $B \approx 144.1°$, and $C \approx 15.3°$.

If you know the measures of three sides of a triangle, you can find the area of the triangle by using the Law of Cosines and the formula $K = \frac{1}{2}bc \sin A$.

Example 3 **Find the area of $\triangle ABC$ if $a = 4$, $b = 7$, and $c = 9$.**

First, solve for A by using the Law of Cosines.

$$
\begin{aligned}
a^2 &= b^2 + c^2 - 2bc \cos A\\
4^2 &= 7^2 + 9^2 - 2(7)(9) \cos A\\
\frac{4^2 - 7^2 - 9^2}{-2(7)(9)} &= \cos A\\
\cos^{-1}\left(\frac{4^2 - 7^2 - 9^2}{-2(7)(9)}\right) &= A && \textit{Definition of } \cos^{-1}\\
25.2087653 &\approx A && \textit{Use a calculator.}
\end{aligned}
$$

So, $A \approx 25.2°$.

Then, find the area.

$K = \frac{1}{2}bc \sin A$

$K \approx \frac{1}{2}(7)(9) \sin 25.2°$

$K \approx 13.41204768$

The area of the triangle is about 13.4 square units.

If you know the measures of three sides of any triangle, you can also use **Hero's Formula** to find the area of the triangle.

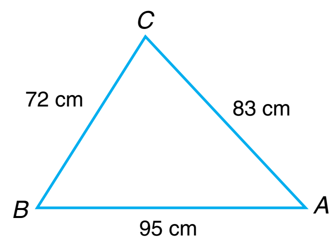
Hero's Formula

If the measures of the sides of a triangle are a, b, and c, then the area, K, of the triangle is found as follows.

$$K = \sqrt{s(s-a)(s-b)(s-c)} \text{ where } s = \frac{1}{2}(a+b+c)$$

s is called the semiperimeter.

Example 4 **Find the area of $\triangle ABC$. Round to the nearest tenth.**



First, find the semiperimeter of $\triangle ABC$.

$s = \frac{1}{2}(a + b + c)$

$s = \frac{1}{2}(72 + 83 + 95)$

$s = 125$

Now, apply Hero's Formula.

$K = \sqrt{s(s-a)(s-b)(s-c)}$

$K = \sqrt{125(125-72)(125-83)(125-95)}$

$K = \sqrt{8{,}347{,}500}$

$K \approx 2889.204043$ *Use a calculator.*

The area of the triangle is about 2889.2 square centimeters.

CHECK FOR UNDERSTANDING

Communicating Mathematics

Read and study the lesson to answer each question.

1. **Describe** two situations where the Law of Cosines is needed to solve a triangle.
2. **Give an example** of three measurements of sides that do not form a triangle.
3. **Explain** how the Pythagorean Theorem is a special case of the Law of Cosines.
4. *Math Journal* **Draw and label** a right triangle that can be solved using the trigonometric functions. Draw and label a triangle that can be solved using the Law of Sines, but not the Law of Cosines. Draw and label a triangle that can be solved using the Law of Cosines. Solve each triangle.

Guided Practice

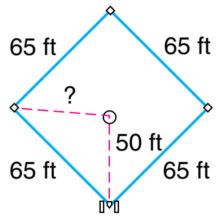
Solve each triangle. Round to the nearest tenth.

5. $a = 32$, $b = 38$, $c = 46$
6. $a = 25$, $b = 30$, $C = 160°$
7. The sides of a triangle are 18 inches, 21 inches, and 14 inches. Find the measure of the angle with the greatest measure.

Find the area of each triangle. Round to the nearest tenth.

8. $a = 2$, $b = 7$, $c = 8$
9. $a = 25$, $b = 13$, $c = 17$

10. **Softball** In slow-pitch softball, the diamond is a square that is 65 feet on each side. The distance between the pitcher's mound and home plate is 50 feet. How far does the pitcher have to throw the softball from the pitcher's mound to third base to stop a player who is trying to steal third base?



EXERCISES

Practice

Solve each triangle. Round to the nearest tenth.

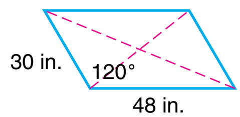
11. $b = 7, c = 10, A = 51°$
12. $a = 5, b = 6, c = 7$
13. $a = 4, b = 5, c = 7$
14. $a = 16, c = 12, B = 63°$
15. $a = 11.4, b = 13.7, c = 12.2$
16. $C = 79.3°, a = 21.5, b = 13$
17. The sides of a triangle measure 14.9 centimeters, 23.8 centimeters, and 36.9 centimeters. Find the measure of the angle with the least measure.
18. **Geometry** Two sides of a parallelogram measure 60 centimeters and 40 centimeters. If one angle of the parallelogram measures 132°, find the length of each diagonal.

interNET CONNECTION

Graphing Calculator Programs
For a graphing calculator program that determines the area of a triangle, given the lengths of all sides of the triangle, visit **www.amc.glencoe.com**

Find the area of each triangle. Round to the nearest tenth.

19. $a = 4, b = 6, c = 8$
20. $a = 17, b = 13, c = 19$
21. $a = 20, b = 30, c = 40$
22. $a = 33, b = 51, c = 42$
23. $a = 174, b = 138, c = 188$
24. $a = 11.5, b = 13.7, c = 12.2$
25. **Geometry** The lengths of two sides of a parallelogram are 48 inches and 30 inches. One angle measures 120°.
    a. Find the length of the longer diagonal.
    b. Find the area of the parallelogram.

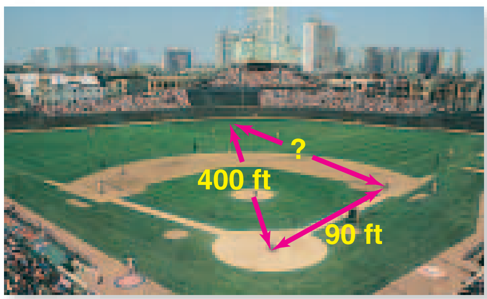


26. **Geometry** The side of a rhombus is 15 centimeters long, and the length of its longer diagonal is 24.6 centimeters.
    a. Find the area of the rhombus.
    b. Find the measures of the angles of the rhombus.

Applications and Problem Solving

Real World Application

27. **Baseball** In baseball, dead center field is the farthest point in the outfield on the straight line through home plate and second base. The distance between consecutive bases is 90 feet. In Wrigley Field in Chicago, dead center field is 400 feet from home plate. How far is dead center field from first base?



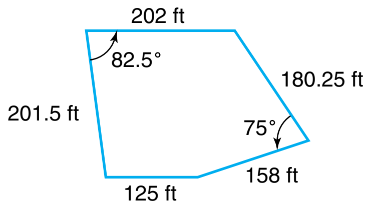
28. **Critical Thinking** The lengths of the sides of a triangle are 74 feet, 38 feet, and 88 feet. What is the length of the altitude drawn to the longest side?

29. **Air Travel** The distance between Miami and Orlando is about 220 miles. A pilot flying from Miami to Orlando starts the flight $10°$ off course to avoid a storm.

a. After flying in this direction for 100 miles, how far is the plane from Orlando?

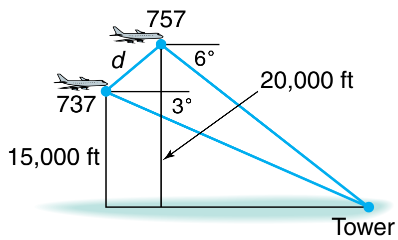
b. If the pilot adjusts his course after 100 miles, how much farther is the flight than a direct route?

30. **Critical Thinking** Find the area of the pentagon at the right.



31. **Soccer** A soccer player is standing 35 feet from one post of the goal and 40 feet from the other post. Another soccer player is standing 30 feet from one post of the same goal and 20 feet from the other post. If the goal is 24 feet wide, which player has a greater angle to make a shot on goal?

32. **Air Traffic Control** A 757 passenger jet and a 737 passenger jet are on their final approaches to San Diego International Airport.



a. The 757 is 20,000 feet from the ground, and the angle of depression to the tower is $6°$. Find the distance between the 757 and the tower.

b. The 737 is 15,000 feet from the ground, and the angle of depression to the tower is $3°$. What is the distance between the 737 and the tower?

c. How far apart are the jets?

Mixed Review

33. Determine the number of possible solutions for $\triangle ABC$ if $A = 63.2°$, $b = 18$ and $a = 17$. *(Lesson 5-7)*

34. **Landmarks** The San Jacinto Column in Texas is 570 feet tall and, at a particular time, casts a shadow 700 feet long. Find the angle of elevation to the sun at that time. *(Lesson 5-5)*

35. Find the reference angle for $-775°$. *(Lesson 5-1)*

36. Find the value of k so that the remainder of $(x^3 - 7x^2 - kx + 6)$ divided by $(x - 3)$ is 0. *(Lesson 4-3)*

37. Find the slope of the line through points at $(2t, t)$ and $(5t, 5t)$. *(Lesson 1-3)*

38. **SAT/ACT Practice** Find an expression equivalent to $\left(\frac{2x^2}{y}\right)^3$.

A $\frac{8x^6}{y^3}$ **B** $\frac{64x^6}{y^3}$ **C** $\frac{6x^5}{y^3}$ **D** $\frac{8x^5}{y^3}$ **E** $\frac{2x^5}{y^4}$

Extra Practice See p. A35.

CONTENTS

GRAPHING CALCULATOR EXPLORATION

5-8B Solving Triangles

An Extension of Lesson 5-8

OBJECTIVE

- Use a program to solve triangles.

The following program allows you to enter the coordinates of the vertices of a triangle in the coordinate plane and display as output the side lengths and the angle measures in degrees.

```
PROGRAM: SOLVTRI
: Disp "INPUT VERTEX A"
: Input A: Input P
: Disp "INPUT VERTEX B"
: Input B: Input Q
: Disp "INPUT VERTEX C"
: Input C: Input S
: √((A − B)² + (P − Q)²) → E
: √((B − C)² + (Q − S)²) → F
: √((A − C)² + (P − S)²) → G
: cos⁻¹ ((F² − E² − G²)/(−2EG)) → M
: cos⁻¹ ((G² − E² − F²)/(−2EF)) → N
: cos⁻¹ ((E² − F² − G²)/(−2FG)) → O
: Disp "SIDE AB" : Disp E
: Disp "SIDE BC" : Disp F
: Disp "SIDE AC" : Disp G
: Pause
: Disp "ANGLE A" : Disp M
: Disp "ANGLE B" : Disp N
: Disp "ANGLE C" : Disp O
: Stop
```

- To solve a triangle using the program, you first need to place the triangle in a coordinate plane, determine the coordinates of the vertices, and then input the coordinates when prompted by the program.
- When you place the triangle in the coordinate plane, it is a good idea to choose a side whose length is given and place that side on the positive x-axis so that its left endpoint is at (0, 0). You can use the length of the side to determine the coordinates of the other endpoint.
- To locate the third vertex, you can use the given information about the triangle to write equations whose graphs intersect at the third vertex. Graph the equations and use intersect on the **CALC** menu to find the coordinates of the third vertex.
- You are now ready to run the program.

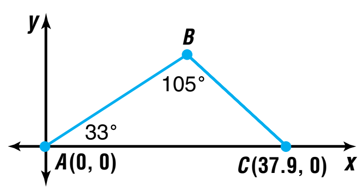
interNET CONNECTION

You can download this program by visiting our Web site at **www.amc.glencoe.com**

Example

1 **Use the program to solve $\triangle ABC$ if $A = 33°$, $B = 105°$, and $b = 37.9$.**

Before you use the calculator, do some advance planning. Make a sketch of how you can place the triangle in the coordinate plane to have the third vertex above the x-axis. One possibility is shown at the right.



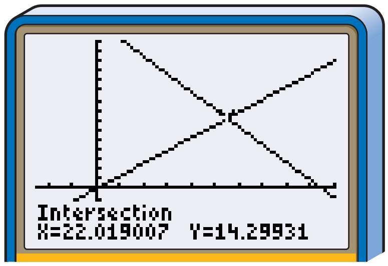
(continued on the next page)

The slope of $\overline{AB}$ is $\frac{y-0}{x-0}$ or $\frac{y}{x}$. This is tan 33°. So, $\frac{y}{x} = \tan 33°$ or $y = (\tan 33°)x$.

The slope of $\overline{BC}$ is $\frac{y-0}{x-37.9}$. This is tangent of (180° − 42°).

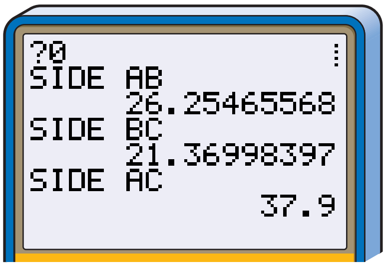
So, $\tan 138° = \frac{y}{x-37.9}$ or $y = (\tan 138°)(x - 37.9)$.

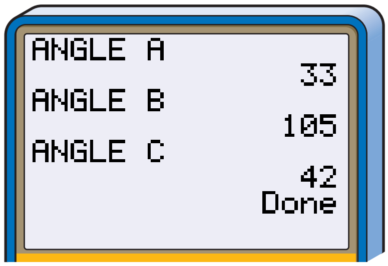
Enter the equations (without the degree signs) on the **Y=** list, and select appropriate window settings. Graph the equations and use intersect in the **CALC** menu to find the coordinates of vertex *B*. When the calculator displays the coordinates at the bottom of the screen, go immediately to the program. Be sure you do nothing to alter the values of *x* and *y* that were displayed.



[-10, 40] scl: 5 by [-10, 40] scl: 5

When the program prompts you to input vertex *A*, enter 0 and 0. For vertex *B*, press X,T,θ,n ENTER ALPHA **[Y]** ENTER. For vertex *C*, enter the numbers 37.9 and 0.The calculator will display the side lengths of the triangle and pause. To display the angle measures, press ENTER.





The side lengths and angle measures agree with those in the text. Therefore, $a \approx 21.4$, $c \approx 26.3$, and $C = 42°$. *Compare the results with those in Example 1 of Lesson 5-6, on page 314.*

TRY THESE

1. Solve △*ABC* given that *AC* = 6, *BC* = 8, and *A* = 35°. (*Hint*: Place $\overline{AC}$ so that the endpoints have coordinates (0, 0) and (6, 0). Use the graphs of $y = (\tan 35°)x$ and $y = \sqrt{5^2 - (x-6)^2}$ to determine the coordinates of vertex *B*.)

2. Use **SOLVTRI** to solve the triangles in Example 2b of Lesson 5-7.

WHAT DO YOU THINK?

3. What law about triangles is the basis of the program **SOLVTRI**?

4. Suppose you want to use **SOLVTRI** to solve the triangle in Example 2a of Lesson 5-7. How would you place the triangle in the coordinate plane?