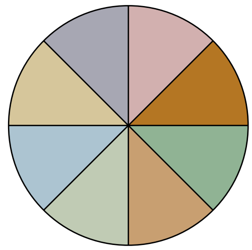
6-1 Angles and Radian Measure

OBJECTIVES

- Change from radian measure to degree measure, and vice versa.
- Find the length of an arc given the measure of the central angle.
- Find the area of a sector.

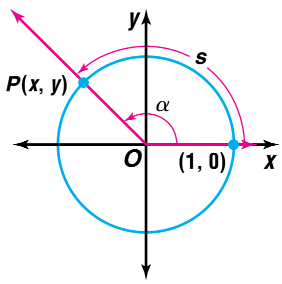
Real World Application

BUSINESS Junjira Putiwuthigool owns a business in Changmai, Thailand, that makes ornate umbrellas and fans. Ms. Putiwuthigool has an order for three dozen umbrellas having a diameter of 2 meters. Bamboo slats that support each circular umbrella divide the umbrella into 8 sections or sectors. Each section will be covered with a different color fabric. How much fabric of each color will Ms. Putiwuthigool need to complete the order? *This problem will be solved in Example 6.*



There are many real-world applications, such as the one described above, which can be solved more easily using an angle measure other than the degree. This other unit is called the **radian.**

The definition of radian is based on the concept of the unit circle. Recall that the unit circle is a circle of radius 1 whose center is at the origin of a rectangular coordinate system.



A point $P(x, y)$ is on the unit circle if and only if its distance from the origin is 1. Thus, for each point $P(x, y)$ on the unit circle, the distance from the origin is represented by the following equation.

$$\sqrt{(x-0)^2 + (y-0)^2} = 1$$

If each side of this equation is squared, the result is an equation of the unit circle.

$$x^2 + y^2 = 1$$

Consider an angle α in standard position, shown above. Let $P(x, y)$ be the point of intersection of its terminal side with the unit circle. The radian measure of an angle in standard position is defined as the length of the corresponding arc on the unit circle. Thus, the measure of angle α is s radians. Since $C = 2\pi r$, a full revolution correponds to an angle of $2\pi(1)$ or 2π radians.

There is an important relationship between radian and degree measure. Since an angle of one complete revolution can be represented either by 360° or by 2π radians, $360° = 2\pi$ radians. Thus, $180° = \pi$ radians, and $90° = \frac{\pi}{2}$ radians.

The following formulas relate degree and radian measures.

Degree/ Radian Conversion Formulas	1 radian $= \frac{180}{\pi}$ degrees or about 57.3° 1 degree $= \frac{\pi}{180}$ radians or about 0.017 radian

Angles expressed in radians are often written in terms of π. The term *radians* is also usually omitted when writing angle measures. However, the degree symbol is always used in this book to express the measure of angles in degrees.

Example 1 **a. Change 330° to radian measure in terms of π.**

$$330° = 330° \times \frac{\pi}{180°} \qquad \textit{1 degree} = \frac{\pi}{180°}$$
$$= \frac{11\pi}{6}$$

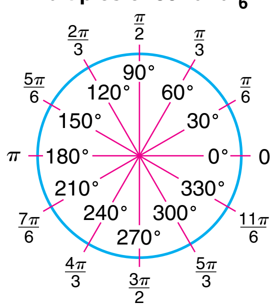
b. Change $\frac{2\pi}{3}$ radians to degree measure.

$$\frac{2\pi}{3} = \frac{2\pi}{3} \times \frac{180°}{\pi} \qquad \textit{1 radian} = \frac{180°}{\pi}$$
$$= 120°$$

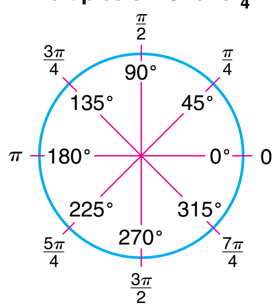
Angles whose measures are multiples of 30° and 45° are commonly used in trigonometry. These angle measures correspond to radian measures of $\frac{\pi}{6}$ and $\frac{\pi}{4}$, respectively. The diagrams below can help you make these conversions mentally.

You may want to memorize these radian measures and their degree equivalents to simplify your work in trigonometry.

Multiples of 30° and $\frac{\pi}{6}$



Multiples of 45° and $\frac{\pi}{4}$



These equivalent values are summarized in the chart below.

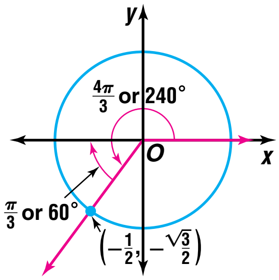
Degrees	0	30	45	60	90	120	135	150	180	210	225	240	270	300	315	330
Radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$

You can use reference angles and the unit circle to determine trigonometric values for angle measures expressed as radians.

Example **2** **Evaluate $\cos \frac{4\pi}{3}$.**

The reference angle for $\frac{4\pi}{3}$ is $\frac{4\pi}{3} - \pi$ or $\frac{\pi}{3}$.

Since $\frac{\pi}{3} = 60°$, the terminal side of the angle intersects the unit circle at a point with coordinates of $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$.



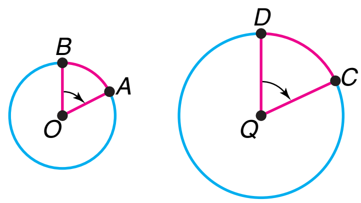
Because the terminal side of this angle is in the third quadrant, both coordinates are negative. The point of intersection has coordinates $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$.

Therefore, $\cos \frac{4\pi}{3} = -\frac{1}{2}$.

Look Back

You can refer to Lesson 5-3 to review reference angles and unit circles used to determine values of trigonometric functions.

Radian measure can be used to find the length of a **circular arc.** A circular arc is a part of a circle. The arc is often defined by the **central angle** that intercepts it. A central angle of a circle is an angle whose vertex lies at the center of the circle.

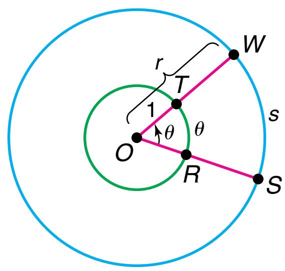


If two central angles in different circles are congruent, the ratio of the lengths of their intercepted arcs is equal to the ratio of the measures of their radii.

For example, given circles O and Q, if $\angle O \cong \angle Q$, then $\frac{m\overset{\frown}{AB}}{m\overset{\frown}{CD}} = \frac{OA}{QC}$.

Let O be the center of two concentric circles, let r be the measure of the radius of the larger circle, and let the smaller circle be a unit circle. A central angle of θ radians is drawn in the two circles that intercept $\overset{\frown}{RT}$ on the unit circle and $\overset{\frown}{SW}$ on the other circle. Suppose $\overset{\frown}{SW}$ is s units long. $\overset{\frown}{RT}$ is θ units long since it is an arc of a unit circle intercepted by a central angle of θ radians. Thus, we can write the following proportion.

$$\frac{s}{\theta} = \frac{r}{1} \text{ or } s = r\theta$$



We say that an arc subtends its central angle.

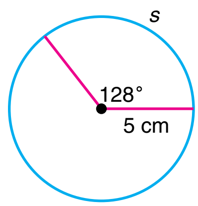
Length of an Arc	The length of any circular arc s is equal to the product of the measure of the radius of the circle r and the radian measure of the central angle θ that it subtends. $s = r\theta$

Example 3 **Given a central angle of 128°, find the length of its intercepted arc in a circle of radius 5 centimeters. Round to the nearest tenth.**

First, convert the measure of the central angle from degrees to radians.

$128° = 128° \times \frac{\pi}{180°}$ *1 degree = $\frac{\pi}{180}$*

$= \frac{32}{45}\pi$ or $\frac{32\pi}{45}$



Then, find the length of the arc.

$s = r\theta$

$s = 5\left(\frac{32\pi}{45}\right)$ *$r = 5$, $\theta = \frac{32\pi}{45}$*

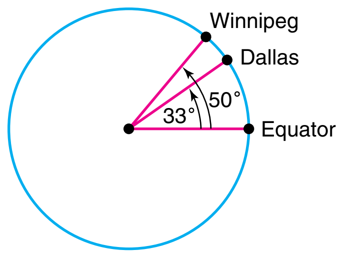
$s \approx 11.17010721$ *Use a calculator.*

The length of the arc is about 11.2 centimeters.

You can use radians to compute distances between two cities that lie on the same longitude line.

Example 4 Real World Application

GEOGRAPHY **Winnipeg, Manitoba, Canada, and Dallas, Texas, lie along the 97° W longitude line. The latitude of Winnipeg is 50° N, and the latitude of Dallas is 33° N. The radius of Earth is about 3960 miles. Find the approximate distance between the two cities.**



The length of the arc between Dallas and Winnipeg is the distance between the two cities. The measure of the central angle subtended by this arc is 50° − 33° or 17°.

$17° = 17° \times \frac{\pi}{180°}$ *1 degree = $\frac{\pi}{180}$*

$= \frac{17\pi}{180}$

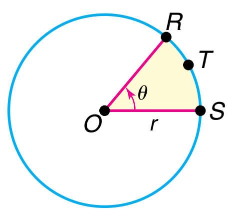
$s = r\theta$

$s = 3960\left(\frac{17\pi}{180}\right)$ *$r = 3960$, $\theta = \frac{17\pi}{180}$*

$s \approx 1174.955652$ *Use a calculator.*

The distance between the two cities is about 1175 miles.

A **sector** of a circle is a region bounded by a central angle and the intercepted arc. For example, the shaded portion in the figure is a sector of circle O. The ratio of the area of a sector to the area of a circle is equal to the ratio of its arc length to the circumference.



Let A represent the area of the sector.

$$\frac{A}{\pi r^2} = \frac{\text{length of } \widehat{RTS}}{2\pi r}$$

$$\frac{A}{\pi r^2} = \frac{r\theta}{2\pi r} \quad \textit{The length of } \widehat{RTS} \textit{ is } r\theta.$$

$$A = \frac{1}{2}r^2\theta \quad \textit{Solve for } A.$$

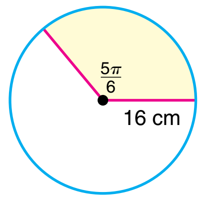
Area of a Circular Sector

If θ is the measure of the central angle expressed in radians and r is the measure of the radius of the circle, then the area of the sector, A, is as follows.

$$A = \frac{1}{2}r^2\theta$$

Examples

5 **Find the area of a sector if the central angle measures $\frac{5\pi}{6}$ radians and the radius of the circle is 16 centimeters. Round to the nearest tenth.**



$A = \frac{1}{2}r^2\theta$ *Formula for the area of a circular sector*

$A = \frac{1}{2}(16^2)\left(\frac{5\pi}{6}\right)$ $r = 16,\ \theta = \frac{5\pi}{6}$

$A \approx 335.1032164$ *Use a calculator.*

The area of the sector is about 335.1 square centimeters.

6 **BUSINESS** **Refer to the application at the beginning of the lesson. How much fabric of each color will Ms. Putiwuthigool need to complete the order?**

Real World Application

There are 2π radians in a complete circle and 8 equal sections or sectors in the umbrella. Therefore, the measure of each central angle is $\frac{2\pi}{8}$ or $\frac{\pi}{4}$ radians. If the diameter of the circle is 2 meters, the radius is 1 meter. Use these values to find the area of each sector.

$A = \frac{1}{2}r^2\theta$

$A = \frac{1}{2}(1^2)\left(\frac{\pi}{4}\right)$ $r = 1,\ \theta = \frac{\pi}{4}$

$A \approx 0.3926990817$ *Use a calculator.*

Since there are 3 dozen or 36 umbrellas, multiply the area of each sector by 36. Ms. Putiwuthigool needs about 14.1 square meters of each color of fabric. *This assumes that the pieces can be cut with no waste and that no extra material is needed for overlapping.*

CHECK FOR UNDERSTANDING

Communicating Mathematics

Read and study the lesson to answer each question.

1. **Draw** a unit circle and a central angle with a measure of $\frac{3\pi}{4}$ radians.
2. **Describe** the angle formed by the hands of a clock at 3:00 in terms of degrees and radians.

3. **Explain** how you could find the radian measure of a central angle subtended by an arc that is 10 inches long in a circle with a radius of 8 inches.

4. **Demonstrate** that if the radius of a circle is doubled and the measure of a central angle remains the same, the length of the arc is doubled and the area of the sector is quadrupled.

Guided Practice

Change each degree measure to radian measure in terms of π.

5. $240°$

6. $570°$

Change each radian measure to degree measure. Round to the nearest tenth, if necessary.

7. $\frac{3\pi}{2}$

8. -1.75

Evaluate each expression.

9. $\sin \frac{3\pi}{4}$

10. $\tan \frac{11\pi}{6}$

Given the measurement of a central angle, find the length of its intercepted arc in a circle of radius 15 inches. Round to the nearest tenth.

11. $\frac{5\pi}{6}$

12. $77°$

Find the area of each sector given its central angle θ and the radius of the circle. Round to the nearest tenth.

13. $\theta = \frac{2\pi}{3}, r = 1.4$

14. $\theta = 54°, r = 6$

15. **Physics** A pendulum with length of 1.4 meters swings through an angle of 30°. How far does the bob at the end of the pendulum travel as it goes from left to right?

EXERCISES

Practice

Change each degree measure to radian measure in terms of π.

16. $135°$

17. $210°$

18. $300°$

19. $-450°$

20. $-75°$

21. $1250°$

Change each radian measure to degree measure. Round to the nearest tenth, if necessary.

22. $\frac{7\pi}{12}$

23. $\frac{11\pi}{3}$

24. 17

25. -3.5

26. $-\frac{\pi}{6.2}$

27. 17.5

Evaluate each expression.

28. $\sin \frac{5\pi}{3}$

29. $\tan \frac{7\pi}{6}$

30. $\cos \frac{5\pi}{4}$

31. $\sin \frac{7\pi}{6}$

32. $\tan \frac{14\pi}{3}$

33. $\cos \left(-\frac{19\pi}{6}\right)$

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Given the measurement of a central angle, find the length of its intercepted arc in a circle of radius 14 centimeters. Round to the nearest tenth.

34. $\frac{2\pi}{3}$ **35.** $\frac{5\pi}{12}$ **36.** $150°$

37. $282°$ **38.** $\frac{3\pi}{11}$ **39.** $320°$

40. The diameter of a circle is 22 inches. If a central angle measures 78°, find the length of the intercepted arc.

41. An arc is 70.7 meters long and is intercepted by a central angle of $\frac{5\pi}{4}$ radians. Find the diameter of the circle.

42. An arc is 14.2 centimeters long and is intercepted by a central angle of 60°. What is the radius of the circle?

Find the area of each sector given its central angle θ and the radius of the circle. Round to the nearest tenth.

43. $\theta = \frac{5\pi}{12}, r = 10$ **44.** $\theta = 90°, r = 22$ **45.** $\theta = \frac{\pi}{8}, r = 7$

46. $\theta = \frac{4\pi}{7}, r = 12.5$ **47.** $\theta = 225°, r = 6$ **48.** $\theta = 82°, r = 7.3$

49. A sector has arc length of 6 feet and central angle of 1.2 radians.
- **a.** Find the radius of the circle.
- **b.** Find the area of the sector.

50. A sector has a central angle of 135° and arc length of 114 millimeters.
- **a.** Find the radius of the circle.
- **b.** Find the area of the sector.

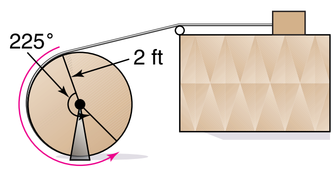
51. A sector has area of 15 square inches and central angle of 0.2 radians.
- **a.** Find the radius of the circle.
- **b.** Find the arc length of the sector.

52. A sector has area of 15.3 square meters. The radius of the circle is 3 meters.
- **a.** Find the radian measure of the central angle.
- **b.** Find the degree measure of the central angle.
- **c.** Find the arc length of the sector.

Applications and Problem Solving

Real World Application

53. Mechanics A wheel has a radius of 2 feet. As it turns, a cable connected to a box winds onto the wheel.
- **a.** How far does the box move if the wheel turns 225° in a counterclockwise direction?
- **b.** Find the number of degrees the wheel must be rotated to move the box 5 feet.

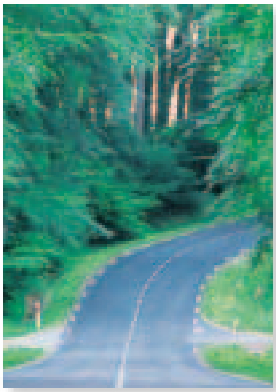


54. Critical Thinking Two gears are interconnected. The smaller gear has a radius of 2 inches, and the larger gear has a radius of 8 inches. The smaller gear rotates 330°. Through how many radians does the larger gear rotate?

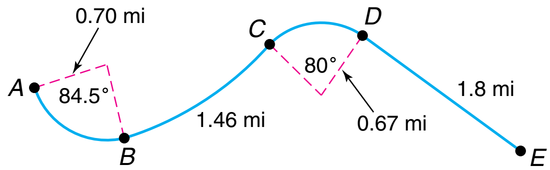
55. Physics A pendulum is 22.9 centimeters long, and the bob at the end of the pendulum travels 10.5 centimeters. Find the degree measure of the angle through which the pendulum swings.

56. **Geography** Minneapolis, Minnesota; Arkadelphia, Arkansas; and Alexandria, Louisiana lie on the same longitude line. The latitude of Minneapolis is 45° N, the latitude of Arkadelphia is 34° N, and the latitude of Alexandria is 31° N. The radius of Earth is about 3960 miles.

a. Find the approximate distance between Minneapolis and Arkadelphia.
b. What is the approximate distance between Minneapolis and Alexandria?
c. Find the approximate distance between Arkadelphia and Alexandria.



57. **Civil Engineering** The figure below shows a stretch of roadway where the curves are arcs of circles.



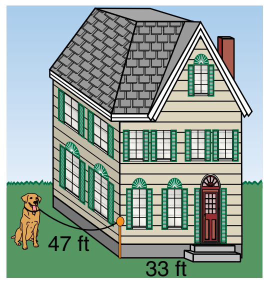
Find the length of the road from point A to point E.

58. **Mechanics** A single pulley is being used to pull up a weight. Suppose the diameter of the pulley is $2\frac{1}{2}$ feet.

a. How far will the weight rise if the pulley turns 1.5 rotations?
b. Find the number of degrees the pulley must be rotated to raise the weight $4\frac{1}{2}$ feet.

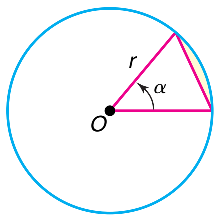
59. **Pet Care** A rectangular house is 33 feet by 47 feet. A dog is placed on a leash that is connected to a pole at the corner of the house.

a. If the leash is 15 feet long, find the area the dog has to play.
b. If the owner wants the dog to have 750 square feet to play, how long should the owner make the leash?



60. **Biking** Rafael rides his bike 3.5 kilometers. If the radius of the tire on his bike is 32 centimeters, determine the number of radians that a spot on the tire will travel during the trip.

61. **Critical Thinking** A *segment* of a circle is the region bounded by an arc and its chord. Consider any minor arc. If α is the radian measure of the central angle and r is the radius of the circle, write a formula for the area of the segment.

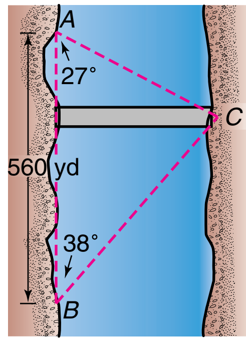


Mixed Review

62. The lengths of the sides of a triangle are 6 inches, 8 inches, and 12 inches. Find the area of the triangle. *(Lesson 5-8)*

63. Determine the number of possible solutions of $\triangle ABC$ if $A = 152°$, $b = 12$, and $a = 10.2$. If solutions exist, solve the triangle. *(Lesson 5-7)*

64. Surveying Two surveyors are determining measurements to be used to build a bridge across a canyon. The two surveyors stand 560 yards apart on one side of the canyon and sight a marker C on the other side of the canyon at angles of 27° and 38°. Find the length of the bridge if it is built through point C as shown. *(Lesson 5-6)*



65. Suppose θ is an angle in standard position and $\tan \theta > 0$. State the quadrants in which the terminal side of θ can lie. *(Lesson 5-3)*

66. Population The population for Forsythe County, Georgia, has experienced significant growth in recent years. *(Lesson 4-8)*

Year	1970	1980	1990	1998
Population	17,000	28,000	44,000	86,000

Source: U.S. Census Bureau

a. Write a model that relates the population of Forsythe County as a function of the number of years since 1970.

b. Use the model to predict the population in the year 2020.

67. Use the Upper Bound Theorem to find an integral upper bound and the Lower Bound Theorem to find a lower bound of the zeros of $f(x) = x^4 - 3x^3 - 2x^2 + 6x + 10$. *(Lesson 4-5)*

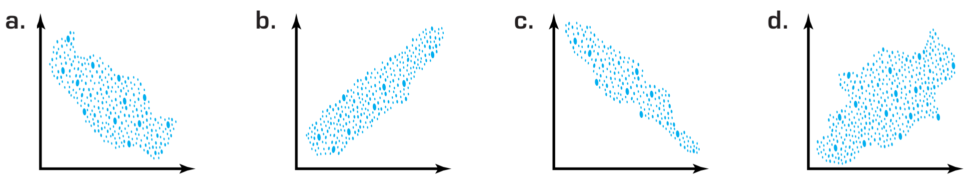
68. Use synthetic division to determine if $x + 2$ is a factor of $x^3 + 6x^2 + 12x + 12$. Explain. *(Lesson 4-3)*

69. Determine whether the graph of $x^2 + y^2 = 16$ is symmetric with respect to the x-axis, the y-axis, the line $y = x$, or the line $y = -x$. *(Lesson 3-1)*

70. Solve the system of equations algebraically. *(Lesson 2-2)*

$$4x - 2y + 3z = -6$$
$$3x + 3y - 2z = 2$$
$$5x - 4y - 3z = -75$$

71. Which scatter plot shows data that has a strongly positive correlation? *(Lesson 1-6)*



72. SAT Practice If $p > 0$ and $q < 0$, which quantity must be positive?

A $p + q$

B $p - q$

C $q - p$

D $p \times q$

E $p \div q$

Extra Practice See p. A36.