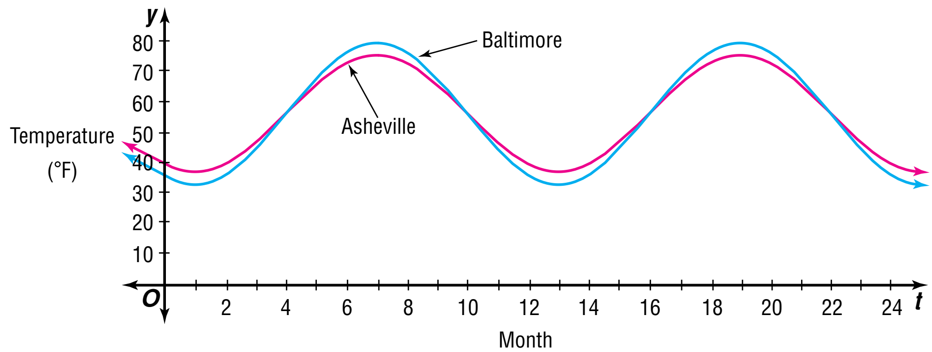
6-3 Graphing Sine and Cosine Functions

OBJECTIVE
- Use the graphs of the sine and cosine functions.

Real World Application

METEOROLOGY The average monthly temperatures for a city demonstrate a repetitious behavior. For cities in the Northern Hemisphere, the average monthly temperatures are usually lowest in January and highest in July. The graph below shows the average monthly temperatures (°F) for Baltimore, Maryland, and Asheville, North Carolina, with January represented by 1.



Model for Baltimore's temperature: $y = 54.4 + 22.5 \sin\left[\frac{\pi}{6}(t - 4)\right]$

Model for Asheville's temperature: $y = 54.5 + 18.5 \sin\left[\frac{\pi}{6}(t - 4)\right]$

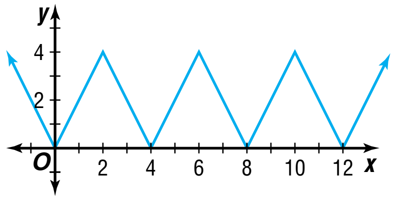
In these equations, t denotes the month with January represented by $t = 1$. What is the average temperature for each city for month 13? Which city has the greater fluctuation in temperature?
These problems will be solved in Example 5.

Each year, the graph for Baltimore will be about the same. This is also true for Asheville. If the values of a function are the same for each given interval of the domain (in this case, 12 months or 1 year), the function is said to be **periodic.** The interval is the **period** of the function.

Periodic Function and Period	A function is *periodic* if, for some real number α, $f(x + \alpha) = f(x)$ for each x in the domain of f. The least positive value of α for which $f(x) = f(x + \alpha)$ is the *period* of the function.

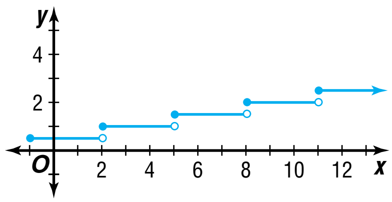
Example 1 **Determine if each function is periodic. If so, state the period.**

a.



The values of the function repeat for each interval of 4 units. The function is periodic, and the period is 4.

b.

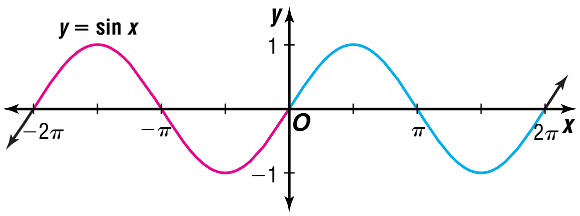


The values of the function do not repeat. The function is not periodic.

Consider the sine function. First evaluate $y = \sin x$ for domain values between -2π and 2π in multiples of $\frac{\pi}{4}$.

x	-2π	$-\frac{7\pi}{4}$	$-\frac{3\pi}{2}$	$-\frac{5\pi}{4}$	$-\pi$	$-\frac{3\pi}{4}$	$-\frac{\pi}{2}$	$-\frac{\pi}{4}$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
$\sin x$	0	$\frac{\sqrt{2}}{2}$	1	$\frac{\sqrt{2}}{2}$	0	$-\frac{\sqrt{2}}{2}$	-1	$-\frac{\sqrt{2}}{2}$	0	$\frac{\sqrt{2}}{2}$	1	$\frac{\sqrt{2}}{2}$	0	$-\frac{\sqrt{2}}{2}$	-1	$-\frac{\sqrt{2}}{2}$	0

To graph $y = \sin x$, plot the coordinate pairs from the table and connect them to form a smooth curve. Notice that the range values for the domain interval $-2\pi < x < 0$ (shown in red) repeat for the domain interval between $0 < x < 2\pi$ (shown in blue). The sine function is a periodic function.



By studying the graph and its repeating pattern, you can determine the following properties of the graph of the sine function.

Properties of the Graph of $y = \sin x$

1. The period is 2π.
2. The domain is the set of real numbers.
3. The range is the set of real numbers between -1 and 1, inclusive.
4. The x-intercepts are located at πn, where n is an integer.
5. The y-intercept is 0.
6. The maximum values are $y = 1$ and occur when $x = \frac{\pi}{2} + 2\pi n$, where n is an integer.
7. The minimum values are $y = -1$ and occur when $x = \frac{3\pi}{2} + 2\pi n$, where n is an integer.

Examples

2 **Find $\sin \frac{9\pi}{2}$ by referring to the graph of the sine function.**

Because the period of the sine function is 2π and $\frac{9\pi}{2} > 2\pi$, rewrite $\frac{9\pi}{2}$ as a sum involving 2π.

$$\frac{9\pi}{2} = 4\pi + \frac{\pi}{2}$$

$$= 2\pi(2) + \frac{\pi}{2} \quad \textit{This is a form of } \frac{\pi}{2} + 2\pi n.$$

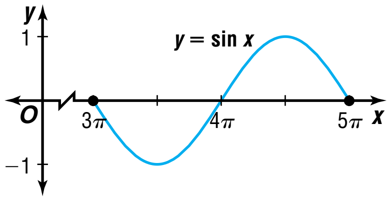
So, $\sin \frac{9\pi}{2} = \sin \frac{\pi}{2}$ or 1.

3 **Find the values of θ for which $\sin \theta = 0$ is true.**

Since $\sin \theta = 0$ indicates the x-intercepts of the function, $\sin \theta = 0$ if $\theta = n\pi$, where n is any integer.

4 **Graph $y = \sin x$ for $3\pi \leq x \leq 5\pi$.**

The graph crosses the x-axis at 3π, 4π, and 5π. It has its maximum value of 1 at $x = \frac{9\pi}{2}$, and its minimum value of -1 at $x = \frac{7\pi}{2}$.
Use this information to sketch the graph.



5 **METEOROLOGY** **Refer to the application at the beginning of the lesson.**

Real World Application

a. What is the average temperature for each city for month 13?
Month 13 is January of the second year. To find the average temperature of this month, substitute this value into each equation.

Baltimore

$$y = 54.4 + 22.5 \sin\left[\frac{\pi}{6}(t - 4)\right]$$

$$y = 54.4 + 22.5 \sin\left[\frac{\pi}{6}(13 - 4)\right]$$

$$y = 54.4 + 22.5 \sin \frac{3\pi}{2}$$

$$y = 54.4 + 22.5(-1)$$

$$y = 31.9$$

Asheville

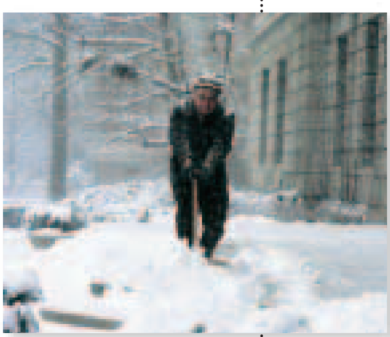
$$y = 54.5 + 18.5 \sin\left[\frac{\pi}{6}(t - 4)\right]$$

$$y = 54.5 + 18.5 \sin\left[\frac{\pi}{6}(13 - 4)\right]$$

$$y = 54.5 + 18.5 \sin \frac{3\pi}{2}$$

$$y = 54.5 + 18.5(-1)$$

$$y = 36.0$$

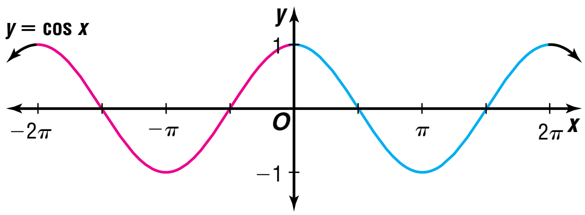


In January, the average temperature for Baltimore is 31.9°, and the average temperature for Asheville is 36.0°.

b. Which city has the greater fluctuation in temperature? Explain.
The average temperature for January is lower in Baltimore than in Asheville. The average temperature for July is higher in Baltimore than in Asheville. Therefore, there is a greater fluctuation in temperature in Baltimore than in Asheville.

Now, consider the graph of $y = \cos x$.

x	-2π	$-\frac{7\pi}{4}$	$-\frac{3\pi}{2}$	$-\frac{5\pi}{4}$	$-\pi$	$-\frac{3\pi}{4}$	$-\frac{\pi}{2}$	$-\frac{\pi}{4}$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
$\cos x$	1	$\frac{\sqrt{2}}{2}$	0	$-\frac{\sqrt{2}}{2}$	-1	$-\frac{\sqrt{2}}{2}$	0	$\frac{\sqrt{2}}{2}$	1	$\frac{\sqrt{2}}{2}$	0	$-\frac{\sqrt{2}}{2}$	-1	$-\frac{\sqrt{2}}{2}$	0	$\frac{\sqrt{2}}{2}$	1

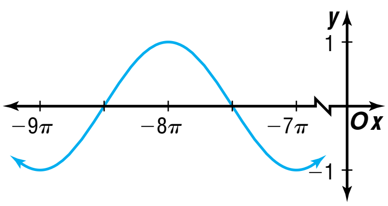


By studying the graph and its repeating pattern, you can determine the following properties of the graph of the cosine function.

Properties of the Graph of $y = \cos x$

1. The period is 2π.
2. The domain is the set of real numbers.
3. The range is the set of real numbers between -1 and 1, inclusive.
4. The x-intercepts are located at $\frac{\pi}{2} + \pi n$, where n is an integer.
5. The y-intercept is 1.
6. The maximum values are $y = 1$ and occur when $x = \pi n$, where n is an even integer.
7. The minimum values are $y = -1$ and occur when $x = \pi n$, where n is an odd integer.

Example 6 **Determine whether the graph represents $y = \sin x$, $y = \cos x$, or neither.**



The maximum value of 1 occurs when $x = -8\pi$. *maximum of 1 when $x = \pi n \to \cos x$*

The minimum value of -1 occurs at -9π and -7π. *minimum of -1 when $x = \pi n \to \cos x$*

The x-intercepts are $-\frac{17\pi}{2}$ and $-\frac{15\pi}{2}$.

These are characteristics of the cosine function. The graph is $y = \cos x$.

CHECK FOR UNDERSTANDING

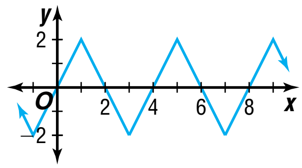
Communicating Mathematics

Read and study the lesson to answer each question.

1. **Counterexample** Sketch the graph of a periodic function that is neither the sine nor cosine function. State the period of the function.
2. **Name** three values of x that would result in the maximum value for $y = \sin x$.
3. **Explain** why the cosine function is a periodic function.
4. *Math Journal* **Draw** the graphs for the sine function and the cosine function. Compare and contrast the two graphs.

Guided Practice

5. **Determine** if the function is periodic. If so, state the period.



Find each value by referring to the graph of the sine or the cosine function.

6. $\cos\left(-\frac{\pi}{2}\right)$

7. $\sin \frac{5\pi}{2}$

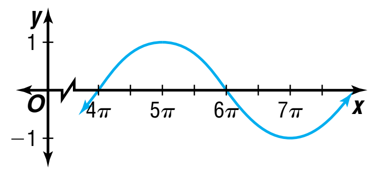
8. Find the values of θ for which $\sin \theta = -1$ is true.

Graph each function for the given interval.

9. $y = \cos x$, $5\pi \leq x \leq 7\pi$

10. $y = \sin x$, $-4\pi \leq x \leq -2\pi$

11. Determine whether the graph represents $y = \sin x$, $y = \cos x$, or neither. Explain.



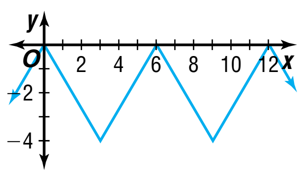
12. **Meteorology** The equation $y = 49 + 28 \sin\left[\frac{\pi}{6}(t - 4)\right]$ models the average monthly temperature for Omaha, Nebraska. In this equation, t denotes the number of months with January represented by 1. Compare the average monthly temperature for April and October.

EXERCISES

Practice

Determine if each function is periodic. If so state the period.

13.



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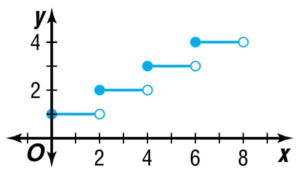

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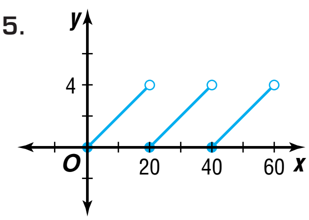


16. $y = |x + 5|$

17. $y = x^2$

18. $y = \frac{1}{x}$

Find each value by referring to the graph of the sine or the cosine function.

19. $\cos 8\pi$

20. $\sin 11\pi$

21. $\cos \frac{\pi}{2}$

22. $\sin\left(-\frac{3\pi}{2}\right)$

23. $\sin \frac{7\pi}{2}$

24. $\cos(-3\pi)$

25. What is the value of $\sin \pi + \cos \pi$?

26. Find the value of $\sin 2\pi - \cos 2\pi$.

Find the values of θ for which each equation is true.

27. $\cos \theta = -1$

28. $\sin \theta = 1$

29. $\cos \theta = 0$

30. Under what conditions does $\cos \theta = 1$?

Graph each function for the given interval.

31. $y = \sin x,\ -5\pi \leq x \leq -3\pi$

32. $y = \cos x,\ 8\pi \leq x \leq 10\pi$

33. $y = \cos x,\ -5\pi \leq x \leq -3\pi$

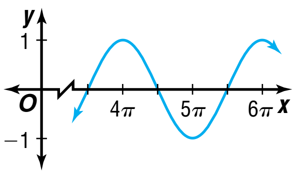
34. $y = \sin x,\ \frac{9\pi}{2} \leq x \leq \frac{13\pi}{2}$

35. $y = \cos x,\ -\frac{7\pi}{2} \leq x \leq -\frac{3\pi}{2}$

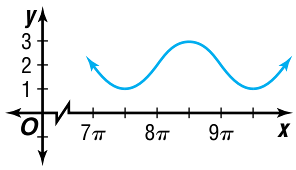
36. $y = \sin x,\ \frac{7\pi}{2} \leq x \leq \frac{11\pi}{2}$

Determine whether each graph is $y = \sin x$, $y = \cos x$, or neither. Explain.

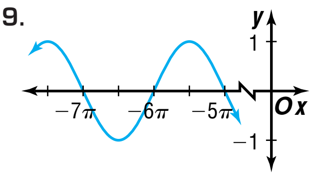
37.



38.



39.



40. Describe a transformation that would change the graph of the sine function to the graph of the cosine function.

41. Name any lines of symmetry for the graph of $y = \sin x$.

42. Name any lines of symmetry for the graph of $y = \cos x$.

43. Use the graph of the sine function to find the values of θ for which each statement is true.

a. $\csc \theta = 1$ b. $\csc \theta = -1$ c. $\csc \theta$ is undefined.

44. Use the graph of the cosine function to find the values of θ for which each statement is true.

a. $\sec \theta = 1$ b. $\sec \theta = -1$ c. $\sec \theta$ is undefined.

Graphing Calculator

Use a graphing calculator to graph the sine and cosine functions on the same set of axes for $0 \leq x \leq 2\pi$. Use the graphs to find the values of x, if any, for which each of the following is true.

45. $\sin x = -\cos x$

46. $\sin x \leq \cos x$

47. $\sin x \cos x > 1$

48. $\sin x \cos x \leq 0$

49. $\sin x + \cos x = 1$

50. $\sin x - \cos x = 0$

Applications and Problem Solving

Real World Application

51. **Meteorology** The equation $y = 43 + 31 \sin \left[\frac{\pi}{6}(t - 4)\right]$ models the average monthly temperatures for Minneapolis, Minnesota. In this equation, t denotes the number of months with January represented by 1.

a. What is the difference between the average monthly temperatures for July and January? What is the relationship between this difference and the coefficient of the sine term?

b. What is the sum of the average monthly temperatures for July and January? What is the relationship between this sum and value of constant term?

52. **Critical Thinking** Consider the graph of $y = 2 \sin x$.

a. What are the x-intercepts of the graph?

b. What is the maximum value of y?

c. What is the minimum value of y?

d. What is the period of the function?

e. Graph the function.

f. How does the 2 in the equation affect the graph?

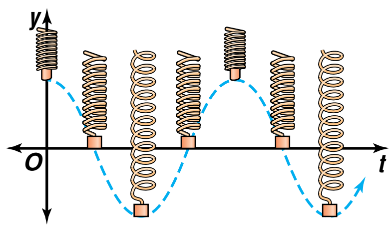
53. **Medicine** The equation $P = 100 + 20 \sin 2\pi t$ models a person's blood pressure P in millimeters of mercury. In this equation, t is time in seconds. The blood pressure oscillates 20 millimeters above and below 100 millimeters, which means that the person's blood pressure is 120 over 80. This function has a period of 1 second, which means that the person's heart beats 60 times a minute.

a. Find the blood pressure at $t = 0$, $t = 0.25$, $t = 0.5$, $t = 0.75$, and $t = 1$.

b. During the first second, when was the blood pressure at a maximum?

c. During the first second, when was the blood pressure at a minimum?

54. **Physics** The motion of a weight on a spring can be described by a modified cosine function. The weight suspended from a spring is at its equilibrium point when it is at rest. When pushed a certain distance above the equilibrium point, the weight oscillates above and below the equilibrium point. The time that it takes for the weight to oscillate from the highest point to the lowest point and back to the highest point is its period. The equation $v = 3.5 \cos \left(t\sqrt{\frac{k}{m}}\right)$ models the vertical displacement v of the weight in relationship to the equilibrium point at any time t if it is initially pushed up 3.5 centimeters. In this equation, k is the elasticity of the spring and m is the mass of the weight.



a. Suppose $k = 19.6$ and $m = 1.99$. Find the vertical displacement after 0.9 second and after 1.7 seconds.

b. When will the weight be at the equilibrium point for the first time?

c. How long will it take the weight to complete one period?

55. **Critical Thinking** Consider the graph of $y = \cos 2x$.

a. What are the x-intercepts of the graph?

b. What is the maximum value of y?

c. What is the minimum value of y?

d. What is the period of the function?

e. Sketch the graph.

56. **Ecology** In predator-prey relationships, the number of animals in each category tends to vary periodically. A certain region has pumas as predators and deer as prey. The equation $P = 500 + 200 \sin [0.4(t - 2)]$ models the number of pumas after t years. The equation $D = 1500 + 400 \sin (0.4t)$ models the number of deer after t years. How many pumas and deer will there be in the region for each value of t?

a. $t = 0$ b. $t = 10$ c. $t = 25$

Mixed Review

57. **Technology** A computer CD-ROM is rotating at 500 revolutions per minute. Write the angular velocity in radians per second. *(Lesson 6-2)*

58. Change -1.5 radians to degree measure. *(Lesson 6-1)*

59. Find the values of x in the interval $0° \leq x \leq 360°$ for which $\sin x = \frac{\sqrt{2}}{2}$. *(Lesson 5-5)*

60. Solve $\frac{2}{x + 2} = \frac{x}{2 - x} + \frac{x^2 + 4}{x^2 - 4}$. *(Lesson 4-6)*

61. Find the number of possible positive real zeros and the number of negative real zeros of $f(x) = 2x^3 + 3x^2 - 11x - 6$. Then determine the rational roots. *(Lesson 4-4)*

62. Use the Remainder Theorem to find the remainder when $x^3 + 2x^2 - 9x + 18$ is divided by $x - 1$. State whether the binomial is a factor of the polynomial. *(Lesson 4-3)*

63. Determine the equations of the vertical and horizontal asymptotes, if any, of $g(x) = \frac{x^2}{x^2 + x}$. *(Lesson 3-7)*

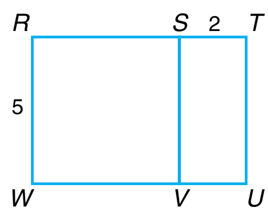
64. Use the graph of the parent function $f(x) = x^3$ to describe the graph of the related function $g(x) = -3x^3$. *(Lesson 3-2)*

65. Find the value of $\begin{vmatrix} -2 & 4 & -1 \\ 1 & -1 & 0 \\ -3 & 4 & 5 \end{vmatrix}$. *(Lesson 2-5)*

66. Use a reflection matrix to find the coordinates of the vertices of $\triangle ABC$ reflected over the y-axis for vertices A (3, 2), B (2, -4), and C (1, 6). *(Lesson 2-4)*

67. Graph $x = \frac{3}{2}y$. *(Lesson 1-3)*

68. **SAT/ACT Practice** How much less is the perimeter of square $RSVW$ than the perimeter of rectangle $RTUW$?



A 2 units B 4 units

C 9 units D 12 units

E 20 units

Extra Practice See p. A36.

HISTORY of MATHEMATICS

FUNCTIONS

Mathematicians and statisticians use functions to express relationships among sets of numbers. When you use a spreadsheet or a graphing calculator, writing an expression as a function is crucial for calculating values in the spreadsheet or for graphing the function.

Early Evidence In about 2000 B.C., the Babylonians used the idea of function in making tables of values for n and $n^3 + n^2$, for $n = 1, 2, \ldots, 30$. Their work indicated that they believed they could show a correspondence between these two sets of values. The following is an example of a Babylonian table.

n	$n^3 + n^2$
1	2
2	12
⋮	⋮
30	?

The Renaissance In about 1637, **René Descartes** may have been the first person to use the term "function." He defined a function as a power of x, such as x^2 or x^3, where the power was a positive integer. About 55 years later, **Gottfried von Leibniz** defined a function as anything that related to a curve, such as a point on a curve or the slope of a curve. In 1718, **Johann Bernoulli** thought of a function as a relationship between a variable and some constants. Later in that same century, **Leonhard Euler's** notion of a function was an equation or formula with variables and constants. Euler also expanded the notion of function to include not only the written expression, but the graphical representation of the relationship as well. He is credited with the modern standard notation for function, $f(x)$.

Johann Bernoulli

Modern Era The 1800s brought **Joseph Lagrange's** idea of function. He limited the meaning of a function to a power series. An example of a power series is $x + x^2 + x^3 + \cdots$, where the three dots indicate that the pattern continues forever. In 1822, **Jean Fourier** determined that any function can be represented by a trigonometric series. **Peter Gustav Dirichlet** used the terminology *y is a function of x* to mean that each first element in the set of ordered pairs is different. Variations of his definition can be found in mathematics textbooks today, including this one.

Georg Cantor and others working in the late 1800s and early 1900s are credited with extending the concept of function from ordered pairs of numbers to ordered pairs of elements.

Today engineers like Julia Chang use functions to calculate the efficiency of equipment used in manufacturing. She also uses functions to determine the amount of hazardous chemicals generated during the manufacturing process. She uses spreadsheets to find many values of these functions.

ACTIVITIES

1. Make a table of values for the Babylonian function, $f(n) = n^3 + n^2$. Use values of n from 1 to 30, inclusive. Then, graph this function using paper and pencil, graphing software, or a graphing calculator. Describe the graph.
2. Research other functions used by notable mathematicians mentioned in this article. You may choose to explore trigonometric series.
3. internet CONNECTION Find out more about personalities referenced in this article and others who contributed to the history of functions. Visit **www.amc.glencoe.com**