6-4

Amplitude and Period of Sine and Cosine Functions

OBJECTIVES

- Find the amplitude and period for sine and cosine functions.
- Write equations of sine and cosine functions given the amplitude and period.

Real World Application

BOATING A signal buoy between the coast of Hilton Head Island, South Carolina, and Savannah, Georgia, bobs up and down in a minor squall. From the highest point to the lowest point, the buoy moves a distance of $3\frac{1}{2}$ feet. It moves from its highest point down to its lowest point and back to its highest point every 14 seconds. Find an equation of the motion for the buoy assuming that it is at its equilibrium point at $t = 0$ and the buoy is on its way down at that time. What is the height of the buoy at 8 seconds and at 17 seconds? *This problem will be solved in Example 5.*

Recall from Chapter 3 that changes to the equation of the parent graph can affect the appearance of the graph by dilating, reflecting, and/or translating the original graph. In this lesson, we will observe the vertical and horizontal expanding and compressing of the parent graphs of the sine and cosine functions.

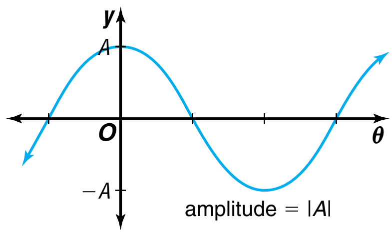
Let's consider an equation of the form $y = A \sin \theta$. We know that the maximum absolute value of $\sin \theta$ is 1. Therefore, for every value of the product of $\sin \theta$ and A, the maximum value of $A \sin \theta$ is $|A|$. Similarly, the maximum value of $A \cos \theta$ is $|A|$. The absolute value of A is called the **amplitude** of the functions $y = A \sin \theta$ and $y = A \cos \theta$.

Amplitude of Sine and Cosine Functions

The amplitude of the functions $y = A \sin \theta$ and $y = A \cos \theta$ is the absolute value of A, or $|A|$.

The amplitude can also be described as the absolute value of one-half the difference of the maximum and minimum function values.

$$|A| = \left|\frac{A - (-A)}{2}\right|$$



Example 1

a. State the amplitude for the function $y = 4 \cos \theta$.

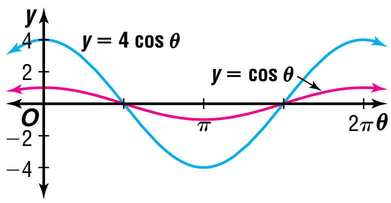
b. Graph $y = 4 \cos \theta$ and $y = \cos \theta$ on the same set of axes.

c. Compare the graphs.

a. According to the definition of amplitude, the amplitude of $y = A \cos \theta$ is $|A|$. So the amplitude of $y = 4 \cos \theta$ is $|4|$ or 4.

b. Make a table of values. Then graph the points and draw a smooth curve.

θ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
$\cos\theta$	1	$\frac{\sqrt{2}}{2}$	0	$-\frac{\sqrt{2}}{2}$	-1	$-\frac{\sqrt{2}}{2}$	0	$\frac{\sqrt{2}}{2}$	1
$4\cos\theta$	4	$2\sqrt{2}$	0	$-2\sqrt{2}$	-4	$-2\sqrt{2}$	0	$2\sqrt{2}$	4



c. The graphs cross the θ-axis at $\theta = \frac{\pi}{2}$ and $\theta = \frac{3\pi}{2}$. Also, both functions reach their maximum value at $\theta = 0$ and $\theta = 2\pi$ and their minimum value at $\theta = \pi$. But the maximum and minimum values of the function $y = \cos\theta$ are 1 and -1, and the maximum and minimum values of the function $y = 4\cos\theta$ are 4 and -4. The graph of $y = 4\cos\theta$ is vertically expanded.

GRAPHING CALCULATOR EXPLORATION

- Select the radian mode.
- Use the domain and range values below to set the viewing window.

$-4.7 \le x \le 4.8$, **Xscl: 1** $\quad -3 \le y \le 3$, **Yscl: 1**

TRY THESE

1. Graph each function on the same screen.
   a. $y = \sin x$ **b.** $y = \sin 2x$ **c.** $y = \sin 3x$

WHAT DO YOU THINK?

2. Describe the behavior of the graph of $f(x) = \sin kx$, where $k > 0$, as k increases.
3. Make a conjecture about the behavior of the graph of $f(x) = \sin kx$, if $k < 0$. Test your conjecture.

Consider an equation of the form $y = \sin k\theta$, where k is any positive integer. Since the period of the sine function is 2π, the following identity can be developed.

$$y = \sin k\theta$$
$$y = \sin(k\theta + 2\pi) \quad \textit{Definition of periodic function}$$
$$y = \sin k\left(\theta + \frac{2\pi}{k}\right) \quad k\theta + 2\pi = k\left(\theta + \frac{2\pi}{k}\right)$$

Therefore, the period of $y = \sin k\theta$ is $\frac{2\pi}{k}$. Similarly, the period of $y = \cos k\theta$ is $\frac{2\pi}{k}$.

Period of Sine and Cosine Functions

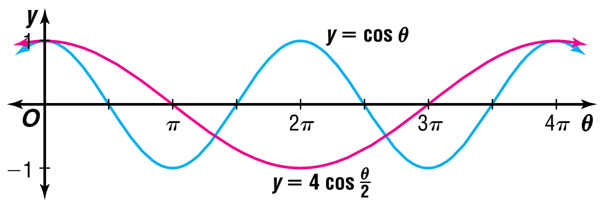
The period of the functions $y = \sin k\theta$ and $y = \cos k\theta$ is $\frac{2\pi}{k}$, where $k > 0$.

Example 2 **a. State the period for the function $y = \cos \frac{\theta}{2}$.**

b. Graph $y = \cos \frac{\theta}{2}$ and $y = \cos \theta$.

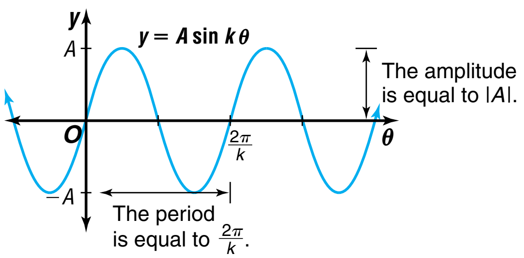
a. The definition of the period of $y = \cos k\theta$ is $\frac{2\pi}{k}$. Since $\cos \frac{\theta}{2}$ equals $\cos\left(\frac{1}{2}\theta\right)$, the period is $\frac{2\pi}{\frac{1}{2}}$ or 4π.

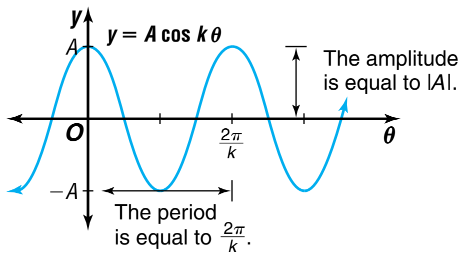
b.



Notice that the graph of $y = \cos \frac{\theta}{2}$ is horizontally expanded.

The graphs of $y = A \sin k\theta$ and $y = A \cos k\theta$ are shown below.



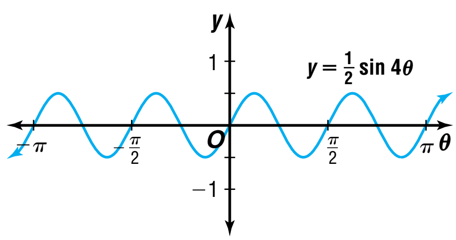


You can use the parent graph of the sine and cosine functions and the amplitude and period to sketch graphs of $y = A \sin k\theta$ and $y = A \cos k\theta$.

Example 3 **State the amplitude and period for the function $y = \frac{1}{2} \sin 4\theta$. Then graph the function.**

Since $A = \frac{1}{2}$, the amplitude is $\left|\frac{1}{2}\right|$ or $\frac{1}{2}$. Since $k = 4$, the period is $\frac{2\pi}{4}$ or $\frac{\pi}{2}$.

Use the basic shape of the sine function and the amplitude and period to graph the equation.



We can write equations for the sine and cosine functions if we are given the amplitude and period.

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Example 4 **Write an equation of the cosine function with amplitude 9.8 and period 6π.**

The form of the equation will be $y = A \cos k\theta$. First find the possible values of A for an amplitude of 9.8.

$$|A| = 9.8$$
$$A = 9.8 \text{ or } -9.8$$

Since there are two values of A, two possible equations exist.

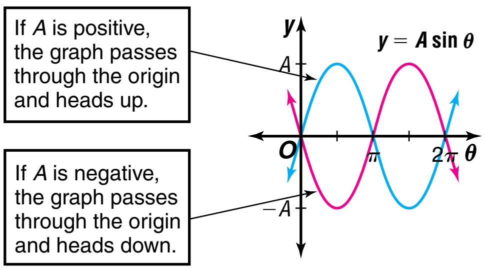
Now find the value of k when the period is 6π.

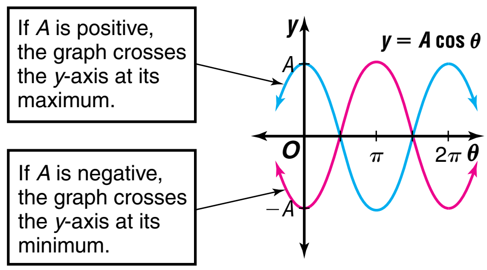
$\frac{2\pi}{k} = 6\pi$ *The period of a cosine function is $\frac{2\pi}{k}$.*

$k = \frac{2\pi}{6\pi}$ or $\frac{1}{3}$

The possible equations are $y = 9.8 \cos\left(\frac{1}{3}\theta\right)$ or $y = -9.8 \cos\left(\frac{1}{3}\theta\right)$.

Many real-world situations have periodic characteristics that can be described with the sine and cosine functions. When you are writing an equation to describe a situation, remember the characteristics of the sine and cosine graphs. If you know the function value when $x = 0$ and whether the function is increasing or decreasing, you can choose the appropriate function to write an equation for the situation.





Example 5 **BOATING** **Refer to the application at the beginning of the lesson.**

Real World Application

a. Find an equation for the motion of the buoy.

b. Determine the height of the buoy at 8 seconds and at 17 seconds.

a. At $t = 0$, the buoy is at equilibrium and is on its way down. This indicates a reflection of the sine function and a negative value of A. The general form of the equation will be $y = A \sin kt$, where A is negative and t is the time in seconds.

$A = -\left(\frac{1}{2} \times 3\frac{1}{2}\right)$ $\qquad$ $\frac{2\pi}{k} = 14$

$A = -\frac{7}{4}$ or -1.75 $\qquad$ $k = \frac{2\pi}{14}$ or $\frac{\pi}{7}$

An equation for the motion of the buoy is $y = -1.75 \sin \frac{\pi}{7}t$.

Graphing Calculator Tip

To find the value of y, use a calculator in radian mode.

b. Use this equation to find the location of the buoy at the given times.

At 8 seconds

$y = -1.75 \sin\left(\frac{\pi}{7} \times 8\right)$

$y \approx 0.7592965435$

At 8 seconds, the buoy is about 0.8 feet above the equilibrium point.

At 17 seconds

$y = -1.75 \sin\left(\frac{\pi}{7} \times 17\right)$

$y \approx -1.706123846$

At 17 seconds, the buoy is about 1.7 feet below the equilibrium point.

The period represents the amount of time that it takes to complete one cycle. The number of cycles per unit of time is known as the **frequency.** The period (time per cycle) and frequency (cycles per unit of time) are reciprocals of each other.

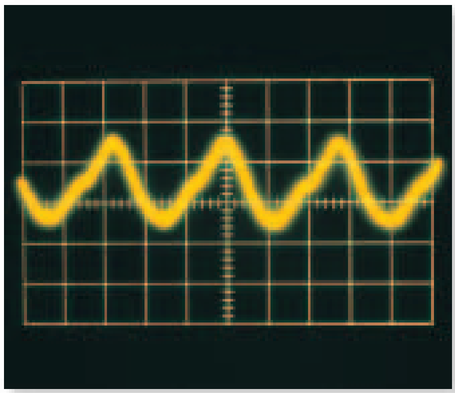
$$\text{period} = \frac{1}{\text{frequency}} \qquad \text{frequency} = \frac{1}{\text{period}}$$

The *hertz* is a unit of frequency. One hertz equals one cycle per second.

Example 6 Real World Application

MUSIC **Write an equation of the sine function that represents the initial behavior of the vibrations of the note G above middle C having amplitude 0.015 and a frequency of 392 hertz.**

The general form of the equation will be $y = A \sin kt$, where t is the time in seconds. Since the amplitude is 0.015, $A = \pm 0.015$.



The period is the reciprocal of the frequency or $\frac{1}{392}$. Use this value to find k.

$\frac{2\pi}{k} = \frac{1}{392}$ *The period* $\frac{2\pi}{k}$ *equals* $\frac{1}{392}$.

$k = 2\pi(392)$ or 784π

One sine function that represents the vibration is $y = 0.015 \sin(784\pi \times t)$.

CHECK FOR UNDERSTANDING

Communicating Mathematics

Read and study the lesson to answer each question.

1. **Write** a sine function that has a greater maximum value than the function $y = 4 \sin 2\theta$.
2. **Describe** the relationship between the graphs of $y = 3 \sin \theta$ and $y = -3 \sin \theta$.

3. **Determine** which function has the greatest period.

   A. $y = 5 \cos 2\theta$ B. $y = 3 \cos 5\theta$ C. $y = 7 \cos \frac{\theta}{2}$ D. $y = \cos \theta$

4. **Explain** the relationship between period and frequency.

5. *Math Journal* **Draw** the graphs for $y = \cos \theta$, $y = 3 \cos \theta$, and $y = \cos 3\theta$. Compare and contrast the three graphs.

Guided Practice

6. State the amplitude for $y = -2.5 \cos \theta$. Then graph the function.

7. State the period for $y = \sin 4\theta$. Then graph the function.

State the amplitude and period for each function. Then graph each function.

8. $y = 10 \sin 2\theta$
9. $y = 3 \cos 2\theta$
10. $y = 0.5 \sin \frac{\theta}{6}$
11. $y = -\frac{1}{5} \cos \frac{\theta}{4}$

Write an equation of the sine function with each amplitude and period.

12. amplitude $= 0.8$, period $= \pi$
13. amplitude $= 7$, period $= \frac{\pi}{3}$

Write an equation of the cosine function with each amplitude and period.

14. amplitude $= 1.5$, period $= 5\pi$
15. amplitude $= \frac{3}{4}$, period $= 6$

16. **Music** Write a sine equation that represents the initial behavior of the vibrations of the note D above middle C having an amplitude of 0.25 and a frequency of 294 hertz.

EXERCISES

Practice

State the amplitude for each function. Then graph each function.

17. $y = 2 \sin \theta$
18. $y = -\frac{3}{4} \cos \theta$
19. $y = 1.5 \sin \theta$

State the period for each function. Then graph each function.

20. $y = \cos 2\theta$
21. $y = \cos \frac{\theta}{4}$
22. $y = \sin 6\theta$

State the amplitude and period for each function. Then graph each function.

23. $y = 5 \cos \theta$
24. $y = -2 \cos 0.5\theta$
25. $y = -\frac{2}{5} \sin 9\theta$
26. $y = 8 \sin 0.5\theta$
27. $y = -3 \sin \frac{\pi}{2}\theta$
28. $y = \frac{2}{3} \cos \frac{3\pi}{7}\theta$
29. $y = 3 \sin 2\theta$
30. $y = 3 \cos 0.5\theta$
31. $y = -\frac{1}{3} \cos 3\theta$
32. $y = \frac{1}{3} \sin \frac{\theta}{3}$
33. $y = -4 \sin \frac{\theta}{2}$
34. $y = -2.5 \cos \frac{\theta}{5}$

35. The equation of the vibrations of the note F above middle C is represented by $y = 0.5 \sin 698\pi t$. Determine the amplitude and period for the function.

Write an equation of the sine function with each amplitude and period.

36. amplitude $= 0.4$, period $= 10\pi$

37. amplitude $= 35.7$, period $= \frac{\pi}{4}$

38. amplitude $= \frac{1}{4}$, period $= \frac{\pi}{3}$

39. amplitude $= 0.34$, period $= 0.75\pi$

40. amplitude $= 4.5$, period $= \frac{5\pi}{4}$

41. amplitude $= 16$, period $= 30$

Write an equation of the cosine function with each amplitude and period.

42. amplitude $= 5$, period $= 2\pi$

43. amplitude $= \frac{5}{8}$, period $= \frac{\pi}{7}$

44. amplitude $= 7.5$, period $= 6\pi$

45. amplitude $= 0.5$, period $= 0.3\pi$

46. amplitude $= \frac{2}{5}$, period $= \frac{3}{5}\pi$

47. amplitude $= 17.9$, period $= 16$

48. Write the possible equations of the sine and cosine functions with amplitude 1.5 and period $\frac{\pi}{2}$.

Write an equation for each graph.

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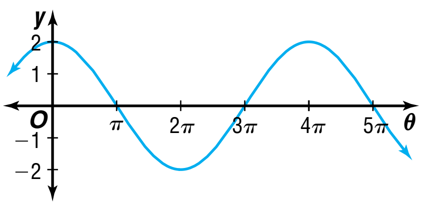

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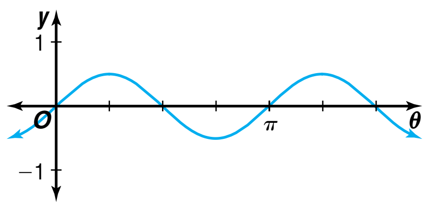

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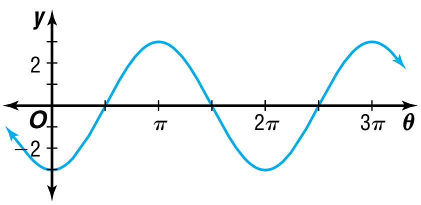

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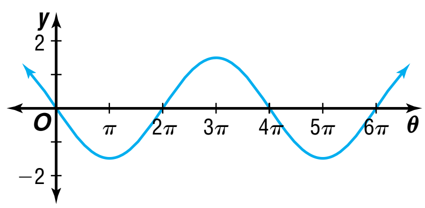


53. Write an equation for a sine function with amplitude 3.8 and frequency 120 hertz.

54. Write an equation for a cosine function with amplitude 15 and frequency 36 hertz.

Graphing Calculator

55. Graph these functions on the same screen of a graphing calculator. Compare the graphs.

a. $y = \sin x$ **b.** $y = \sin x + 1$ **c.** $y = \sin x + 2$

CONTENTS

Applications and Problem Solving

56. **Boating** A buoy in the harbor of San Juan, Puerto Rico, bobs up and down. The distance between the highest and lowest point is 3 feet. It moves from its highest point down to its lowest point and back to its highest point every 8 seconds.

 a. Find the equation of the motion for the buoy assuming that it is at its equilibrium point at $t = 0$ and the buoy is on its way down at that time.

 b. Determine the height of the buoy at 3 seconds.

 c. Determine the height of the buoy at 12 seconds.

57. **Critical Thinking** Consider the graph of $y = 2 + \sin \theta$.

 a. What is the maximum value of y?

 b. What is the minimum value of y?

 c. What is the period of the function?

 d. Sketch the graph.

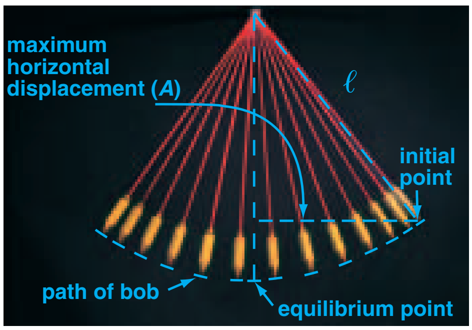
58. **Music** Musical notes are classified by frequency. The note middle C has a frequency of 262 hertz. The note C above middle C has a frequency of 524 hertz. The note C below middle C has a frequency of 131 hertz.

 a. Write an equation of the sine function that represents middle C if its amplitude is 0.2.

 b. Write an equation of the sine function that represents C above middle C if its amplitude is one half that of middle C.

 c. Write an equation of the sine function that represents C below middle C if its amplitude is twice that of middle C.

59. **Physics** For a pendulum, the equation representing the horizontal displacement of the bob is $y = A \cos \left(t\sqrt{\frac{g}{\ell}}\right)$. In this equation, A is the maximum horizontal distance that the bob moves from the equilibrium point, t is the time, g is the acceleration due to gravity, and ℓ is the length of the pendulum. The acceleration due to gravity is 9.8 meters per second squared.



 a. A pendulum has a length of 6 meters and its bob has a maximum horizontal displacement to the right of 1.5 meters. Write an equation that models the horizontal displacement of the bob if it is at its maximum distance to the right when $t = 0$.

 b. Find the location of the bob at 4 seconds.

 c. Find the location of the bob at 7.9 seconds.

60. **Critical Thinking** Consider the graph of $y = \cos(\theta + \pi)$.

 a. Write an expression for the x-intercepts of the graph.

 b. What is the y-intercept of the graph?

 c. What is the period of the function?

 d. Sketch the graph.

61. **Physics** Three different weights are suspended from three different springs. Each spring has an elasticity coefficient of 18.5. The equation for the vertical displacement is $y = 1.5 \cos\left(t\sqrt{\frac{k}{m}}\right)$, where t is time, k is the elasticity coefficient, and m is the mass of the weight.

a. The first weight has a mass of 0.4 kilogram. Find the period and frequency of this spring.

b. The second weight has a mass of 0.6 kilogram. Find the period and frequency of this spring.

c. The third weight has a mass of 0.8 kilogram. Find the period and frequency of this spring.

d. As the mass increases, what happens to the period?

e. As the mass increases, what happens to the frequency?

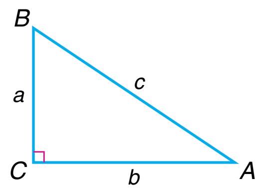
Mixed Review

62. Find $\cos\left(-\frac{5\pi}{2}\right)$ by referring to the graph of the cosine function. *(Lesson 6-3)*

63. Determine the angular velocity if 84 revolutions are completed in 6 seconds. *(Lesson 6-2)*

64. Given a central angle of $73°$, find the length of its intercepted arc in a circle of radius 9 inches. *(Lesson 6-1)*

65. Solve the triangle if $a = 15.1$ and $b = 19.5$. Round to the nearest tenth. *(Lesson 5-5)*



66. **Physics** The period of a pendulum can be determined by the formula $T = 2\pi\sqrt{\frac{\ell}{g}}$, where T represents the period, ℓ represents the length of the pendulum, and g represents the acceleration due to gravity. Determine the length of the pendulum if the pendulum has a period on Earth of 4.1 seconds and the acceleration due to gravity at Earth's surface is 9.8 meters per second squared. *(Lesson 4-7)*

67. Find the discriminant of $3m^2 + 5m + 10 = 0$. Describe the nature of the roots. *(Lesson 4-2)*

68. **Manufacturing** Icon, Inc. manufactures two types of computer graphics cards, Model 28 and Model 74. There are three stations, A, B, and C, on the assembly line. The assembly of a Model 28 graphics card requires 30 minutes at station A, 20 minutes at station B, and 12 minutes at station C. Model 74 requires 15 minutes at station A, 30 minutes at station B, and 10 minutes at station C. Station A can be operated for no more than 4 hours a day, station B can be operated for no more than 6 hours a day, and station C can be operated for no more than 8 hours. *(Lesson 2-7)*

a. If the profit on Model 28 is \$100 and on Model 74 is \$60, how many of each model should be assembled each day to provide maximum profit?

b. What is the maximum daily profit?

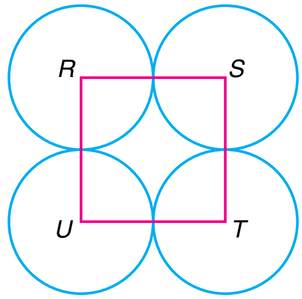
69. Use a reflection matrix to find the coordinates of the vertices of a quadrilateral reflected over the x-axis if the coordinates of the vertices of the quadrilateral are located at $(-2, -1)$, $(1, -1)$, $(3, -4)$, and $(-3, -2)$. *(Lesson 2-4)*

70. Graph $g(x) = \begin{cases} -3x \text{ if } x < -2 \\ 2 \text{ if } -2 \le x < 3. \\ x + 1 \text{ if } x \ge 3 \end{cases}$ *(Lesson 1-7)*

71. **Fund-Raising** The regression equation of a set of data is $y = 14.7x + 140.1$, where y represents the money collected for a fund-raiser and x represents the number of members of the organization. Use the equation to predict the amount of money collected by 20 members. *(Lesson 1-6)*

72. Given that x is an integer, state the relation representing $y = x^2$ and $-4 \le x \le -2$ by listing a set of ordered pairs. Then state whether this relation is a function. *(Lesson 1-1)*

73. **SAT/ACT Practice** Points $RSTU$ are the centers of four congruent circles. If the area of square $RSTU$ is 100, what is the sum of the areas of the four circles?



A 25π
B 50π
C 100π
D 200π
E 400π

MID-CHAPTER QUIZ

1. Change $\frac{5\pi}{6}$ radians to degree measure. (Lesson 6-1)

2. **Mechanics** A pulley with diameter 0.5 meter is being used to lift a box. How far will the box weight rise if the pulley is rotated through an angle of $\frac{5\pi}{3}$ radians? (Lesson 6-1)

3. Find the area of a sector if the central angle measures $\frac{2\pi}{5}$ radians and the radius of the circle is 8 feet. (Lesson 6-1)

4. Determine the angular displacement in radians of 7.8 revolutions. (Lesson 6-2)

5. Determine the angular velocity if 8.6 revolutions are completed in 7 seconds. (Lesson 6-2)

6. Determine the linear velocity of a point rotating at an angular velocity of 8π radians per second at a distance of 3 meters from the center of the rotating object. (Lesson 6-2)

7. Find $\sin\left(-\frac{7\pi}{2}\right)$ by referring to the graph of the sine function. (Lesson 6-3)

8. Graph $y = \cos x$ for $7\pi \le x \le 9\pi$. (Lesson 6-3)

9. State the amplitude and period for the function $y = -7 \cos \frac{\theta}{3}$. Then graph the function. (Lesson 6-4)

10. Find the possible equations of the sine function with amplitude 5 and period $\frac{\pi}{3}$. (Lesson 6-4)

Extra Practice See p. A36.