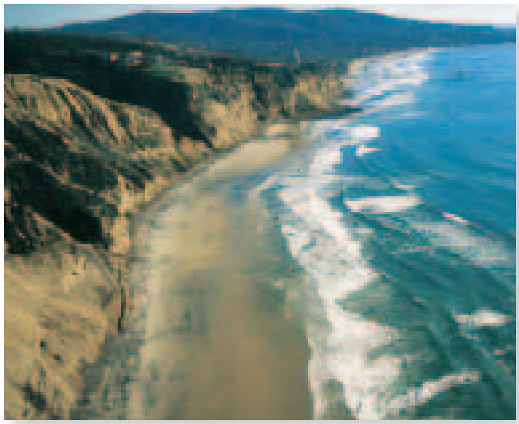
6-5 Translations of Sine and Cosine Functions

OBJECTIVES

- Find the phase shift and the vertical translation for sine and cosine functions.
- Write the equations of sine and cosine functions given the amplitude, period, phase shift, and vertical translation.
- Graph compound functions.



TIDES One day in March in San Diego, California, the first low tide occurred at 1:45 A.M., and the first high tide occurred at 7:44 A.M. Approximately 12 hours and 24 minutes or 12.4 hours after the first low tide occurred, the second low tide occurred. The equation that models these tides is

$$h = 2.9 + 2.2 \sin\left(\frac{\pi}{6.2}t - \frac{4.85\pi}{6.2}\right),$$

where t represents the number of hours since midnight and h represents the height of the water. Draw a graph that models the cyclic nature of the tide. *This problem will be solved in Example 4.*

In Chapter 3, you learned that the graph of $y = (x - 2)^2$ is a horizontal translation of the parent graph of $y = x^2$. Similarly, graphs of the sine and cosine functions can be translated horizontally.

GRAPHING CALCULATOR EXPLORATION

- Select the radian mode.
- Use the domain and range values below to set the viewing window.

  $-4.7 \leq x \leq 4.8$, **Xscl: 1** $\quad -3 \leq y \leq 3$, **Yscl: 1**

TRY THESE

1. Graph each function on the same screen.

a. $y = \sin x$ **b.** $y = \sin\left(x + \frac{\pi}{4}\right)$

c. $y = \sin\left(x + \frac{\pi}{2}\right)$

WHAT DO YOU THINK?

2. Describe the behavior of the graph of $f(x) = \sin(x + c)$, where $c > 0$, as c increases.

3. Make a conjecture about what happens to the graph of $f(x) = \sin(x + c)$ if $c < 0$ and continues to decrease. Test your conjecture.

A horizontal translation or shift of a trigonometric function is called a **phase shift.** Consider the equation of the form $y = A \sin(k\theta + c)$, where $A, k, c \neq 0$. To find a zero of the function, find the value of θ for which $A \sin(k\theta + c) = 0$. Since $\sin 0 = 0$, solving $k\theta + c = 0$ will yield a zero of the function.

$$k\theta + c = 0$$
$$\theta = -\frac{c}{k} \quad \textit{Solve for } \theta.$$

Therefore, $y = 0$ when $\theta = -\frac{c}{k}$. The value of $-\frac{c}{k}$ is the phase shift.

When $c > 0$: The graph of $y = A \sin (k\theta + c)$ is the graph of $y = A \sin k\theta$, shifted $\left|\frac{c}{k}\right|$ to the left.

When $c < 0$: The graph of $y = A \sin (k\theta + c)$ is the graph of $y = A \sin k\theta$, shifted $\left|\frac{c}{k}\right|$ to the right.

Phase Shift of Sine and Cosine Functions

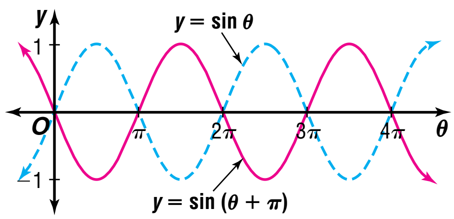
The phase shift of the functions $y = A \sin (k\theta + c)$ and $y = A \cos (k\theta + c)$ is $-\frac{c}{k}$, where $k > 0$.
If $c > 0$, the shift is to the left.
If $c < 0$, the shift is to the right.

Example 1 **State the phase shift for each function. Then graph the function.**

a. $y = \sin (\theta + \pi)$

The phase shift of the function is $-\frac{c}{k}$ or $-\frac{\pi}{1}$, which equals $-\pi$.

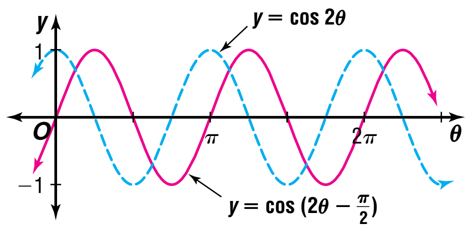
To graph $y = \sin (\theta + \pi)$, consider the graph of $y = \sin \theta$. Graph this function and then shift the graph $-\pi$.



b. $y = \cos \left(2\theta - \frac{\pi}{2}\right)$

The phase shift of the function is $-\frac{c}{k}$ or $-\left(\frac{-\frac{\pi}{2}}{2}\right)$, which equals $\frac{\pi}{4}$.

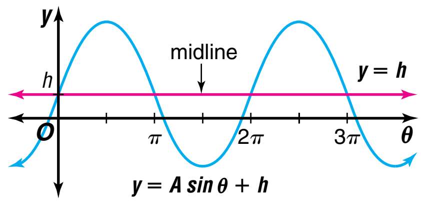
To graph $y = \cos \left(2\theta - \frac{\pi}{2}\right)$, consider the graph of $y = \cos 2\theta$. The graph of $y = \cos 2\theta$ has amplitude of 1 and a period of $\frac{2\pi}{2}$ or π. Graph this function and then shift the graph $\frac{\pi}{4}$.



In Chapter 3, you also learned that the graph of $y = x^2 - 2$ is a vertical translation of the parent graph of $y = x^2$. Similarly, graphs of the sine and cosine functions can be translated vertically.

When a constant is added to a sine or cosine function, the graph is shifted upward or downward. If (x, y) are the coordinates of $y = \sin x$, then $(x, y + d)$ are the coordinates of $y = \sin x + d$.

A new horizontal axis known as the **midline** becomes the reference line or equilibrium point about which the graph oscillates. For the graph of $y = A \sin \theta + h$, the midline is the graph of $y = h$.



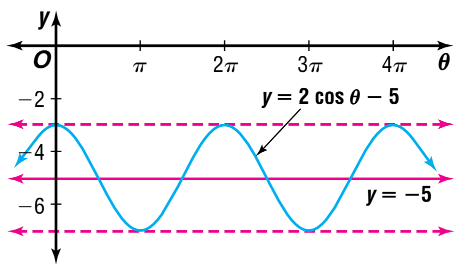
Vertical Shift of Sine and Cosine Functions

The vertical shift of the functions $y = A \sin (k\theta + c) + h$ and $y = A \cos (k\theta + c) + h$ is h. If $h > 0$, the shift is upward. If $h < 0$, the shift is downward. The midline is $y = h$.

Example 2 **State the vertical shift and the equation of the midline for the function $y = 2 \cos \theta - 5$. Then graph the function.**

The vertical shift is 5 units downward. The midline is the graph of $y = -5$.

To graph the function, draw the midline, the graph of $y = -5$. Since the amplitude of the function is $|2|$ or 2, draw dashed lines parallel to the midline which are 2 units above and below the midline. That is, $y = -3$ and $y = -7$. Then draw the cosine curve.



In general, use the following steps to graph any sine or cosine function.

Graphing Sine and Cosine Functions

1. Determine the vertical shift and graph the midline.
2. Determine the amplitude. Use dashed lines to indicate the maximum and minimum values of the function.
3. Determine the period of the function and graph the appropriate sine or cosine curve.
4. Determine the phase shift and translate the graph accordingly.

Example 3 **State the amplitude, period, phase shift, and vertical shift for $y = 4\cos\left(\frac{\theta}{2} + \pi\right) - 6$. Then graph the function.**

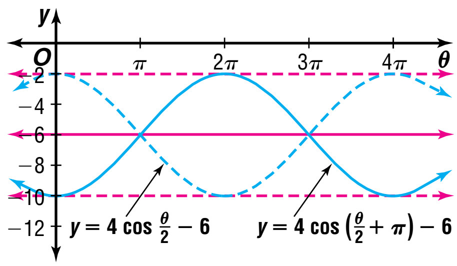
The amplitude is $|4|$ or 4. The period is $\frac{2\pi}{\frac{1}{2}}$ or 4π. The phase shift is $-\frac{\pi}{\frac{1}{2}}$ or -2π. The vertical shift is -6. Using this information, follow the steps for graphing a cosine function.

Step 1 Draw the midline which is the graph of $y = -6$.

Step 2 Draw dashed lines parallel to the midline, which are 4 units above and below the midline.

Step 3 Draw the cosine curve with period of 4π.

Step 4 Shift the graph 2π units to the left.



You can use information about amplitude, period, and translations of sine and cosine functions to model real-world applications.

Example 4 Real World Application

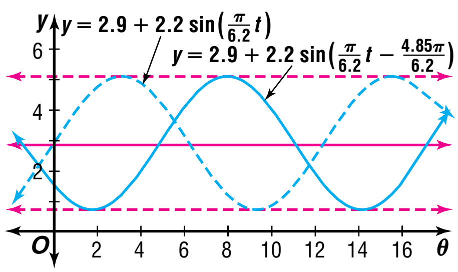
TIDES **Refer to the application at the beginning of the lesson. Draw a graph that models the San Diego tide.**

The vertical shift is 2.9. Draw the midline $y = 2.9$.

The amplitude is $|2.2|$ or 2.2. Draw dashed lines parallel to and 2.2 units above and below the midline.

The period is $\frac{2\pi}{\frac{\pi}{6.2}}$ or 12.4. Draw the sine curve with a period of 12.4.

Shift the graph $-\frac{\frac{-4.85\pi}{6.2}}{\frac{\pi}{6.2}}$ or 4.85 units.



You can write an equation for a trigonometric function if you are given the amplitude, period, phase shift, and vertical shift.

Example 5 **Write an equation of a sine function with amplitude 4, period π, phase shift $-\frac{\pi}{8}$, and vertical shift 6.**

The form of the equation will be $y = A \sin (k\theta + c) + h$. Find the values of A, k, c, and h.

A: $|A| = 4$
$A = 4$ or -4

k: $\frac{2\pi}{k} = \pi$ *The period is π.*
$k = 2$

c: $-\frac{c}{k} = -\frac{\pi}{8}$ *The phase shift is $-\frac{\pi}{8}$.*
$-\frac{c}{2} = -\frac{\pi}{8}$ *$k = 2$*
$c = \frac{\pi}{4}$

h: $h = 6$

Substitute these values into the general equation. The possible equations are $y = 4 \sin \left(2\theta + \frac{\pi}{4}\right) + 6$ and $y = -4 \sin \left(2\theta + \frac{\pi}{4}\right) + 6$.

Compound functions may consist of sums or products of trigonometric functions. Compound functions may also include sums and products of trigonometric functions and other functions.

Here are some examples of compound functions.

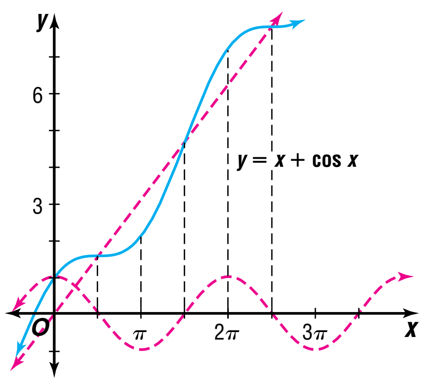
$y = \sin x \cdot \cos x$ *Product of trigonometric functions*
$y = \cos x + x$ *Sum of a trigonometric function and a linear function*

You can graph compound functions involving addition by graphing each function separately on the same coordinate axes and then adding the ordinates. After you find a few of the critical points in this way, you can sketch the rest of the curve of the function of the compound function.

Example 6 **Graph $y = x + \cos x$.**

First graph $y = \cos x$ and $y = x$ on the same axis. Then add the corresponding ordinates of the function. Finally, sketch the graph.

x	$\cos x$	$x + \cos x$
0	1	1
$\frac{\pi}{2}$	0	$\frac{\pi}{2} + 0 \approx 1.57$
π	-1	$\pi - 1 \approx 2.14$
$\frac{3\pi}{2}$	0	$\frac{3\pi}{2} \approx 4.71$
2π	1	$2\pi + 1 \approx 7.28$
$\frac{5\pi}{2}$	0	$\frac{5\pi}{2} \approx 7.85$
3π	-1	$3\pi - 1 \approx 8.42$



CONTENTS

CHECK FOR UNDERSTANDING

Communicating Mathematics

Read and study the lesson to answer each question.

1. **Compare and contrast** the graphs $y = \sin x + 1$ and $y = \sin(x + 1)$.
2. **Name** the function whose graph is the same as the graph of $y = \cos x$ with a phase shift of $\frac{\pi}{2}$.
3. **Analyze** the function $y = A \sin(k\theta + c) + h$. Which variable could you increase or decrease to have each of the following effects on the graph?
   a. stretch the graph vertically
   b. translate the graph downward vertically
   c. shrink the graph horizontally
   d. translate the graph to the left.
4. **Explain** how to graph $y = \sin x + \cos x$.
5. **You Decide** Marsha and Jamal are graphing $y = \cos\left(\frac{\pi}{6}\theta - \frac{\pi}{2}\right)$. Marsha says that the phase shift of the graph is $\frac{\pi}{2}$. Jamal says that the phase shift is 3. Who is correct? Explain.

Guided Practice

6. State the phase shift for $y = 3\cos\left(\theta - \frac{\pi}{2}\right)$. Then graph the function.
7. State the vertical shift and the equation of the midline for $y = \sin 2\theta + 3$. Then graph the function.

State the amplitude, period, phase shift, and vertical shift for each function. Then graph the function.

8. $y = 2\sin(2\theta + \pi) - 5$
9. $y = 3 - \frac{1}{2}\cos\left(\frac{\theta}{2} - \frac{\pi}{4}\right)$
10. Write an equation of a sine function with amplitude 20, period 1, phase shift 0, and vertical shift 100.
11. Write an equation of a cosine function with amplitude 0.6, period 12.4, phase shift -2.13, and vertical shift 7.
12. Graph $y = \sin x - \cos x$.

13. **Health** If a person has a blood pressure of 130 over 70, then the person's blood pressure oscillates between the maximum of 130 and a minimum of 70.
    a. Write the equation for the midline about which this person's blood pressure oscillates.
    b. If the person's pulse rate is 60 beats a minute, write a sine equation that models his or her blood pressure using t as time in seconds.
    c. Graph the equation.

EXERCISES

Practice

State the phase shift for each function. Then graph each function.

14. $y = \sin(\theta - 2\pi)$
15. $y = \sin(2\theta + \pi)$
16. $y = 2\cos\left(\frac{\theta}{4} + \frac{\pi}{2}\right)$

State the vertical shift and the equation of the midline for each function. Then graph each function.

17. $y = \sin \frac{\theta}{2} + \frac{1}{2}$ **18.** $y = 5 \cos \theta - 4$ **19.** $y = 7 + \cos 2\theta$

20. State the horizontal and vertical shift for $y = -8 \sin (2\theta - 4\pi) - 3$.

State the amplitude, period, phase shift, and vertical shift for each function. Then graph the function.

21. $y = 3 \cos \left(\theta - \frac{\pi}{2}\right)$

22. $y = 6 \sin \left(\theta + \frac{\pi}{3}\right) + 2$

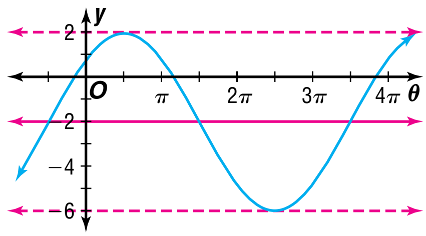
23. $y = -2 + \sin \left(\frac{\theta}{3} - \frac{\pi}{12}\right)$

24. $y = 20 + 5 \cos (3\theta + \pi)$

25. $y = \frac{1}{4} \cos \frac{\theta}{2} - 3$

26. $y = 10 \sin \left(\frac{\theta}{4} - 4\pi\right) -5$

27. State the amplitude, period, phase shift, and vertical shift of the sine curve shown at the right.



Write an equation of the sine function with each amplitude, period, phase shift, and vertical shift.

28. amplitude $= 7$, period $= 3\pi$, phase shift $= \pi$, vertical shift $= -7$

29. amplitude $= 50$, period $= \frac{3\pi}{4}$, phase shift $= \frac{\pi}{2}$, vertical shift $= -25$

30. amplitude $= \frac{3}{4}$, period $= \frac{\pi}{5}$, phase shift $= \pi$, vertical shift $= \frac{1}{4}$

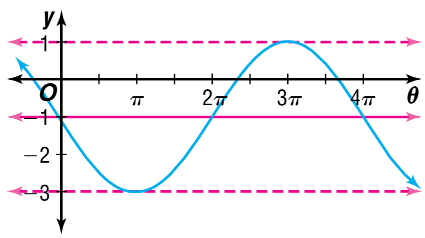
Write an equation of the cosine function with each amplitude, period, phase shift, and vertical shift.

31. amplitude $= 3.5$, period $= \frac{\pi}{2}$, phase shift $= \frac{\pi}{4}$, vertical shift $= 7$

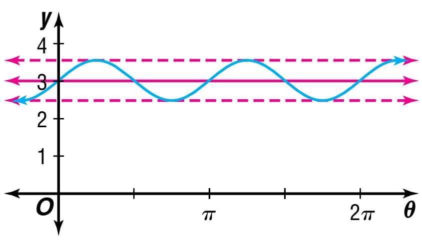
32. amplitude $= \frac{4}{5}$, period $= \frac{\pi}{6}$, phase shift $= \frac{\pi}{3}$, vertical shift $= \frac{7}{5}$

33. amplitude $= 100$, period $= 45$, phase shift $= 0$, vertical shift $= -110$

34. Write a cosine equation for the graph at the right.



35. Write a sine equation for the graph at the right.



Graph each function.

36. $y = \sin x + x$ **37.** $y = \cos x - \sin x$ **38.** $y = \sin x + \sin 2x$

39. On the same coordinate plane, graph each function.

a. $y = 2 \sin x$ **b.** $y = 3 \cos x$ **c.** $y = 2 \sin x + 3 \cos x$

40. Use the graphs of $y = \cos 2x$ and $y = \cos 3x$ to graph $y = \cos 2x - \cos 3x$.

Applications and Problem Solving

Real World Application

41. Biology In the wild, predators such as wolves need prey such as sheep to survive. The population of the wolves and the sheep are cyclic in nature. Suppose the population of the wolves W is modeled by $W = 2000 + 1000 \sin \left(\frac{\pi t}{6}\right)$ and population of the sheep S is modeled by $S = 10{,}000 + 5000 \cos \left(\frac{\pi t}{6}\right)$ where t is the time in months.

a. What are the maximum number and the minimum number of wolves?

b. What are the maximum number and the minimum number of sheep?

c. Use a graphing calculator to graph both equations for values of t from 0 to 24.

d. During which months does the wolf population reach a maximum?

e. During which months does the sheep population reach a maximum?

f. What is the relationship of the maximum population of the wolves and the maximum population of the sheep? Explain.

42. Critical Thinking Use the graphs of $y = x$ and $y = \cos x$ to graph $y = x \cos x$.

43. Entertainment As you ride a Ferris wheel, the height that you are above the ground varies periodically. Consider the height of the center of the wheel to be the equilibrium point. Suppose the diameter of a Ferris Wheel is 42 feet and travels at a rate of 3 revolutions per minute. At the highest point, a seat on the Ferris wheel is 46 feet above the ground.

a. What is the lowest height of a seat?

b. What is the equation of the midline?

c. What is the period of the function?

d. Write a sine equation to model the height of a seat that was at the equilibrium point heading upward when the ride began.

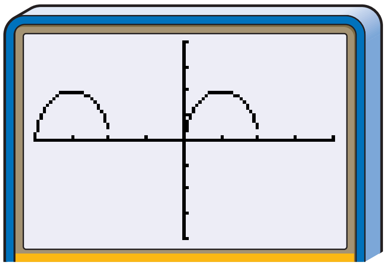
e. According to the model, when will the seat reach the highest point for the first time?

f. According to the model, what is the height of the seat after 10 seconds?

44. Electronics In electrical circuits, the voltage and current can be described by sine or cosine functions. If the graphs of these functions have the same period, but do not pass through their zero points at the same time, they are said to have a *phase difference*. For example, if the voltage is 0 at 90° and the current is 0 at 180°, they are 90° out of phase. Suppose the voltage across an inductor of a circuit is represented by $y = 2 \cos 2x$ and the current across the component is represented by $y = \cos \left(2x - \frac{\pi}{2}\right)$. What is the phase relationship between the signals?

45. Critical Thinking The windows for the following calculator screens are set at **[−2π, 2π] scl: 0.5π** by **[−2, 2] scl: 0.5**. Without using a graphing calculator, use the equations below to identify the graph on each calculator screen.

$y = \cos x^2$ $\qquad y = \sqrt{\sin x}$ $\qquad y = \dfrac{\cos x}{x}$ $\qquad y = \sin \sqrt{x}$

a.


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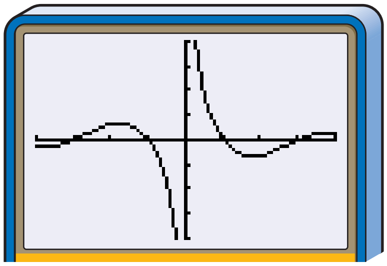

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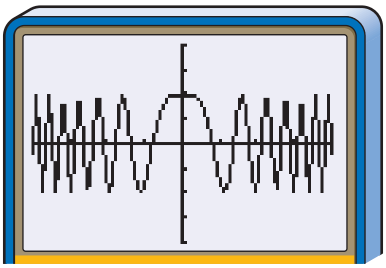
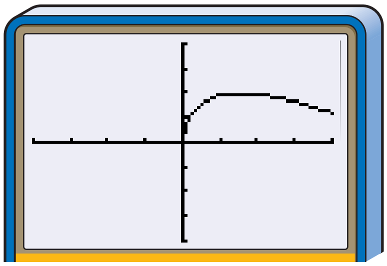


d.


Mixed Review

46. Music Write an equation of the sine function that represents the initial behavior of the vibrations of the note D above middle C having amplitude 0.25 and a frequency of 294 hertz. *(Lesson 6-4)*

47. Determine the linear velocity of a point rotating at an angular velocity of 19.2 radians per second at a distance of 7 centimeters from the center of the rotating object. *(Lesson 6-2)*

48. Graph $y = \dfrac{x-3}{x-2}$. *(Lesson 3-7)*

49. Find the inverse of $f(x) = \dfrac{3}{x-1}$. *(Lesson 3-4)*

50. Find matrix X in the equation $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 3 & 5 \\ -3 & -5 \end{bmatrix} = X$. *(Lesson 2-3)*

51. Solve the system of equations. *(Lesson 2-1)*

$3x + 5y = 4$

$14x - 35y = 21$

52. Graph $y \leq |x + 4|$. *(Lesson 1-8)*

53. Write the standard form of the equation of the line through the point at $(3, -2)$ that is parallel to the graph of $3x - y + 7 = 0$. *(Lesson 1-5)*

54. SAT Practice **Grid-In** A swimming pool is 75 feet long and 42 feet wide. If 7.48 gallons equals 1 cubic foot, how many gallons of water are needed to raise the level of the water 4 inches?

Extra Practice See p. A37.