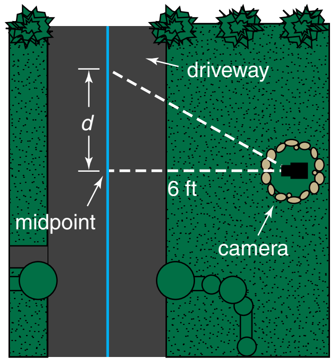
6-7 Graphing Other Trigonometric Functions

OBJECTIVES

- Graph tangent, cotangent, secant, and cosecant functions.
- Write equations of trigonometric functions.

Real World Application

SECURITY A security camera scans a long, straight driveway that serves as an entrance to an historic mansion. Suppose a line is drawn down the center of the driveway. The camera is located 6 feet to the right of the midpoint of the line. Let d represent the distance along the line from its midpoint. If t is time in seconds and the camera points at the midpoint at $t = 0$, then $d = 6 \tan\left(\frac{\pi}{30}t\right)$ models the point being scanned. In this model, the distance below the midpoint is a negative. Graph the equation for $-15 \le t \le 15$. Find the location the camera is scanning at 5 seconds. What happens when $t = 15$?



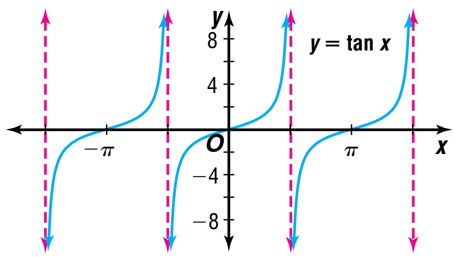
This problem will be solved in Example 4.

You have learned to graph variations of the sine and cosine functions. In this lesson, we will study the graphs of the tangent, cotangent, secant, and cosecant functions. Consider the tangent function. First evaluate $y = \tan x$ for multiples of $\frac{\pi}{4}$ in the interval $-\frac{3\pi}{2} \le x \le \frac{3\pi}{2}$.

x	$-\frac{3\pi}{2}$	$-\frac{5\pi}{4}$	$-\pi$	$-\frac{3\pi}{4}$	$-\frac{\pi}{2}$	$-\frac{\pi}{4}$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$
$\tan x$	undefined	-1	0	1	undefined	-1	0	1	undefined	-1	0	1	undefined

Look Back
You can refer to Lesson 3-7 to review asymptotes.

To graph $y = \tan x$, draw the asymptotes and plot the coordinate pairs from the table. Then draw the curves.



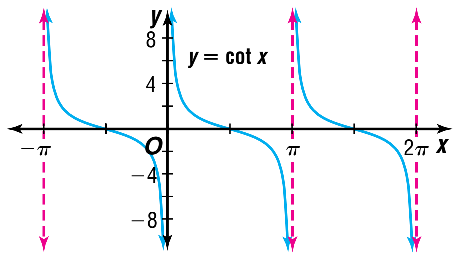
Notice that the range values for the interval $-\frac{3\pi}{2} \le x \le -\frac{\pi}{2}$ repeat for the intervals $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$ and $\frac{\pi}{2} \le x \le \frac{3\pi}{2}$. So, the tangent function is a periodic function. Its period is π.

By studying the graph and its repeating pattern, you can determine the following properties of the graph of the tangent function.

Properties of the Graph $y = \tan x$	1. The period is π. 2. The domain is the set of real numbers except $\frac{\pi}{2}n$, where n is an odd integer. 3. The range is the set of real numbers. 4. The x-intercepts are located at πn, where n is an integer. 5. The y-intercept is 0. 6. The asymptotes are $x = \frac{\pi}{2}n$, where n is an odd integer.

Now consider the graph of $y = \cot x$ in the interval $-\pi \leq x \leq 3\pi$.

x	$-\pi$	$-\frac{3\pi}{4}$	$-\frac{\pi}{2}$	$-\frac{\pi}{4}$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
cot x	undefined	1	0	−1	undefined	1	0	−1	undefined	1	0	−1	undefined



By studying the graph and its repeating pattern, you can determine the following properties of the graph of the cotangent function.

Properties of the Graph of $y = \cot x$	1. The period is π. 2. The domain is the set of real numbers except πn, where n is an integer. 3. The range is the set of real numbers. 4. The x-intercepts are located at $\frac{\pi}{2}n$, where n is an odd integer. 5. There is no y-intercept. 6. The asymptotes are $x = \pi n$, where n is an integer.

Example 1 **Find each value by referring to the graphs of the trigonometric functions.**

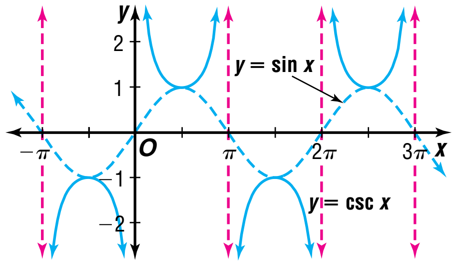
a. $\tan \frac{9\pi}{2}$

Since $\frac{9\pi}{2} = \frac{\pi}{2}(9)$, $\tan \frac{9\pi}{2}$ is undefined.

b. $\cot \frac{7\pi}{2}$

Since $\frac{7\pi}{2} = \frac{\pi}{2}(7)$ and 7 is an odd integer, $\cot \frac{7\pi}{2} = 0$.

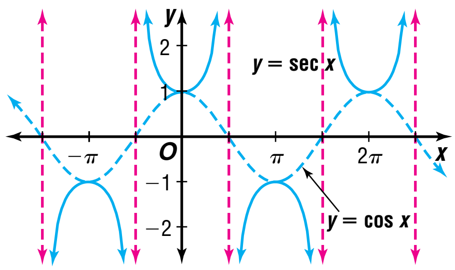
The sine and cosecant functions have a reciprocal relationship. To graph the cosecant, first graph the sine function and the asymptotes of the cosecant function. By studying the graph of the cosecant and its repeating pattern, you can determine the following properties of the graph of the cosecant function.



Properties of the Graph of $y = \csc x$

1. The period is 2π.
2. The domain is the set of real numbers except πn, where n is an integer.
3. The range is the set of real numbers greater than or equal to 1 or less than or equal to -1.
4. There are no x-intercepts.
5. There are no y-intercepts.
6. The asymptotes are $x = \pi n$, where n is an integer.
7. $y = 1$ when $x = \frac{\pi}{2} + 2\pi n$, where n is an integer.
8. $y = -1$ when $x = \frac{3\pi}{2} + 2\pi n$, where n is an integer.

The cosine and secant functions have a reciprocal relationship. To graph the secant, first graph the cosine function and the asymptotes of the secant function. By studying the graph and its repeating pattern, you can determine the following properties of the graph of the secant function.



Properties of the Graph of $y = \sec x$

1. The period is 2π.
2. The domain is the set of real numbers except $\frac{\pi}{2}n$, where n is an odd integer.
3. The range is the set of real numbers greater than or equal to 1 or less than or equal to -1.
4. There are no x-intercepts.
5. The y-intercept is 1.
6. The asymptotes are $x = \frac{\pi}{2}n$, where n is an odd integer.
7. $y = 1$ when $x = \pi n$, where n is an even integer.
8. $y = -1$ when $x = \pi n$, where n is an odd integer.

Example 2 **Find the values of θ for which each equation is true.**

a. $\csc \theta = 1$

From the pattern of the cosecant function, $\csc \theta = 1$ if $\theta = \frac{\pi}{2} + 2\pi n$, where n is an integer.

b. $\sec \theta = -1$

From the pattern of the secant function, $\sec \theta = -1$ if $\theta = \pi n$, where n is an odd integer.

The period of $y = \sin k\theta$ or $y = \cos k\theta$ is $\frac{2\pi}{k}$. Likewise, the period of $y = \csc k\theta$ or $y = \sec k\theta$ is $\frac{2\pi}{k}$. However, since the period of the tangent or cotangent function is π, the period of $y = \tan k\theta$ or $y = \cot k\theta$ is $\frac{\pi}{k}$. In each case, $k > 0$.

Period of Trigonometric Functions

The period of functions $y = \sin k\theta$, $y = \cos k\theta$, $y = \csc k\theta$, and $y = \sec k\theta$ is $\frac{2\pi}{k}$, where $k > 0$.

The period of functions $y = \tan k\theta$ and $y = \cot k\theta$ is $\frac{\pi}{k}$, where $k > 0$.

The phase shift and vertical shift work the same way for all trigonometric functions. For example, the phase shift of the function $y = \tan (k\theta + c) + h$ is $-\frac{c}{k}$, and its vertical shift is h.

Examples 3 **Graph $y = \csc\left(\frac{\theta}{2} - \frac{\pi}{4}\right) + 2$.**

The period is $\frac{2\pi}{\frac{1}{2}}$ or 4π. The phase shift is $-\frac{-\frac{\pi}{4}}{\frac{1}{2}}$ or $\frac{\pi}{2}$. The vertical shift is 2.

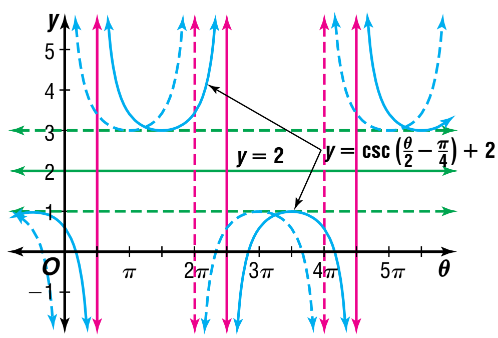
Use this information to graph the function.

Step 1 Draw the midline which is the graph of $y = 2$.

Step 2 Draw dashed lines parallel to the midline, which are 1 unit above and below the midline.

Step 3 Draw the cosecant curve with period of 4π.

Step 4 Shift the graph $\frac{\pi}{2}$ units to the right.



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Real World Application

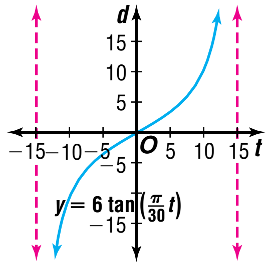
4 **SECURITY** **Refer to the application at the beginning of the lesson.**

a. Graph the equation $y = 6 \tan\left(\frac{\pi}{30}t\right)$.

b. Find the location the camera is scanning after 5 seconds.

c. What happens when $t = 15$?

a. The period is $\frac{\pi}{\frac{\pi}{30}}$ or 30. There are no horizontal or vertical shifts. Draw the asymptotes at $t = -15$ and $t = 15$. Graph the equation.



b. Evaluate the equation at $t = 5$.

$d = 6 \tan\left(\frac{\pi}{30}t\right)$

$d = 6 \tan\left[\frac{\pi}{30}(5)\right]$ *$t = 5$*

$d \approx 3.464101615$ *Use a calculator.*

The camera is scanning a point that is about 3.5 feet above the center of the driveway.

c. At $\tan\left[\frac{\pi}{30}(15)\right]$ or $\tan \frac{\pi}{2}$, the function is undefined. Therefore, the camera will not scan any part of the driveway when $t = 15$. It will be pointed in a direction that is parallel with the driveway.

You can write an equation of a trigonometric function if you are given the period, phase shift, and vertical translation.

Example 5 **Write an equation for a secant function with period π, phase shift $\frac{\pi}{3}$, and vertical shift -3.**

The form of the equation will be $y = \sec(k\theta + c) + h$. Find the values of k, c, and h.

***k*:** $\frac{2\pi}{k} = \pi$ *The period is π.*

$k = 2$

***c*:** $-\frac{c}{k} = \frac{\pi}{3}$ *The phase shift is $\frac{\pi}{3}$.*

$-\frac{c}{2} = \frac{\pi}{3}$ *$k = 2$*

$c = -\frac{2\pi}{3}$

***h*:** $h = -3$

Substitute these values into the general equation. The equation is

$y = \sec\left(2\theta - \frac{2\pi}{3}\right) - 3.$

CHECK FOR UNDERSTANDING

Communicating Mathematics

Read and study the lesson to answer each question.

1. **Name** three values of θ that would result in $y = \cot \theta$ being undefined.
2. **Compare** the asymptotes and periods of $y = \tan \theta$ and $y = \sec \theta$.
3. **Describe** two different phase shifts of the secant function that would make it appear to be the cosecant function.

Guided Practice

Find each value by referring to the graphs of the trigonometric functions.

4. $\tan 4\pi$
5. $\csc\left(-\frac{7\pi}{2}\right)$

Find the values of θ for which each equation is true.

6. $\sec \theta = -1$
7. $\cot \theta = 1$

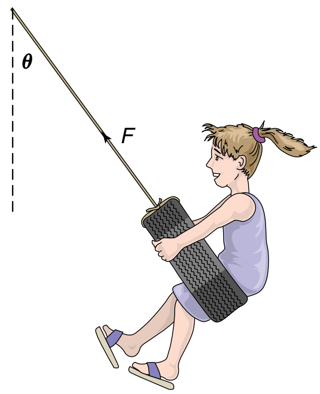
Graph each function.

8. $y = \tan\left(\theta + \frac{\pi}{4}\right)$
9. $y = \sec(2\theta + \pi) - 1$

Write an equation for the given function given the period, phase shift, and vertical shift.

10. cosecant function, period $= 3\pi$, phase shift $= \frac{\pi}{3}$, vertical shift $= -4$
11. cotangent function, period $= 2\pi$, phase shift $= -\frac{\pi}{4}$, vertical shift $= 0$

12. **Physics** A child is swinging on a tire swing. The tension on the rope is equal to the downward force on the end of the rope times $\sec \theta$, where θ is the angle formed by a vertical line and the rope.
    a. The downward force in newtons equals the mass of the child and the swing in kilograms times the acceleration due to gravity (9.8 meters per second squared). If the mass of the child and the tire is 73 kilograms, find the downward force.
    b. Write an equation that represents the tension on the rope as the child swings back and forth.
    c. Graph the equation for $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$.
    d. What is the least amount of tension on the rope?
    e. What happens to the tension on the rope as the child swings higher and higher?



EXERCISES

Practice

Find each value by referring to the graphs of the trigonometric functions.

13. $\cot\left(\frac{5\pi}{2}\right)$
14. $\tan(-8\pi)$
15. $\sec\left(\frac{9\pi}{2}\right)$
16. $\csc\left(-\frac{5\pi}{2}\right)$
17. $\sec 7\pi$
18. $\cot(-5\pi)$

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19. What is the value of csc (-6π)?

20. Find the value of tan (10π).

Find the values of θ for which each equation is true.

21. $\tan \theta = 0$

22. $\sec \theta = 1$

23. $\csc \theta = -1$

24. $\tan \theta = 1$

25. $\tan \theta = -1$

26. $\cot \theta = -1$

27. What are the values of θ for which sec θ is undefined?

28. Find the values of θ for which cot θ is undefined.

Graph each function.

29. $y = \cot\left(\theta - \frac{\pi}{2}\right)$

30. $y = \sec \frac{\theta}{3}$

31. $y = \csc \theta + 5$

32. $y = \tan\left(\frac{\theta}{2} - \frac{\pi}{4}\right) + 1$

33. $y = \csc(2\theta + \pi) - 3$

34. $y = \sec\left(\frac{\theta}{3} + \frac{\pi}{6}\right) - 2$

35. Graph $y = \cos \theta$ and $y = \sec \theta$. In the interval of -2π and 2π, what are the values of θ where the two graphs are tangent to each other?

Write an equation for the given function given the period, phase shift, and vertical shift.

36. tangent function, period $= 2\pi$, phase shift $= 0$, vertical shift $= -6$

37. cotangent function, period $= \frac{\pi}{2}$, phase shift $= \frac{\pi}{8}$, vertical shift $= 7$

38. secant function, period $= \pi$, phase shift $= -\frac{\pi}{4}$, vertical shift $= -10$

39. cosecant function, period $= 3\pi$, phase shift $= \pi$, vertical shift $= -1$

40. cotangent function, period $= 5\pi$, phase shift $= -\pi$, vertical shift $= 12$

41. cosecant function, period $= \frac{\pi}{3}$, phase shift $= -\frac{\pi}{2}$, vertical shift $= -5$

42. Write a secant function with a period of 3π, a phase shift of π units to the left, and a vertical shift of 8 units downward.

43. Write a tangent function with a period of $\frac{\pi}{2}$, a phase shift of $\frac{\pi}{4}$ to the right, and a vertical shift of 7 units upward.

Applications and Problem Solving

Real World Application

44. **Security** A security camera is scanning a long straight fence along one side of a military base. The camera is located 10 feet from the center of the fence. If d represents the distance along the fence from the center and t is time in seconds, then $d = 10 \tan \frac{\pi}{40} t$ models the point being scanned.

a. Graph the equation for $-20 \leq t \leq 20$.

b. Find the location the camera is scanning at 3 seconds.

c. Find the location the camera is scanning at 15 seconds.

45. **Critical Thinking** Graph $y = \csc \theta$, $y = 3 \csc \theta$, and $y = -3 \csc \theta$. Compare and contrast the graphs.

46. **Physics** A wire is used to hang a painting from a nail on a wall as shown at the right. The tension on each half of the wire is equal to half the downward force times $\sec \frac{\theta}{2}$.

a. The downward force in newtons equals the mass of the painting in kilograms times 9.8. If the mass of the painting is 7 kilograms, find the downward force.

b. Write an equation that represents the tension on each half of the wire.

c. Graph the equation for $0 \leq \theta \leq \pi$.

d. What is the least amount of tension on each side of the wire?

e. As the measure of θ becomes greater, what happens to the tension on each side of the wire?

47. **Electronics** The current I measured in amperes that is flowing through an alternating current at any time t in seconds is modeled by $I = 220 \sin \left(60\pi t - \frac{\pi}{6}\right)$.

a. What is the amplitude of the current?

b. What is the period of the current?

c. What is the phase shift of this sine function?

d. Find the current when $t = 60$.

48. **Critical Thinking** Write a tangent function that has the same graph as $y = \cot \theta$.

Mixed Review

49. **Tides** In Daytona Beach, Florida, the first high tide was 3.99 feet at 12:03 A.M. The first low tide of 0.55 foot occurred at 6:24 A.M. The second high tide occurred at 12:19 P.M. *(Lesson 6-6)*

a. Find the amplitude of a sinusoidal function that models the tides.

b. Find the vertical shift of the sinusoidal function that models the tides.

c. What is the period of the sinusoidal function that models the tides?

d. Write a sinusoidal function to model the tides, using t to represent the number of hours in decimals since midnight.

e. According to your model, determine the height of the water at noon.

50. Graph $y = 2 \cos \frac{\theta}{2}$. *(Lesson 6-4)*

51. If a central angle of a circle with radius 18 centimeters measures $\frac{\pi}{3}$, find the length (in terms of π) of its intercepted arc. *(Lesson 6-1)*

52. Solve $\triangle ABC$ if $A = 62°31'$, $B = 75°18'$, and $a = 57.3$. Round angle measures to the nearest minute and side measures to the nearest tenth. *(Lesson 5-6)*

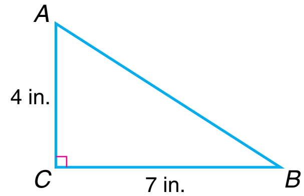
53. **Entertainment** A utility pole is braced by a cable attached to the top of the pole and anchored in a concrete block at the ground level 4 meters from the base of the pole. The angle between the cable and the ground is 73°. *(Lesson 5-4)*

a. Draw a diagram of the problem.

b. If the pole is perpendicular with the ground, what is the height of the pole?

c. Find the length of the cable.

54. Find the values of the sine, cosine, and tangent for $\angle A$. *(Lesson 5-2)*



55. Solve $\frac{x^2 - 4}{x^2 - 3x - 10} \leq 0$. *(Lesson 4-6)*

56. If r varies directly as t and $t = 6$ when $r = 0.5$, find r when $t = 10$. *(Lesson 3-8)*

57. Solve the system of inequalities by graphing. *(Lesson 2-6)*
$3x + 2y < 8$
$y < 2x + 1$
$-2y < -x + 4$

58. **Nutrition** The fat grams and Calories in various frozen pizzas are listed below. Use a graphing calculator to find the equation of the regression line and the Pearson product-moment correlation value. *(Lesson 1-6)*

Pizza	Fat (grams)	Calories
Cheese Pizza	14	270
Party Pizza	17	340
Pepperoni French Bread Pizza	22	430
Hamburger French Bread Pizza	19	410
Deluxe French Bread Pizza	20	420
Pepperoni Pizza	19	360
Sausage Pizza	18	360
Sausage and Pepperoni Pizza	18	340
Spicy Chicken Pizza	16	360
Supreme Pizza	18	308
Vegetable Pizza	13	300
Pizza Roll-Ups	13	250

59. **SAT/ACT Practice** The distance from City A to City B is 150 miles. From City A to City C is 90 miles. Which of the following is necessarily true?

A The distance from B to C is 60 miles.

B Six times the distance from A to B equals 10 times the distance from A to C.

C The distance from B to C is 240 miles.

D The distance from A to B exceeds by 30 miles twice the distance from A to C.

E Three times the distance from A to C exceeds by 30 miles twice the distance from A to B.

Extra Practice See p. A37.

GRAPHING CALCULATOR EXPLORATION

6-7B Sound Beats

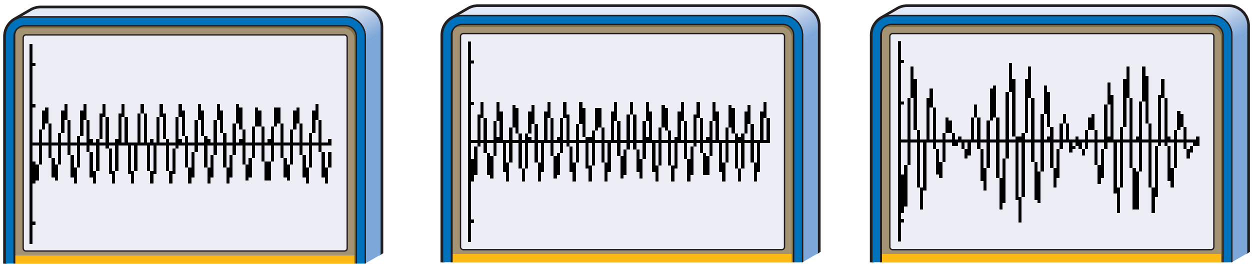
An Extension of Lesson 6-7

OBJECTIVE

- Use a graphing calculator to model beat effects produced by waves of almost equal frequencies.

The frequency of a wave is defined as the reciprocal of the period of the wave. If you listen to two steady sounds that have almost the same frequencies, you can detect an effect known as *beat.* Used in this sense, the word refers to a regular variation in sound intensity. This meaning is very different from another common meaning of the word, which you use when you are speaking about the rhythm of music for dancing.

A beat effect can be modeled mathematically by combination of two sine waves. The loudness of an actual combination of two steady sound waves of almost equal frequency depends on the amplitudes of the component sound waves. The first two graphs below picture two sine waves of almost equal frequencies. The amplitudes are equal, and the graphs, on first inspection, look almost the same. However, when the functions shown by the graphs are added, the resulting third graph is not what you would get by stretching either of the original graphs by a factor of 2, but is instead something quite different.



TRY THESE

1. Graph $f(x) = \sin(5\pi x) + \sin(4.79\pi x)$ using a window **[0, 10π] scl:π** by **[−2.5, 2.5] scl:1**. Which of the graphs shown above does the graph resemble?
2. Change the window settings for the independent variable to have **Xmax** $= 200\pi$. How does the appearance of the graph change?
3. For the graph in Exercise 2, use value on the **CALC** menu to find the value of $f(x)$ when $x = 187.158$.
4. Does your graph of Exercise 2 show negative values of y when x is close to 187.158?
5. Use value on the **CALC** menu to find $f(191.5)$. Does your result have any bearing on your answer for Exercise 4? Explain.

WHAT DO YOU THINK?

6. What aspect of the calculator explains your observations in Exercises 3-5?
7. Write two sine functions with almost equal frequencies. Graph the sum of the two functions. Discuss any interesting features of the graph.
8. Do functions that model beat effects appear to be periodic functions? Do your graphs prove that your answer is correct?