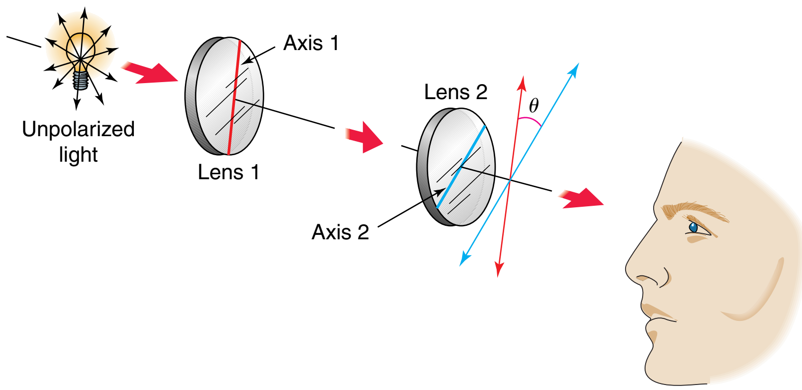
7-1 Basic Trigonometric Identities

OBJECTIVE

- Identify and use reciprocal identities, quotient identities, Pythagorean identities, symmetry identities, and opposite-angle identities.

Real World Application

OPTICS Many sunglasses have polarized lenses that reduce the intensity of light. When unpolarized light passes through a polarized lens, the intensity of the light is cut in half. If the light then passes through another polarized lens with its axis at an angle of θ to the first, the intensity of the light is again diminished.



The intensity of the emerging light can be found by using the formula $I = I_0 - \frac{I_0}{\csc^2 \theta}$, where I_0 is the intensity of the light incoming to the second polarized lens, I is the intensity of the emerging light, and θ is the angle between the axes of polarization. Simplify this expression and determine the intensity of light emerging from a polarized lens with its axis at a 30° angle to the original. *This problem will be solved in Example 5.*

In algebra, variables and constants usually represent real numbers. The values of trigonometric functions are also real numbers. Therefore, the language and operations of algebra also apply to trigonometry. Algebraic expressions involve the operations of addition, subtraction, multiplication, division, and exponentiation. These operations are used to form trigonometric expressions. Each expression below is a trigonometric expression.

$$\cos x - x \qquad \sin^2 a + \cos^2 a \qquad \frac{1 - \sec A}{\tan A}$$

A statement of equality between two expressions that is true for *all* values of the variable(s) for which the expressions are defined is called an **identity.** For example, $x^2 - y^2 = (x - y)(x + y)$ is an algebraic identity. An identity involving trigonometric expressions is called a **trigonometric identity.**

If you can show that a specific value of the variable in an equation makes the equation false, then you have produced a *counterexample.* It only takes one counterexample to prove that an equation is not an identity.

Example 1 **Prove that $\sin x \cos x = \tan x$ is *not* a trigonometric identity by producing a counterexample.**

Suppose $x = \frac{\pi}{4}$.

$$\sin x \cos x \stackrel{?}{=} \tan x$$

$$\sin \frac{\pi}{4} \cos \frac{\pi}{4} \stackrel{?}{=} \tan \frac{\pi}{4} \quad \textit{Replace } x \textit{ with } \frac{\pi}{4}.$$

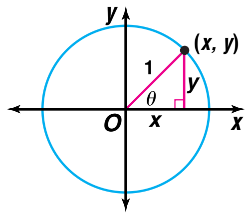
$$\left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) \stackrel{?}{=} 1$$

$$\frac{1}{2} \neq 1$$

Since evaluating each side of the equation for the same value of x produces an inequality, the equation is not an identity.

Although producing a counterexample can show that an equation is not an identity, proving that an equation is an identity generally takes more work. Proving that an equation is an identity requires showing that the equality holds for *all* values of the variable where each expression is defined. Several fundamental trigonometric identities can be verified using geometry.

Recall from Lesson 5-3 that the trigonometric functions can be defined using the unit circle. From the unit circle, $\sin \theta = \frac{y}{1}$, or y and $\csc \theta = \frac{1}{y}$. That is, $\sin \theta = \frac{1}{\csc \theta}$. Identities derived in this manner are called **reciprocal identities.**



Reciprocal Identities

The following trigonometric identities hold for all values of θ where each expression is defined.

$$\sin \theta = \frac{1}{\csc \theta} \qquad \cos \theta = \frac{1}{\sec \theta}$$

$$\csc \theta = \frac{1}{\sin \theta} \qquad \sec \theta = \frac{1}{\cos \theta}$$

$$\tan \theta = \frac{1}{\cot \theta} \qquad \cot \theta = \frac{1}{\tan \theta}$$

Returning to the unit circle, we can say that $\frac{\sin \theta}{\cos \theta} = \frac{y}{x} = \tan \theta$. This is an example of a **quotient identity.**

Quotient Identities

The following trigonometric identities hold for all values of θ where each expression is defined.

$$\frac{\sin \theta}{\cos \theta} = \tan \theta \qquad \frac{\cos \theta}{\sin \theta} = \cot \theta$$

Since the triangle in the unit circle on the previous page is a right triangle, we may apply the Pythagorean Theorem: $y^2 + x^2 = 1^2$, or $\sin^2 \theta + \cos^2 \theta = 1$. Other identities can be derived from this one.

$$\sin^2 \theta + \cos^2 \theta = 1$$

$$\frac{\sin^2 \theta}{\cos^2 \theta} + \frac{\cos^2 \theta}{\cos^2 \theta} = \frac{1}{\cos^2 \theta}$$ *Divide each side by $\cos^2 \theta$.*

$$\tan^2 \theta + 1 = \sec^2 \theta$$ *Quotient and reciprocal identities*

Likewise, the identity $1 + \cot^2 \theta = \csc^2 \theta$ can be derived by dividing each side of the equation $\sin^2 \theta + \cos^2 \theta = 1$ by $\sin^2 \theta$. These are the **Pythagorean identities.**

Pythagorean Identities

The following trigonometric identities hold for all values of θ where each expression is defined.

$\sin^2 \theta + \cos^2 \theta = 1$ $\quad \tan^2 \theta + 1 = \sec^2 \theta$ $\quad 1 + \cot^2 \theta = \csc^2 \theta$

You can use the identities to help find the values of trigonometric functions.

Example 2 **Use the given information to find the trigonometric value.**

a. If $\sec \theta = \frac{3}{2}$, find $\cos \theta$.

$$\cos \theta = \frac{1}{\sec \theta}$$ *Choose an identity that involves $\cos \theta$ and $\sec \theta$.*

$$= \frac{1}{\frac{3}{2}} \text{ or } \frac{2}{3}$$ *Substitute $\frac{3}{2}$ for $\sec \theta$ and evaluate.*

b. If $\csc \theta = \frac{4}{3}$, find $\tan \theta$.

Since there are no identities relating $\csc \theta$ and $\tan \theta$, we must use two identities, one relating $\csc \theta$ and $\cot \theta$ and another relating $\cot \theta$ and $\tan \theta$.

$$\csc^2 \theta = 1 + \cot^2 \theta$$ *Pythagorean identity*

$$\left(\frac{4}{3}\right)^2 = 1 + \cot^2 \theta$$ *Substitute $\frac{4}{3}$ for $\csc \theta$.*

$$\frac{16}{9} = 1 + \cot^2 \theta$$

$$\frac{7}{9} = \cot^2 \theta$$

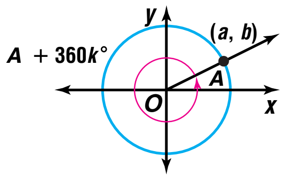
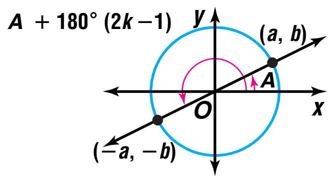
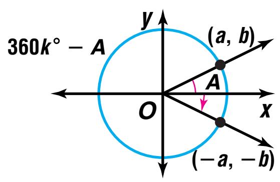
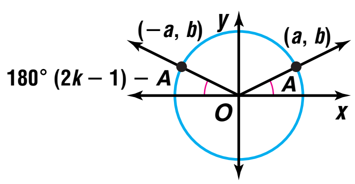
$$\pm\frac{\sqrt{7}}{3} = \cot \theta$$ *Take the square root of each side.*

Now find $\tan \theta$.

$$\tan \theta = \frac{1}{\cot \theta}$$ *Reciprocal identity*

$$= \pm\frac{3\sqrt{7}}{7}, \text{ or about } \pm 1.134$$

To determine the sign of a function value, you need to know the quadrant in which the angle terminates. The signs of function values in different quadrants are related according to the symmetries of the unit circle. Since we can determine the values of tan A, cot A, sec A, and csc A in terms of sin A and/or cos A with the reciprocal and quotient identities, we only need to investigate sin A and cos A.

Case	Relationship between angles A and B	Diagram	Conclusion
1	The angles differ by a multiple of 360°. $B - A = 360k°$ or $B = A + 360\,k°$		Since A and $A + 360k°$ are coterminal, they share the same value of sine and cosine.
2	The angles differ by an odd multiple of 180°. $B - A = 180°(2k - 1)$ or $B = A + 180°(2k - 1)$		Since A and $A + 180°(2k - 1)$ have terminal sides in diagonally opposite quadrants, the values of both sine and cosine change sign.
3	The sum of the angles is a multiple of 360°. $A + B = 360k°$ or $B = 360k° - A$		Since A and $360k° - A$ lie in vertically adjacent quadrants, the sine values are opposite but the cosine values are the same.
4	The sum of the angles is an odd multiple of 180°. $A + B = 180°\,(2k - 1)$ or $B = 180°(2k - 1) - A$		Since A and $180°(2k - 1) - A$ lie in horizontally adjacent quadrants, the sine values are the same but the cosine values are opposite.

These general rules for sine and cosine are called **symmetry identities.**

Symmetry Identities	The following trigonometric identities hold for any integer k and all values of A.	
Case 1:	$\sin(A + 360k°) = \sin A$	$\cos(A + 360k°) = \cos A$
Case 2:	$\sin(A + 180°(2k - 1)) = -\sin A$	$\cos(A + 180°(2k - 1)) = -\cos A$
Case 3:	$\sin(360k° - A) = -\sin A$	$\cos(360k° - A) = \cos A$
Case 4:	$\sin(180°(2k - 1) - A) = \sin A$	$\cos(180°(2k - 1) - A) = -\cos A$

To use the symmetry identities with radian measure, replace 180° with π and 360° with 2π.

Example 3 **Express each value as a trigonometric function of an angle in Quadrant I.**

a. $\sin 600°$

Relate 600° to an angle in Quadrant I.

$600° = 60° + 3(180°)$ — *600° and 60° differ by an odd multiple of 180°.*

$\sin 600° = \sin (60° + 3(180°))$ — *Case 2, with A = 60° and k = 2*

$= -\sin 60°$

b. $\sin \frac{19\pi}{4}$

The sum of $\frac{19\pi}{4}$ and $\frac{\pi}{4}$, which is $\frac{20\pi}{4}$ or 5π, is an odd multiple of π.

$\frac{19\pi}{4} = 5\pi - \frac{\pi}{4}$

$\sin \frac{19\pi}{4} = \sin \left(5\pi - \frac{\pi}{4}\right)$ — *Case 4, with $A = \frac{\pi}{4}$ and $k = 3$*

$= \sin \frac{\pi}{4}$

c. $\cos (-410°)$

The sum of $-410°$ and $50°$ is a multiple of 360°.

$-410° = -360° - 50°$

$\cos (-410°) = \cos (-360° - 50°)$ — *Case 3, with A = 50° and k = −1*

$= \cos 50°$

d. $\tan \frac{37\pi}{6}$

$\frac{37\pi}{6}$ and $\frac{\pi}{6}$ differ by a multiple of 2π.

$\frac{37\pi}{6} = 3(2\pi) + \frac{\pi}{6}$ — *Case 1, with $A = \frac{\pi}{6}$ and $k = 3$*

$\tan \frac{37\pi}{6} = \dfrac{\sin \frac{37\pi}{6}}{\cos \frac{37\pi}{6}}$ — *Rewrite using a quotient identity since the symmetry identities are in terms of sine and cosine.*

$= \dfrac{\sin \left(3(2\pi) + \frac{\pi}{6}\right)}{\cos \left(3(2\pi) + \frac{\pi}{6}\right)}$

$= \dfrac{\sin \frac{\pi}{6}}{\cos \frac{\pi}{6}}$ or $\tan \frac{\pi}{6}$ — *Quotient identity*

Case 3 of the Symmetry Identities can be written as the **opposite-angle identities** when $k = 0$.

Opposite-Angle Identities

The following trigonometric identities hold for all values of A.

$$\sin(-A) = -\sin A$$

$$\cos(-A) = \cos A$$

The basic trigonometric identities can be used to simplify trigonometric expressions. Simplifying a trigonometric expression means that the expression is written using the fewest trigonometric functions possible and as algebraically simplified as possible. This may mean writing the expression as a numerical value.

Examples

4 **Simplify $\sin x + \sin x \cot^2 x$.**

$$\begin{aligned} \sin x + \sin x \cot^2 x &= \sin x\,(1 + \cot^2 x) && \textit{Factor.} \\ &= \sin x \csc^2 x && \textit{Pythagorean identity: } 1 + \cot^2 x = \csc^2 x \\ &= \sin x \cdot \frac{1}{\sin^2 x} && \textit{Reciprocal identity} \\ &= \frac{1}{\sin x} \\ &= \csc x && \textit{Reciprocal identity} \end{aligned}$$

5 **OPTICS** **Refer to the application at the beginning of the lesson.**

Real World Application

a. Simplify the formula $I = I_0 - \dfrac{I_0}{\csc^2 \theta}$.

b. Use the simplified formula to determine the intensity of light that passes through a second polarizing lens with axis at 30° to the original.

a. $I = I_0 - \dfrac{I_0}{\csc^2 \theta}$

$I = I_0 - I_0 \sin^2 \theta$ *Reciprocal identity*

$I = I_0(1 - \sin^2 \theta)$ *Factor.*

$I = I_0 \cos^2 \theta$ *$1 - \sin^2 \theta = \cos^2 \theta$*

b. $I = I_0 \cos^2 30°$

$I = I_0\left(\dfrac{\sqrt{3}}{2}\right)^2$

$I = \dfrac{3}{4} I_0$

The light has three-fourths the intensity it had before passing through the second polarizing lens.

CHECK FOR UNDERSTANDING

Communicating Mathematics

Read and study the lesson to answer each question.

1. **Find a counterexample** to show that the equation $1 - \sin x = \cos x$ is not an identity.
2. **Explain** why the Pythagorean and opposite-angle identities are so named.
3. **Write** two reciprocal identities, one quotient identity, and one Pythagorean identity, each of which involves $\cot \theta$.
4. **Prove** that $\tan(-A) = -\tan A$ using the quotient and opposite-angle identities.
5. **You Decide** Claude and Rosalinda are debating whether an equation from their homework assignment is an identity. Claude says that since he has tried ten specific values for the variable and all of them worked, it must be an identity. Rosalinda explained that specific values could only be used as counterexamples to prove that an equation is not an identity. Who is correct? Explain your answer.

Guided Practice

Prove that each equation is *not* a trigonometric identity by producing a counterexample.

6. $\sin \theta + \cos \theta = \tan \theta$
7. $\sec^2 x + \csc^2 x = 1$

Use the given information to determine the exact trigonometric value.

8. $\cos \theta = \frac{2}{3}, 0° < \theta < 90°$; $\sec \theta$
9. $\cot \theta = -\frac{\sqrt{5}}{2}, \frac{\pi}{2} < \theta < \pi$; $\tan \theta$
10. $\sin \theta = -\frac{1}{5}, \pi < \theta < \frac{3\pi}{2}$; $\cos \theta$
11. $\tan \theta = -\frac{4}{7}, 270° < \theta < 360°$; $\sec \theta$

Express each value as a trigonometric function of an angle in Quadrant I.

12. $\cos \frac{7\pi}{3}$
13. $\csc(-330°)$

Simplify each expression.

14. $\frac{\csc \theta}{\cot \theta}$
15. $\cos x \csc x \tan x$
16. $\cos x \cot x + \sin x$

17. **Physics** When there is a current in a wire in a magnetic field, a force acts on the wire. The strength of the magnetic field can be determined using the formula $B = \frac{F \csc \theta}{I\ell}$, where F is the force on the wire, I is the current in the wire, ℓ is the length of the wire, and θ is the angle the wire makes with the magnetic field. Physics texts often write the formula as $F = I\ell B \sin \theta$. Show that the two formulas are equivalent.

EXERCISES

Practice

Prove that each equation is not a trigonometric identity by producing a counterexample.

18. $\sin \theta \cos \theta = \cot \theta$
19. $\frac{\sec \theta}{\tan \theta} = \sin \theta$
20. $\sec^2 x - 1 = \frac{\cos x}{\csc x}$
21. $\sin x + \cos x = 1$
22. $\sin y \tan y = \cos y$
23. $\tan^2 A + \cot^2 A = 1$

24. Find a value of θ for which $\cos\left(\theta + \frac{\pi}{2}\right) \neq \cos\theta + \cos\frac{\pi}{2}$.

Use the given information to determine the exact trigonometric value.

25. $\sin\theta = \frac{2}{5}$, $0° < \theta < 90°$; $\csc\theta$

26. $\tan\theta = \frac{\sqrt{3}}{4}$, $0 < \theta < \frac{\pi}{2}$; $\cot\theta$

27. $\sin\theta = \frac{1}{4}$, $0 < \theta < \frac{\pi}{2}$; $\cos\theta$

28. $\cos\theta = -\frac{2}{3}$, $90° < \theta < 180°$; $\sin\theta$

29. $\csc\theta = \frac{\sqrt{11}}{3}$, $\frac{\pi}{2} < \theta < \pi$; $\cot\theta$

30. $\sec\theta = -\frac{5}{4}$, $90° < \theta < 180°$; $\tan\theta$

31. $\sin\theta = -\frac{1}{3}$, $180° < \theta < 270°$; $\tan\theta$

32. $\tan\theta = \frac{2}{3}$, $\pi < \theta < \frac{3\pi}{2}$; $\cos\theta$

33. $\sec\theta = -\frac{7}{5}$, $180° < \theta < 270°$; $\sin\theta$

34. $\cos\theta = \frac{1}{8}$, $\frac{3\pi}{2} < \theta < 2\pi$; $\tan\theta$

35. $\cot\theta = -\frac{4}{3}$, $270° < \theta < 360°$; $\sin\theta$

36. $\cot\theta = -8$, $\frac{3\pi}{2} < \theta < 2\pi$; $\csc\theta$

37. If A is a second quadrant angle, and $\cos A = -\frac{\sqrt{3}}{4}$, find $\frac{\sec^2 A - \tan^2 A}{2\sin^2 A + 2\cos^2 A}$.

Express each value as a trigonometric function of an angle in Quadrant I.

38. $\sin 390°$

39. $\cos\frac{27\pi}{8}$

40. $\tan\frac{19\pi}{5}$

41. $\csc\frac{10\pi}{3}$

42. $\sec(-1290°)$

43. $\cot(-660°)$

Simplify each expression.

44. $\frac{\sec x}{\tan x}$

45. $\frac{\cot\theta}{\cos\theta}$

46. $\frac{\sin(\theta + \pi)}{\cos(\theta - \pi)}$

47. $(\sin x + \cos x)^2 + (\sin x - \cos x)^2$

48. $\sin x \cos x \sec x \cot x$

49. $\cos x \tan x + \sin x \cot x$

50. $(1 + \cos\theta)(\csc\theta - \cot\theta)$

51. $1 + \cot^2\theta - \cos^2\theta - \cos^2\theta\cot^2\theta$

52. $\frac{\sin x}{1 + \cos x} + \frac{\sin x}{1 - \cos x}$

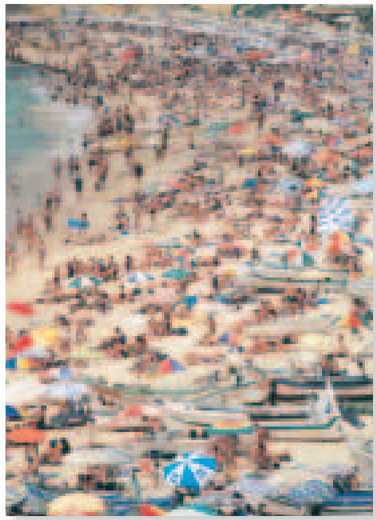
53. $\cos^4\alpha + 2\cos^2\alpha\sin^2\alpha + \sin^4\alpha$

Applications and Problem Solving

54. **Optics** Refer to the equation derived in Example 5. What angle should the axes of two polarizing lenses make in order to block all light from passing through?

55. **Critical Thinking** Use the unit circle definitions of sine and cosine to provide a geometric interpretation of the opposite-angle identities.

56. **Dermatology** It has been shown that skin cancer is related to sun exposure. The rate W at which a person's skin absorbs energy from the sun depends on the energy S, in watts per square meter, provided by the sun, the surface area A exposed to the sun, the ability of the body to absorb energy, and the angle θ between the sun's rays and a line perpendicular to the body. The ability of an object to absorb energy is related to a factor called the *emissivity,* e, of the object. The emissivity can be calculated using the formula $e = \frac{W \sec \theta}{AS}$.



a. Solve this equation for W. Write your answer using only $\sin \theta$ or $\cos \theta$.

b. Find W if $e = 0.80$, $\theta = 40°$, $A = 0.75 \text{ m}^2$, and $S = 1000 \text{ W/m}^2$.

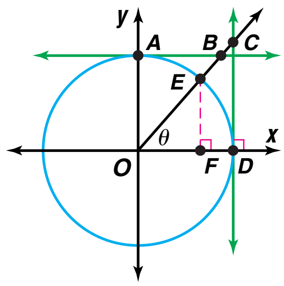
57. **Physics** A skier of mass m descends a θ-degree hill at a constant speed. When Newton's Laws are applied to the situation, the following system of equations is produced.

$$F_N - mg \cos \theta = 0$$
$$mg \sin \theta - \mu_k F_N = 0$$

where g is the acceleration due to gravity, F_N is the normal force exerted on the skier, and μ_k is the coefficient of friction. Use the system to define μ_k as a function of θ.

58. **Geometry** Show that the area of a regular polygon of n sides, each of length a, is given by $A = \frac{1}{4} na^2 \cot \left(\frac{180°}{n}\right)$.

59. **Critical Thinking** The circle at the right is a unit circle with its center at the origin. $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are tangent to the circle. State the segments whose measures represent the ratios $\sin \theta$, $\cos \theta$, $\tan \theta$, $\sec \theta$, $\cot \theta$, and $\csc \theta$. Justify your answers.



Mixed Review

60. Find $\text{Cos}^{-1}\left(-\frac{\sqrt{2}}{2}\right)$. *(Lesson 6-8)*

61. Graph $y = \cos\left(x - \frac{\pi}{6}\right)$. *(Lesson 6-5)*

62. **Physics** A pendulum 20 centimeters long swings $3°30'$ on each side of its vertical position. Find the length of the arc formed by the tip of the pendulum as it swings. *(Lesson 6-1)*

63. Angle C of $\triangle ABC$ is a right angle. Solve the triangle if $A = 20°$ and $c = 35$. *(Lesson 5-4)*

64. Find all the rational roots of the equation $2x^3 + x^2 - 8x - 4 = 0$. *(Lesson 4-4)*

65. Solve $2x^2 + 7x - 4 = 0$ by completing the square. *(Lesson 4-2)*

66. Determine whether $f(x) = 3x^3 + 2x - 5$ is continuous or discontinuous at $x = 5$. *(Lesson 3-5)*

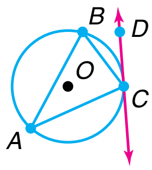
Extra Practice See p. A38.

67. Solve the system of equations algebraically. *(Lesson 2-2)*

$x + y - 2z = 3$
$-4x - y - z = 0$
$-x - 5y + 4z = 11$

68. Write the slope-intercept form of the equation of the line that passes through points at (5, 2) and (−4, 4). *(Lesson 1-4)*

69. **SAT/ACT Practice** Triangle ABC is inscribed in circle O, and $\overleftrightarrow{CD}$ is tangent to circle O at point C. If $m\angle BCD = 40°$, find $m\angle A$.



A 60° **B** 50° **C** 40° **D** 30° **E** 20°

CAREER CHOICES

Cartographer

Do maps fascinate you? Do you like drawing, working with computers, and geography? You may want to consider a career in cartography. As a cartographer, you would make maps, charts, and drawings. Cartography has changed a great deal with modern technology. Computers and satellites have become powerful new tools in making maps. As a cartographer, you may work with manual drafting tools as well as computer software designed for making maps.

The image at the right shows how a cartographer uses a three-dimensional landscape to create a two-dimensional topographic map.

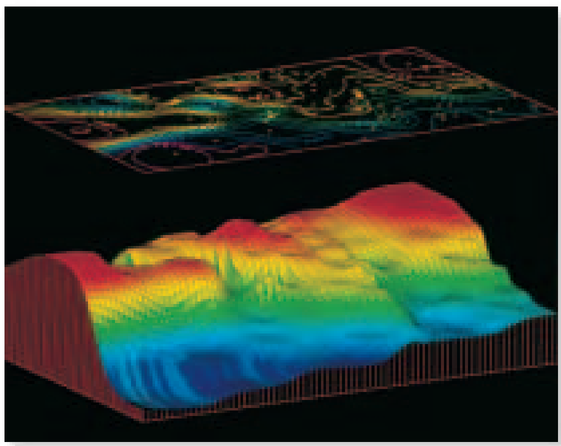
There are several areas of specialization in the field of cartography. Some of these include making maps from political boundaries and natural features, making maps from aerial photographs, and correcting original maps.

Career Overview

Degree Preferred:
bachelor's degree in engineering or a physical science

Related Courses:
mathematics, geography, computer science, mechanical drawing

Outlook:
slower than average through 2006



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