7-3

Sum and Difference Identities

OBJECTIVE

- Use the sum and difference identities for the sine, cosine, and tangent functions.

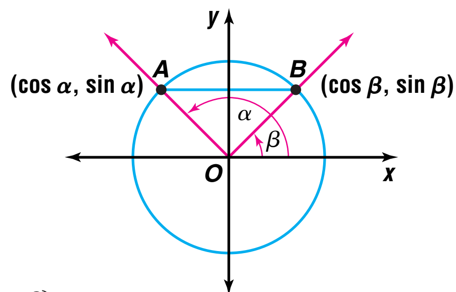
Real World Application

BROADCASTING Have you ever had trouble tuning in your favorite radio station? Does the picture on your TV sometimes appear blurry? Sometimes these problems are caused by *interference.* Interference can result when two waves pass through the same space at the same time. The two kinds of interference are:

- *constructive interference,* which occurs if the amplitude of the sum of the waves is greater than the amplitudes of the two component waves, and
- *destructive interference,* which occurs if the amplitude of the sum is less than the amplitudes of the component waves.

What type of interference results when a signal modeled by the equation $y = 20 \sin(3t + 45°)$ is combined with a signal modeled by the equation $y = 20 \sin(3t + 225°)$? *This problem will be solved in Example 4.*

Consider two angles α and β in standard position. Let the terminal side of α intersect the unit circle at point $A(\cos \alpha, \sin \alpha)$. Let the terminal side of β intersect the unit circle at $B(\cos \beta, \sin \beta)$. We will calculate $(AB)^2$ in two different ways.



$m\angle BOA = \alpha - \beta$

Look Back
You can refer to Lesson 5-8 to review the Law of Cosines.

First use the Law of Cosines.

$(AB)^2 = (OA)^2 + (OB)^2 - 2(OA)(OB) \cos (\alpha - \beta)$

$(AB)^2 = 1^2 + 1^2 - 2(1)(1) \cos (\alpha - \beta)$ *OA = OB = 1*

$(AB)^2 = 2 - 2 \cos (\alpha - \beta)$ *Simplify.*

Now use the distance formula.

$(AB)^2 = (\cos \alpha - \cos \beta)^2 + (\sin \alpha - \sin \beta)^2$

$(AB)^2 = \cos^2 \alpha - 2 \cos \alpha \cos \beta + \cos^2 \beta + \sin^2 \alpha - 2 \sin \alpha \sin \beta + \sin^2 \beta$

$(AB)^2 = (\cos^2 \alpha + \sin^2 \alpha) + (\cos^2 \beta + \sin^2 \beta) - 2(\cos \alpha \cos \beta + \sin \alpha \sin \beta)$

$(AB)^2 = 1 + 1 - 2(\cos \alpha \cos \beta + \sin \alpha \sin \beta)$ *$\cos^2 a + \sin^2 a = 1$*

$(AB)^2 = 2 - 2(\cos \alpha \cos \beta + \sin \alpha \sin \beta)$ *Simplify.*

Set the two expressions for $(AB)^2$ equal to each other.

$2 - 2 \cos (\alpha - \beta) = 2 - 2(\cos \alpha \cos \beta + \sin \alpha \sin \beta)$

$-2 \cos (\alpha - \beta) = -2(\cos \alpha \cos \beta + \sin \alpha \sin \beta)$ *Subtract 2 from each side.*

$\cos (\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$ *Divide each side by −2.*

This equation is known as the **difference identity for cosine.**

The **sum identity for cosine** can be derived by substituting $-\beta$ for β in the difference identity.

$$\begin{aligned}\cos(\alpha + \beta) &= \cos(\alpha - (-\beta))\\ &= \cos\alpha\cos(-\beta) + \sin\alpha\sin(-\beta)\\ &= \cos\alpha\cos\beta + \sin\alpha(-\sin\beta) \quad \textit{cos}(-\beta) = \textit{cos}\,\beta;\ \textit{sin}(-\beta) = -\textit{sin}\,\beta\\ &= \cos\alpha\cos\beta - \sin\alpha\sin\beta\end{aligned}$$

Sum and Difference Identities for the Cosine Function

If α and β represent the measures of two angles, then the following identities hold for all values of α and β.

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

Notice how the addition and subtraction symbols are related in the sum and difference identities.

You can use the sum and difference identities and the values of the trigonometric functions of common angles to find the values of trigonometric functions of other angles. Note that α and β can be expressed in either degrees or radians.

Example 1

a. Show by producing a counterexample that $\cos(x + y) \neq \cos x + \cos y$.

b. Show that the sum identity for cosine is true for the values used in part a.

a. Let $x = \frac{\pi}{3}$ and $y = \frac{\pi}{6}$. First find $\cos(x + y)$ for $x = \frac{\pi}{3}$ and $y = \frac{\pi}{6}$.

$$\begin{aligned}\cos(x + y) &= \cos\left(\frac{\pi}{3} + \frac{\pi}{6}\right) \quad \textit{Replace x with } \frac{\pi}{3} \textit{ and y with } \frac{\pi}{6}.\\ &= \cos\frac{\pi}{2} \quad \frac{\pi}{3} + \frac{\pi}{6} = \frac{\pi}{2}\\ &= 0\end{aligned}$$

Now find $\cos x + \cos y$.

$$\begin{aligned}\cos x + \cos y &= \cos\frac{\pi}{3} + \cos\frac{\pi}{6} \quad \textit{Replace x with } \frac{\pi}{3} \textit{ and y with } \frac{\pi}{6}.\\ &= \frac{1}{2} + \frac{\sqrt{3}}{2} \text{ or } \frac{1 + \sqrt{3}}{2}\end{aligned}$$

So, $\cos(x + y) \neq \cos x + \cos y$.

b. Show that $\cos(x + y) = \cos x\cos y - \sin x\sin y$ for $x = \frac{\pi}{3}$ and $y = \frac{\pi}{6}$.

First find $\cos(x + y)$. From part a, we know that $\cos\left(\frac{\pi}{3} + \frac{\pi}{6}\right) = 0$.

Now find $\cos x\cos y - \sin x\sin y$.

$$\begin{aligned}\cos x\cos y - \sin x\sin y &= \cos\frac{\pi}{3}\cos\frac{\pi}{6} - \sin\frac{\pi}{3}\sin\frac{\pi}{6} \quad \textit{Substitute for x and y.}\\ &= \left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right) - \left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right)\\ &= 0\end{aligned}$$

Thus, the sum identity for cosine is true for $x = \frac{\pi}{3}$ and $y = \frac{\pi}{6}$.

Example 2 **Use the sum or difference identity for cosine to find the exact value of cos 735°.**

$735° = 2(360°) + 15°$ *Symmetry identity, Case 1*

$\cos 735° = \cos 15°$

$\cos 15° = \cos (45° - 30°)$ *45° and 30° are two common angles that differ by 15°.*

$= \cos 45° \cos 30° + \sin 45° \sin 30°$ *Difference identity for cosine*

$$= \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \cdot \frac{1}{2}$$

$$= \frac{\sqrt{6} + \sqrt{2}}{4}$$

Therefore, $\cos 735° = \frac{\sqrt{6} + \sqrt{2}}{4}$.

We can derive sum and difference identities for the sine function from those for the cosine function. Replace α with $\frac{\pi}{2}$ and β with s in the identities for $\cos (\alpha \pm \beta)$. The following equations result.

These equations are examples of cofunction identities.

$$\cos \left(\frac{\pi}{2} + s\right) = -\sin s \qquad \cos \left(\frac{\pi}{2} - s\right) = \sin s$$

Replace s with $\frac{\pi}{2} + s$ in the equation for $\cos \left(\frac{\pi}{2} + s\right)$ and with $\frac{\pi}{2} - s$ in the equation for $\cos \left(\frac{\pi}{2} - s\right)$ to obtain the following equations.

These equations are other cofunction identities.

$$\cos s = \sin \left(\frac{\pi}{2} + s\right) \qquad \cos s = \sin \left(\frac{\pi}{2} - s\right)$$

Replace s with $(\alpha + \beta)$ in the equation for $\cos \left(\frac{\pi}{2} - s\right)$ to derive an identity for the sine of the sum of two real numbers.

$$\cos \left[\frac{\pi}{2} - (\alpha + \beta)\right] = \sin (\alpha + \beta)$$

$$\cos \left[\left(\frac{\pi}{2} - \alpha\right) - \beta\right] = \sin (\alpha + \beta)$$

$$\cos \left(\frac{\pi}{2} - \alpha\right) \cos \beta + \sin \left(\frac{\pi}{2} - \alpha\right) \sin \beta = \sin \left(\alpha + \beta\right)$$ *Identity for cos (α − β)*

$$\sin \alpha \cos \beta + \cos \alpha \sin \beta = \sin (\alpha + \beta)$$ *Substitute.*

This equation is known as the **sum identity for sine.**

The **difference identity for sine** can be derived by replacing β with $(-\beta)$ in the sum identity for sine.

$$\sin [\alpha + (-\beta)] = \sin \alpha \cos (-\beta) + \cos \alpha \sin (-\beta)$$

$$\sin (\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta$$

Sum and Difference Identities for the Sine Function

If α and β represent the measures of two angles, then the following identities hold for all values of α and β.

$$\sin (\alpha \pm \beta) = \sin \alpha \cos \beta \pm \cos \alpha \sin \beta$$

Examples

3 **Find the value of sin $(x - y)$ if $0 < x < \frac{\pi}{2}$, $0 < y < \frac{\pi}{2}$, $\sin x = \frac{9}{41}$, and $\sin y = \frac{7}{25}$.**

In order to use the difference identity for sine, we need to know cos x and cos y. We can use a Pythagorean identity to determine the necessary values.

$\sin^2 \alpha + \cos^2 \alpha = 1 \Rightarrow \cos^2 \alpha = 1 - \sin^2 \alpha$ *Pythagorean identity*

Since we are given that the angles are in Quadrant I, the values of sine and cosine are positive. Therefore, $\cos \alpha = \sqrt{1 - \sin^2 \alpha}$.

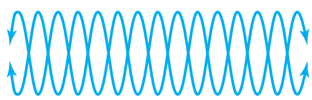
$$\cos x = \sqrt{1 - \left(\frac{9}{41}\right)^2}$$
$$= \sqrt{\frac{1600}{1681}} \text{ or } \frac{40}{41}$$
$$\cos y = \sqrt{1 - \left(\frac{7}{25}\right)^2}$$
$$= \sqrt{\frac{576}{625}} \text{ or } \frac{24}{25}$$

Now substitute these values into the difference identity for sine.

$$\sin (x - y) = \sin x \cos y - \cos x \sin y$$
$$= \left(\frac{9}{41}\right)\left(\frac{24}{25}\right) - \left(\frac{40}{41}\right)\left(\frac{7}{25}\right)$$
$$= -\frac{64}{1025} \text{ or about } -0.0624$$

4 **BROADCASTING** **Refer to the application at the beginning of the lesson. What type of interference results when signals modeled by the equations $y = 20 \sin (3t + 45°)$ and $y = 20 \sin (3t + 225°)$ are combined?**

Real World Application



Add the two sine functions together and simplify.

$$20 \sin (3t + 45°) + 20 \sin (3t + 225°)$$
$$= 20(\sin 3t \cos 45° + \cos 3t \sin 45°) + 20(\sin 3t \cos 225° + \cos 3t \sin 225°)$$
$$= 20\left[(\sin 3t)\left(\frac{\sqrt{2}}{2}\right) + (\cos 3t)\left(\frac{\sqrt{2}}{2}\right)\right] + 20\left[(\sin 3t)\left(-\frac{\sqrt{2}}{2}\right) + (\cos 3t)\left(-\frac{\sqrt{2}}{2}\right)\right]$$
$$= 10\sqrt{2} \sin 3t + 10\sqrt{2} \cos 3t - 10\sqrt{2} \sin 3t - 10\sqrt{2} \cos 3t$$
$$= 0$$

The interference is destructive. The signals cancel each other completely.

You can use the sum and difference identities for the cosine and sine functions to find sum and difference identities for the tangent function.

Sum and Difference Identities for the Tangent Function

If α and β represent the measures of two angles, then the following identities hold for all values of α and β.

$$\tan(\alpha \pm \beta) = \frac{\tan \alpha \pm \tan \beta}{1 \mp \tan \alpha \tan \beta}$$

You will be asked to derive these identities in Exercise 47.

Example 5 **Use the sum or difference identity for tangent to find the exact value of tan 285°.**

$\tan 285° = \tan(240° + 45°)$ — *240° and 45° are common angles whose sum is 285°.*

$= \frac{\tan 240° + \tan 45°}{1 - \tan 240° \tan 45°}$ — *Sum identity for tangent*

$= \frac{\sqrt{3} + 1}{1 - (\sqrt{3})(1)}$ — *Multiply by $\frac{1 + \sqrt{3}}{1 + \sqrt{3}}$ to simplify.*

$= -2 - \sqrt{3}$

You can use sum and difference identities to verify other identities.

Example 6 **Verify that $\csc\left(\frac{3\pi}{2} + A\right) = -\sec A$ is an identity.**

Transform the left side since it is more complicated.

$\csc\left(\frac{3\pi}{2} + A\right) \stackrel{?}{=} -\sec A$

$\frac{1}{\sin\left(\frac{3\pi}{2} + A\right)} \stackrel{?}{=} -\sec A$ — *Reciprocal identity: $\csc x = \frac{1}{\sin x}$*

$\frac{1}{\sin \frac{3\pi}{2} \cos A + \cos \frac{3\pi}{2} \sin A} \stackrel{?}{=} -\sec A$ — *Sum identity for sine*

$\frac{1}{(-1)\cos A + (0)\sin A} \stackrel{?}{=} -\sec A$ — *$\sin \frac{3\pi}{2} = -1$; $\cos \frac{3\pi}{2} = 0$*

$-\frac{1}{\cos A} \stackrel{?}{=} -\sec A$ — *Simplify.*

$-\sec A = -\sec A$ — *Reciprocal identity*

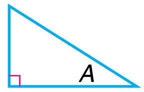
CHECK FOR UNDERSTANDING

Communicating Mathematics

Read and study the lesson to answer each question.

1. **Describe** how you would convince a friend that $\sin(x + y) \neq \sin x + \sin y$.
2. **Explain** how to use the sum and difference identities to find values for the secant, cosecant, and cotangent functions of a sum or difference.

3. **Write** an interpretation of the identity $\sin(90° - A) = \cos A$ in terms of a right triangle.



4. **Derive** a formula for $\cot(\alpha + \beta)$ in terms of $\cot \alpha$ and $\cot \beta$.

Guided Practice

Use sum or difference identities to find the exact value of each trigonometric function.

5. $\cos 165°$
6. $\tan \frac{\pi}{12}$
7. $\sec 795°$

Find each exact value if $0 < x < \frac{\pi}{2}$ and $0 < y < \frac{\pi}{2}$.

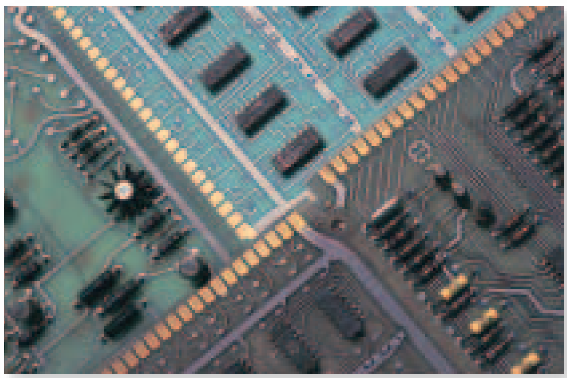
8. $\sin(x - y)$ if $\sin x = \frac{4}{9}$ and $\sin y = \frac{1}{4}$
9. $\tan(x + y)$ if $\csc x = \frac{5}{3}$ and $\cos y = \frac{5}{13}$

Verify that each equation is an identity.

10. $\sin(90° + A) = \cos A$
11. $\tan\left(\theta + \frac{\pi}{2}\right) = -\cot \theta$
12. $\sin(x - y) = \frac{1 - \cot x \tan y}{\csc x \sec y}$

ω is the Greek letter omega.

13. **Electrical Engineering** Analysis of the voltage in certain types of circuits involves terms of the form $\sin(n\omega_0 t - 90°)$, where n is a positive integer, ω_0 is the frequency of the voltage, and t is time. Use an identity to simplify this expression.



EXERCISES

Practice

Use sum or difference identities to find the exact value of each trigonometric function.

14. $\cos 105°$
15. $\sin 165°$
16. $\cos \frac{7\pi}{12}$
17. $\sin \frac{\pi}{12}$
18. $\tan 195°$
19. $\cos\left(-\frac{\pi}{12}\right)$
20. $\tan 165°$
21. $\tan \frac{23\pi}{12}$
22. $\sin 735°$
23. $\sec 1275°$
24. $\csc \frac{5\pi}{12}$
25. $\cot \frac{113\pi}{12}$

Find each exact value if $0 < x < \frac{\pi}{2}$ and $0 < y < \frac{\pi}{2}$.

26. $\sin(x + y)$ if $\cos x = \frac{8}{17}$ and $\sin y = \frac{12}{37}$

27. $\cos(x - y)$ if $\cos x = \frac{3}{5}$ and $\cos y = \frac{4}{5}$

28. $\tan(x - y)$ if $\sin x = \frac{8}{17}$ and $\cos y = \frac{3}{5}$

29. $\cos(x + y)$ if $\tan x = \frac{5}{3}$ and $\sin y = \frac{1}{3}$

30. $\tan(x + y)$ if $\cot x = \frac{6}{5}$ and $\sec y = \frac{3}{2}$

31. $\sec(x - y)$ if $\csc x = \frac{5}{3}$ and $\tan y = \frac{12}{5}$

32. If α and β are two angles in Quadrant I such that $\sin \alpha = \frac{1}{5}$ and $\cos \beta = \frac{2}{7}$, find $\sin(\alpha - \beta)$.

33. If x and y are acute angles such that $\cos x = \frac{1}{3}$ and $\cos y = \frac{3}{4}$, what is the value of $\cos(x + y)$?

Verify that each equation is an identity.

34. $\cos\left(\frac{\pi}{2} + x\right) = -\sin x$

35. $\cos(60° + A) = \sin(30° - A)$

36. $\sin(A + \pi) = -\sin A$

37. $\cos(180° + x) = -\cos x$

38. $\tan(x + 45°) = \frac{1 + \tan x}{1 - \tan x}$

39. $\sin(A + B) = \frac{\tan A + \tan B}{\sec A \sec B}$

40. $\cos(A + B) = \frac{1 - \tan A \tan B}{\sec A \sec B}$

41. $\sec(A - B) = \frac{\sec A \sec B}{1 + \tan A \tan B}$

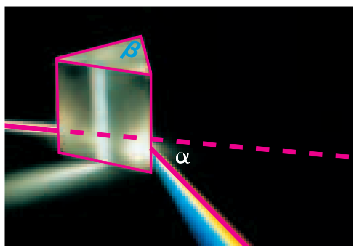
42. $\sin(x + y)\sin(x - y) = \sin^2 x - \sin^2 y$

Applications and Problem Solving

Real World Application

43. Electronics In an electric circuit containing a capacitor, inductor, and resistor the voltage drop across the inductor is given by $V_L = I_0 \omega L \cos\left(\omega t + \frac{\pi}{2}\right)$, where I_0 is the peak current, ω is the frequency, L is the inductance, and t is time. Use the sum identity for cosine to express V_L as a function of $\sin \omega t$.

44. Optics The index of refraction for a medium through which light is passing is the ratio of the velocity of light in free space to the velocity of light in the medium. For light passing symmetrically through a glass prism, the index of refraction n is given by the equation $n = \dfrac{\sin\left[\frac{1}{2}(\alpha + \beta)\right]}{\sin \frac{\beta}{2}}$, where α is the deviation angle and β is the angle of the apex of the prism as shown in the diagram. If $\beta = 60°$, show that $n = \sqrt{3}\sin\frac{\alpha}{2} + \cos\frac{\alpha}{2}$.



45. **Critical Thinking** Simplify the following expression without expanding any of the sums or differences.

$$\sin\left(\frac{\pi}{3} - A\right)\cos\left(\frac{\pi}{3} + A\right) - \cos\left(\frac{\pi}{3} - A\right)\sin\left(\frac{\pi}{3} + A\right)$$

46. **Calculus** In calculus, you will explore the *difference quotient* $\frac{f(x + h) - f(x)}{h}$.

a. Let $f(x) = \sin x$. Write and expand an expression for the difference quotient.

b. Set your answer from part a equal to y. Let $h = 0.1$ and graph.

c. What function has a graph similar to the graph in part b?

47. **Critical Thinking** Derive the sum and difference identities for the tangent function.

48. **Critical Thinking** Consider the following theorem.
If A, B, and C are the angles of a nonright triangle, then $\tan A + \tan B + \tan C = \tan A \tan B \tan C$.

a. Choose values for A, B, and C. Verify that the conclusion is true for your specific values.

b. Prove the theorem.

Mixed Review

49. Verify the identity $\sec^2 x = \frac{1 - \cos^2 x}{1 - \sin^2 x} + \csc^2 x - \cot^2 x$. *(Lesson 7-2)*

50. If $\sin \theta = -\frac{1}{8}$ and $\pi < \theta < \frac{3\pi}{2}$, find $\tan \theta$. *(Lesson 7-1)*

51. Find $\sin\left(\text{Arctan}\sqrt{3}\right)$. *(Lesson 6-8)*

52. Find the values of θ for which $\csc \theta$ is undefined. *(Lesson 6-7)*

53. **Weather** The average seasonal high temperatures for Greensboro, North Carolina, are given in the table. Write a sinusoidal function that models the temperatures, using $t = 1$ to represent winter. *(Lesson 6-6)*

Winter	Spring	Summer	Fall
50°	70°	86°	71°

Source: Rand McNally & Company

54. State the amplitude, period, and phase shift for the function $y = 8 \cos(\theta - 30°)$. *(Lesson 6-5)*

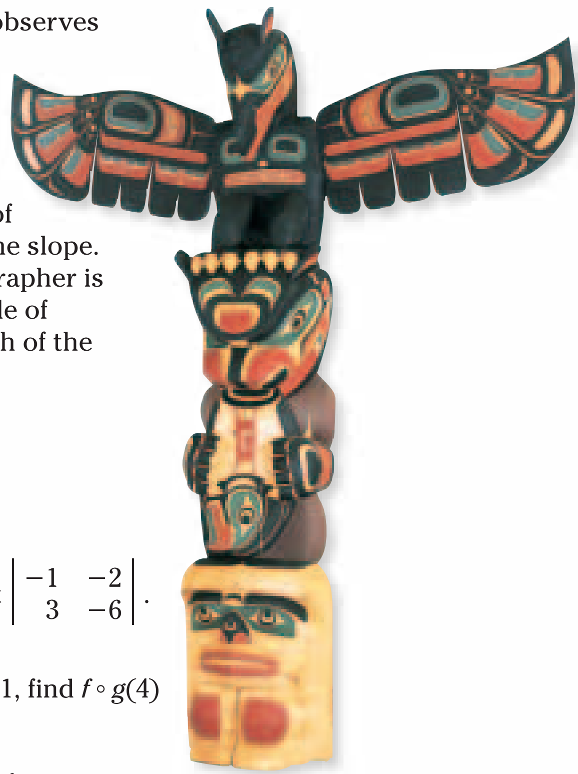
55. Find the value of $\sin(-540°)$. *(Lesson 6-3)*

56. **Geometry** A sector has arc length of 18 feet and a central angle measuring 2.9 radians. Find the radius and the area of the sector. *(Lesson 6-1)*

57. **Navigation** A ship at sea is 70 miles from one radio transmitter and 130 miles from another. The angle formed by the rays from the ship to the transmitters measures 130°. How far apart are the transmitters? *(Lesson 5-8)*

58. Determine the number of possible solutions for a triangle if $A = 120°$, $b = 12$, and $a = 4$. *(Lesson 5-7)*

59. Photography A photographer observes a 35-foot totem pole that stands vertically on a uniformly-sloped hillside and the shadow cast by it at different times of day. At a time when the angle of elevation of the sun is $37°12'$, the shadow of the pole extends directly down the slope. This is the effect that the photographer is seeking. If the hillside has an angle of inclination of $6°40'$, find the length of the shadow. *(Lesson 5-6)*



60. Find the roots of the equation $4x^3 + 3x^2 - x = 0$. *(Lesson 4-1)*

61. Solve $|x + 1| > 4$. *(Lesson 3-3)*

62. Find the value of the determinant $\begin{vmatrix} -1 & -2 \\ 3 & -6 \end{vmatrix}$. *(Lesson 2-5)*

63. If $f(x) = 3x^2 - 4$ and $g(x) = 5x + 1$, find $f \circ g(4)$ and $g \circ f(4)$. *(Lesson 1-2)*

64. SAT Practice What is the value of $(-8)^{62} \div 8^{62}$?

A 1
B 0
C -1
D -8
E -62

MID-CHAPTER QUIZ

Use the given information to determine the exact trigonometric value. (Lesson 7-1)

1. $\sin \theta = \frac{2}{7}, 0 < \theta < \frac{\pi}{2}$; $\cot \theta$

2. $\tan \theta = -\frac{4}{3}, 90° < \theta < 180°$; $\cos \theta$

3. Express $\cos \frac{19\pi}{4}$ as a trigonometric function of an angle in Quadrant I. (Lesson 7-1)

Verify that each equation is an identity. (Lesson 7-2)

4. $\frac{1}{1 + \tan^2 x} + \frac{1}{1 + \cot^2 x} = 1$

5. $\frac{\csc^2 \theta + \sec^2 \theta}{\sec^2 \theta} = \csc^2 \theta$

Verify that each equation is an identity. (Lessons 7-2 and 7-3)

6. $\cot x \sec x \sin x = 2 - \tan x \cos x \csc x$

7. $\tan (\alpha - \beta) = \frac{1 - \cot \alpha \tan \beta}{\cot \alpha + \tan \beta}$

8. Use a sum or difference identity to find the exact value of $\cos 75°$. (Lesson 7-3)

Find each exact value if $0 < x < \frac{\pi}{2}$ and $0 < y < \frac{\pi}{2}$. (Lesson 7-3)

9. $\cos (x + y)$ if $\sin x = \frac{2}{3}$ and $\sin y = \frac{3}{4}$

10. $\tan (x - y)$ if $\tan x = \frac{5}{4}$ and $\sec y = 2$

Extra Practice See p. A38.

GRAPHING CALCULATOR EXPLORATION

7-3B Reduction Identities

OBJECTIVE

- Identify equivalent values for trigonometric functions involving quadrantal angles.

An Extension of Lesson 7-3

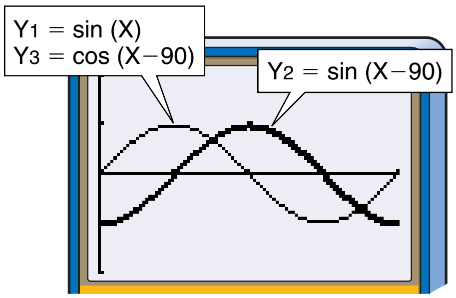
In Chapter 5, you learned that any trigonometric function of an acute angle is equal to the *cofunction* of the complement of the angle. For example, $\sin \alpha = \cos (90° - \alpha)$. This is a part of a large family of identities called the **reduction identities.** These identities involve adding and subtracting the quandrantal angles, 90°, 180°, and 270°, from the angle measure to find equivalent values of the trigonometric function. You can use your knowledge of phase shifts and reflections to find the components of these identities.

Example **Find the values of the sine and cosine functions for $\alpha - 90°$, $\alpha - 180°$, and $\alpha - 270°$ that are equivalent to $\sin \alpha$.**

$\alpha - 90°$

Graph $y = \sin \alpha$, $y = \sin (\alpha - 90°)$, and $y = \cos (\alpha - 90°)$, letting X in degree mode represent α. *You can select different display formats to help you distinguish the three graphs.*

You may recall from Chapter 6 that a phase shift of 90° right for the cosine function results in the sine function.

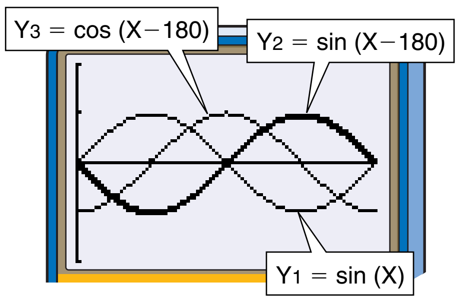


[0, 360] scl:90 by [−2, 2] scl:1

Note that the graph of $y = \cos (X - 90)$ is the same as the graph of $y = \sin X$. This suggests that $\sin \alpha = \cos (\alpha - 90°)$. *Remember that an identity must be proved algebraically. A graph does not prove an identity.*

$\alpha - 180°$

Graph $y = \sin \alpha$, $y = \sin (\alpha - 180°)$, and $y = \cos (\alpha - 180°)$ using X to represent α.



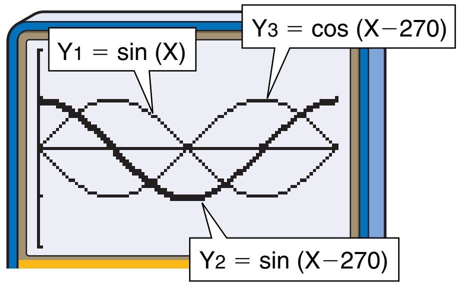
[0, 360] scl:90 by [−2, 2] scl:1

Discount $y = \cos (\alpha - 180°)$ as a possible equivalence because it would involve a phase shift, which would change the actual value of the angle being considered.

Note that the graph of $\sin (\alpha - 180°)$ is a mirror reflection of $\sin \alpha$. Remember that a reflection over the x-axis results in the mapping $(x, y) \rightarrow (x, -y)$. So to obtain a graph that is identical to $y = \sin \alpha$, we need the reflection of $y = \sin (\alpha - 180°)$ over the x-axis, or $y = -\sin (\alpha - 180°)$. Thus, $\sin \alpha = -\sin (\alpha - 180°)$. Graph the two equations to investigate this equality.

$\alpha - 270°$

In this case, $\sin(\alpha - 270°)$ is a phase shift, so ignore it. The graph of $\cos(\alpha - 270°)$ is a reflection of $\sin \alpha$ over the x-axis. So, $\sin \alpha = -\cos(\alpha - 270°)$.



[0, 360] scl:90 by [−2, 2] scl:1

The family of reduction identities also contains the relationships among the other cofunctions of tangent and cotangent and secant and cosecant. In addition to $\alpha - 90°$, $\alpha - 180°$, and $\alpha - 270°$ angle measures, the reduction identities address other measures such as $90° \pm \alpha$, $180° \pm \alpha$, $270° \pm \alpha$, and $360° \pm \alpha$.

TRY THESE

Copy and complete each statement with the proper trigonometric functions.

1. $\cos \alpha = \underline{\ ?\ }(\alpha - 90°) = \underline{\ ?\ }(\alpha - 180°) = \underline{\ ?\ }(\alpha - 270°)$

2. $\tan \alpha = \underline{\ ?\ }(\alpha - 90°) = \underline{\ ?\ }(\alpha - 180°) = \underline{\ ?\ }(\alpha - 270°)$

3. $\cot \alpha = \underline{\ ?\ }(\alpha - 90°) = \underline{\ ?\ }(\alpha - 180°) = \underline{\ ?\ }(\alpha - 270°)$

4. $\sec \alpha = \underline{\ ?\ }(\alpha - 90°) = \underline{\ ?\ }(\alpha - 180°) = \underline{\ ?\ }(\alpha - 270°)$

5. $\csc \alpha = \underline{\ ?\ }(\alpha - 90°) = \underline{\ ?\ }(\alpha - 180°) = \underline{\ ?\ }(\alpha - 270°)$

WHAT DO YOU THINK?

6. Suppose the expressions involving subtraction in Exercises 1-5 were changed to sums.

a. Copy and complete each statement with the proper trigonometric functions.

(1) $\sin \alpha = \underline{\ ?\ }(\alpha + 90°) = \underline{\ ?\ }(\alpha + 180°) = \underline{\ ?\ }(\alpha + 270°)$

(2) $\cos \alpha = \underline{\ ?\ }(\alpha + 90°) = \underline{\ ?\ }(\alpha + 180°) = \underline{\ ?\ }(\alpha + 270°)$

(3) $\tan \alpha = \underline{\ ?\ }(\alpha + 90°) = \underline{\ ?\ }(\alpha + 180°) = \underline{\ ?\ }(\alpha + 270°)$

(4) $\cot \alpha = \underline{\ ?\ }(\alpha + 90°) = \underline{\ ?\ }(\alpha + 180°) = \underline{\ ?\ }(\alpha + 270°)$

(5) $\sec \alpha = \underline{\ ?\ }(\alpha + 90°) = \underline{\ ?\ }(\alpha + 180°) = \underline{\ ?\ }(\alpha + 270°)$

(6) $\csc \alpha = \underline{\ ?\ }(\alpha + 90°) = \underline{\ ?\ }(\alpha + 180°) = \underline{\ ?\ }(\alpha + 270°)$

b. How do the identities in part a compare to those in Exercises 1-5?

7. a. Copy and complete each statement with the proper trigonometric functions.

(1) $\sin \alpha = \underline{\ ?\ }(90° - \alpha) = \underline{\ ?\ }(180° - \alpha) = \underline{\ ?\ }(270° - \alpha)$

(2) $\cos \alpha = \underline{\ ?\ }(90° - \alpha) = \underline{\ ?\ }(180° - \alpha) = \underline{\ ?\ }(270° - \alpha)$

(3) $\tan \alpha = \underline{\ ?\ }(90° - \alpha) = \underline{\ ?\ }(180° - \alpha) = \underline{\ ?\ }(270° - \alpha)$

(4) $\cot \alpha = \underline{\ ?\ }(90° - \alpha) = \underline{\ ?\ }(180° - \alpha) = \underline{\ ?\ }(270° - \alpha)$

(5) $\sec \alpha = \underline{\ ?\ }(90° - \alpha) = \underline{\ ?\ }(180° - \alpha) = \underline{\ ?\ }(270° - \alpha)$

(6) $\csc \alpha = \underline{\ ?\ }(90° - \alpha) = \underline{\ ?\ }(180° - \alpha) = \underline{\ ?\ }(270° - \alpha)$

b. How do the identities in part a compare to those in Exercise 6a?

8. a. How did reduction identities get their name?

b. If you needed one of these identities, but could not remember it, what other type(s) of identities could you use to derive it?