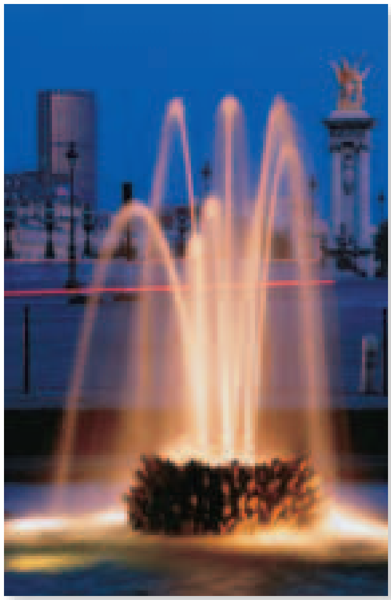
7-4

Double-Angle and Half-Angle Identities

OBJECTIVE

- Use the double- and half-angle identities for the sine, cosine, and tangent functions.

Real World Application



ARCHITECTURE Mike MacDonald is an architect who designs water fountains. One part of his job is determining the placement of the water jets that shoot the water into the air to create arcs. These arcs are modeled by parabolic functions. When a stream of water is shot into the air with velocity v at an angle of θ with the horizontal, the model predicts that the water will travel a horizontal distance of $D = \frac{v^2}{g} \sin 2\theta$ and reach a maximum height of $H = \frac{v^2}{2g} \sin^2 \theta$, where g is the acceleration due to gravity. The ratio of H to D helps determine the total height and width of the fountain. Express $\frac{H}{D}$ as a function of θ. *This problem will be solved in Example 3.*

It is sometimes useful to have identities to find the value of a function of twice an angle or half an angle. We can substitute θ for both α and β in $\sin(\alpha + \beta)$ to find an identity for $\sin 2\theta$.

$$\begin{aligned} \sin 2\theta &= \sin(\theta + \theta) \\ &= \sin\theta\cos\theta + \cos\theta\sin\theta && \textit{Sum identity for sine} \\ &= 2\sin\theta\cos\theta \end{aligned}$$

The same method can be used to find an identity for $\cos 2\theta$.

$$\begin{aligned} \cos 2\theta &= \cos(\theta + \theta) \\ &= \cos\theta\cos\theta - \sin\theta\sin\theta && \textit{Sum identity for cosine} \\ &= \cos^2\theta - \sin^2\theta \end{aligned}$$

If we substitute $1 - \cos^2\theta$ for $\sin^2\theta$ or $1 - \sin^2\theta$ for $\cos^2\theta$, we will have two alternate identities for $\cos 2\theta$.

$$\cos 2\theta = 2\cos^2\theta - 1$$

$$\cos 2\theta = 1 - 2\sin^2\theta$$

These identities may be used if θ is measured in degrees or radians. So, θ may represent either a degree measure or a real number.

The tangent of a double angle can be found by substituting θ for both α and β in $\tan(\alpha + \beta)$.

$$\tan 2\theta = \tan(\theta + \theta)$$

$$= \frac{\tan\theta + \tan\theta}{1 - \tan\theta\tan\theta} \quad \textit{Sum identity for tangent}$$

$$= \frac{2\tan\theta}{1 - \tan^2\theta}$$

Double-Angle Identities

If θ represents the measure of an angle, then the following identities hold for all values of θ.

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$\cos 2\theta = 2\cos^2\theta - 1$$

$$\cos 2\theta = 1 - 2\sin^2\theta$$

$$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$$

Example 1 **If $\sin\theta = \frac{2}{3}$ and θ has its terminal side in the first quadrant, find the exact value of each function.**

a. $\sin 2\theta$

To use the double-angle identity for $\sin 2\theta$, we must first find $\cos\theta$.

$$\sin^2\theta + \cos^2\theta = 1$$

$$\left(\frac{2}{3}\right)^2 + \cos^2\theta = 1 \quad \sin\theta = \frac{2}{3}$$

$$\cos^2\theta = \frac{5}{9}$$

$$\cos\theta = \frac{\sqrt{5}}{3}$$

Then find $\sin 2\theta$.

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$= 2\left(\frac{2}{3}\right)\left(\frac{\sqrt{5}}{3}\right) \quad \sin\theta = \frac{2}{3};\ \cos\theta = \frac{\sqrt{5}}{3}$$

$$= \frac{4\sqrt{5}}{9}$$

b. $\cos 2\theta$

Since we know the values of $\cos\theta$ and $\sin\theta$, we can use any of the double-angle identities for cosine.

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$= \left(\frac{\sqrt{5}}{3}\right)^2 - \left(\frac{2}{3}\right)^2 \quad \cos\theta = \frac{\sqrt{5}}{3};\ \sin\theta = \frac{2}{3}$$

$$= \frac{1}{9}$$

c. tan 2θ

We must find tan θ to use the double-angle identity for tan 2θ.

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$= \frac{\frac{2}{3}}{\frac{\sqrt{5}}{3}} \quad \textit{\sin\theta = \frac{2}{3}, \cos\theta = \frac{\sqrt{5}}{3}}$$

$$= \frac{2}{\sqrt{5}} \text{ or } \frac{2\sqrt{5}}{5}$$

Then find tan 2θ.

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$$

$$= \frac{2\left(\frac{2\sqrt{5}}{5}\right)}{1-\left(\frac{2\sqrt{5}}{5}\right)^2} \quad \tan\theta = \frac{2\sqrt{5}}{5}$$

$$= \frac{\frac{4\sqrt{5}}{5}}{\frac{1}{5}} \text{ or } 4\sqrt{5}$$

d. cos 4θ

Since $4\theta = 2(2\theta)$, use a double-angle identity for cosine again.

$$\cos 4\theta = \cos 2(2\theta)$$

$$= \cos^2(2\theta) - \sin^2(2\theta) \quad \textit{Double-angle identity}$$

$$= \left(\frac{1}{9}\right)^2 - \left(\frac{4\sqrt{5}}{9}\right)^2 \quad \cos 2\theta = \frac{1}{9}, \sin 2\theta = \frac{4\sqrt{5}}{9} \text{ (parts a and b)}$$

$$= -\frac{79}{81}$$

We can solve two of the forms of the identity for cos 2θ for cos θ and sin θ, respectively, and the following equations result.

$\cos 2\theta = 2\cos^2\theta - 1$ **Solve for cos θ.** $\cos\theta = \pm\sqrt{\frac{1+\cos 2\theta}{2}}$

$\cos 2\theta = 1 - 2\sin^2\theta$ **Solve for sin θ.** $\sin\theta = \pm\sqrt{\frac{1-\cos 2\theta}{2}}$

We can replace 2θ with α and θ with $\frac{\alpha}{2}$ to derive the half-angle identities.

$$\tan \frac{\alpha}{2} = \frac{\sin \frac{\alpha}{2}}{\cos \frac{\alpha}{2}}$$

$$= \frac{\pm\sqrt{\frac{1 - \cos \alpha}{2}}}{\pm\sqrt{\frac{1 + \cos \alpha}{2}}} \text{ or } \pm\sqrt{\frac{1 - \cos \alpha}{1 + \cos \alpha}}$$

Half-Angle Identities

If α represents the measure of an angle, then the following identities hold for all values of α.

$$\sin \frac{\alpha}{2} = \pm\sqrt{\frac{1 - \cos \alpha}{2}}$$

$$\cos \frac{\alpha}{2} = \pm\sqrt{\frac{1 + \cos \alpha}{2}}$$

$$\tan \frac{\alpha}{2} = \pm\sqrt{\frac{1 - \cos \alpha}{1 + \cos \alpha}}, \cos \alpha \neq -1$$

Unlike with the double-angles identities, you must determine the sign.

Example 2 **Use a half-angle identity to find the exact value of each function.**

a. $\sin \frac{7\pi}{12}$

$$\sin \frac{7\pi}{12} = \sin \frac{\frac{7\pi}{6}}{2}$$

$$= \sqrt{\frac{1 - \cos \frac{7\pi}{6}}{2}}$$

Use $\sin \frac{\alpha}{2} = \pm\sqrt{\frac{1 - \cos \alpha}{2}}$. Since $\frac{7\pi}{12}$ is in Quadrant II, choose the positive sine value.

$$= \sqrt{\frac{1 - \left(-\frac{\sqrt{3}}{2}\right)}{2}}$$

$$= \frac{\sqrt{2 + \sqrt{3}}}{2}$$

b. $\cos 67.5°$

$$\cos 67.5° = \cos \frac{135°}{2}$$

$$= \sqrt{\frac{1 + \cos 135°}{2}}$$

Use $\cos \frac{\alpha}{2} = \pm\sqrt{\frac{1 + \cos \alpha}{2}}$. Since 67.5° is in Quadrant I, choose the positive cosine value.

$$= \sqrt{\frac{1 - \frac{\sqrt{2}}{2}}{2}}$$

$$= \frac{\sqrt{2 - \sqrt{2}}}{2}$$

Double- and half-angle identities can be used to simplify trigonometric expressions.

Example 3 **ARCHITECTURE** **Refer to the application at the beginning of the lesson.**

Real World Application

a. Find and simplify $\frac{H}{D}$.

b. What is the ratio of the maximum height of the water to the horizontal distance it travels for an angle of 27°?

a. $\frac{H}{D} = \frac{\frac{v^2}{2g}\sin^2\theta}{\frac{v^2}{g}\sin 2\theta}$

$= \frac{\sin^2\theta}{2\sin 2\theta}$ *Simplify.*

$= \frac{\sin^2\theta}{4\sin\theta\cos\theta}$ *$\sin 2\theta = 2\sin\theta\cos\theta$*

$= \frac{1}{4}\cdot\frac{\sin\theta}{\cos\theta}$ *Simplify.*

$= \frac{1}{4}\tan\theta$ *Quotient identity: $\frac{\sin\theta}{\cos\theta} = \tan\theta$*

Therefore, the ratio of the maximum height of the water to the horizontal distance it travels is $\frac{1}{4}\tan\theta$.

b. When $\theta = 27°$, $\frac{H}{D} = \frac{1}{4}\tan 27°$, or about 0.13.

For an angle of 27°, the ratio of the maximum height of the water to the horizontal distance it travels is about 0.13.

The double- and half-angle identities can also be used to verify other identities.

Example 4 **Verify that $\frac{\cos 2\theta}{1+\sin 2\theta} = \frac{\cot\theta - 1}{\cot\theta + 1}$ is an identity.**

$\frac{\cos 2\theta}{1+\sin 2\theta} \stackrel{?}{=} \frac{\cot\theta - 1}{\cot\theta + 1}$

$\frac{\cos 2\theta}{1+\sin 2\theta} \stackrel{?}{=} \frac{\frac{\cos\theta}{\sin\theta} - 1}{\frac{\cos\theta}{\sin\theta} + 1}$ *Reciprocal identity: $\cot\theta = \frac{\cos\theta}{\sin\theta}$*

$\frac{\cos 2\theta}{1+\sin 2\theta} \stackrel{?}{=} \frac{\cos\theta - \sin\theta}{\cos\theta + \sin\theta}$ *Multiply numerator and denominator by $\sin\theta$.*

$\frac{\cos 2\theta}{1+\sin 2\theta} \stackrel{?}{=} \frac{\cos\theta - \sin\theta}{\cos\theta + \sin\theta}\cdot\frac{\cos\theta + \sin\theta}{\cos\theta + \sin\theta}$ *Multiply each side by 1.*

$\frac{\cos 2\theta}{1+\sin 2\theta} \stackrel{?}{=} \frac{\cos^2\theta - \sin^2\theta}{\cos^2\theta + 2\cos\theta\sin\theta + \sin^2\theta}$ *Multiply.*

$$\frac{\cos 2\theta}{1 + \sin 2\theta} \stackrel{?}{=} \frac{\cos^2\theta - \sin^2\theta}{1 + 2\cos\theta\sin\theta}$$ *Simplify.*

$$\frac{\cos 2\theta}{1 + \sin 2\theta} = \frac{\cos 2\theta}{1 + \sin 2\theta}$$ *Double-angle identities:* $\cos^2\theta - \sin^2\theta = \cos 2\theta$, $2\cos\theta\sin\theta = \sin 2\theta$

CHECK FOR UNDERSTANDING

Communicating Mathematics

Read and study the lesson to answer each question.

1. **Write** a paragraph about the conditions under which you would use each of the three identities for $\cos 2\theta$.
2. **Derive** the identity $\sin\frac{\alpha}{2} = \pm\sqrt{\frac{1 - \cos\alpha}{2}}$ from $\cos 2\theta = 1 - 2\sin^2\theta$.
3. **Name** the quadrant in which the terminal side lies.
   a. x is a second quadrant angle. In which quadrant does $2x$ lie?
   b. $\frac{x}{2}$ is a first quadrant angle. In which quadrant does x lie?
   c. $2x$ is a second quadrant angle. In which quadrant does $\frac{x}{2}$ lie?
4. **Provide a counterexample** to show that $\sin 2\theta = 2\sin\theta$ is not an identity.
5. **You Decide** Tamika calculated the exact value of $\sin 15°$ in two different ways. Using the difference identity for sine, $\sin 15°$ was $\frac{\sqrt{6} - \sqrt{2}}{4}$. When she used the half-angle identity, $\sin 15°$ equaled $\frac{\sqrt{2 - \sqrt{3}}}{2}$. Which answer is correct? Explain.

Guided Practice

Use a half-angle identity to find the exact value of each function.

6. $\sin\frac{\pi}{8}$
7. $\tan 165°$

Use the given information to find $\sin 2\theta$, $\cos 2\theta$, and $\tan 2\theta$.

8. $\sin\theta = \frac{2}{5}$, $0° < \theta < 90°$
9. $\tan\theta = \frac{4}{3}$, $\pi < \theta < \frac{3\pi}{2}$

Verify that each equation is an identity.

10. $\tan 2\theta = \frac{2}{\cot\theta - \tan\theta}$
11. $1 + \frac{1}{2}\sin 2A = \frac{\sec A + \sin A}{\sec A}$
12. $\sin\frac{x}{2}\cos\frac{x}{2} = \frac{\sin x}{2}$

13. **Electronics** Consider an AC circuit consisting of a power supply and a resistor. If the current in the circuit at time t is $I_0 \sin \omega t$, then the power delivered to the resistor is $P = I_0^2 R \sin^2 \omega t$, where R is the resistance. Express the power in terms of $\cos 2\omega t$.

EXERCISES

Practice

Use a half-angle identity to find the exact value of each function.

14. $\cos 15°$

15. $\sin 75°$

16. $\tan \frac{5\pi}{12}$

17. $\sin \frac{3\pi}{8}$

18. $\cos \frac{7\pi}{12}$

19. $\tan 22.5°$

20. If θ is an angle in the first quadrant and $\cos \theta = \frac{1}{4}$, find $\tan \frac{\theta}{2}$.

Use the given information to find $\sin 2\theta$, $\cos 2\theta$, and $\tan 2\theta$.

21. $\cos \theta = \frac{4}{5}$, $0° < \theta < 90°$

22. $\sin \theta = \frac{1}{3}$, $0 < \theta < \frac{\pi}{2}$

23. $\tan \theta = -2$, $\frac{\pi}{2} < \theta < \pi$

24. $\sec \theta = -\frac{4}{3}$, $90° < \theta < 180°$

25. $\cot \theta = \frac{3}{2}$, $180° < \theta < 270°$

26. $\csc \theta = -\frac{5}{2}$, $\frac{3\pi}{2} < \theta < 2\pi$

27. If α is an angle in the second quadrant and $\cos \alpha = -\frac{\sqrt{2}}{3}$, find $\tan 2\alpha$.

Verify that each equation is an identity.

28. $\csc 2\theta = \frac{1}{2} \sec \theta \csc \theta$

29. $\cos A - \sin A = \frac{\cos 2A}{\cos A + \sin A}$

30. $(\sin \theta + \cos \theta)^2 - 1 = \sin 2\theta$

31. $\cos x - 1 = \frac{\cos 2x - 1}{2(\cos x + 1)}$

32. $\sec 2\theta = \frac{\cos^2 \theta + \sin^2 \theta}{\cos^2 \theta - \sin^2 \theta}$

33. $\tan \frac{A}{2} = \frac{\sin A}{1 + \cos A}$

34. $\sin 3x = 3 \sin x - 4 \sin^3 x$

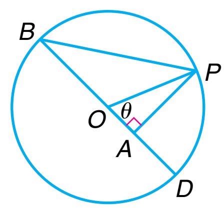
35. $\cos 3x = 4 \cos^3 x - 3 \cos x$

Applications and Problem Solving

Real World Application

36. Architecture Refer to the application at the beginning of the lesson. If the angle of the water is doubled, what is the ratio of the new maximum height to the original maximum height?

37. Critical Thinking Circle O is a unit circle. Use the figure to prove that $\tan \frac{1}{2}\theta = \frac{\sin \theta}{1 + \cos \theta}$.



38. Physics Suppose a projectile is launched with velocity v at an angle θ to the horizontal from the base of a hill that makes an angle α with the horizontal ($\theta > \alpha$). Then the range of the projectile, measured along the slope of the hill, is given by $R = \frac{2v^2 \cos \theta \sin (\theta - \alpha)}{g \cos^2 \alpha}$. Show that if $\alpha = 45°$, then $R = \frac{v^2\sqrt{2}}{g}(\sin 2\theta - \cos 2\theta - 1)$.

interNET CONNECTION

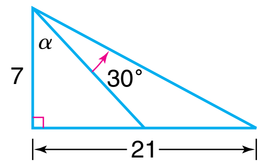
Research For the latitude and longitude of world cities, and the distance between them, visit: **www.amc.glencoe.com**

39. Geography The Mercator projection of the globe is a projection on which the distance between the lines of latitude increases with their distance from the equator. The calculation of the location of a point on this projection involves the expression $\tan\left(45° + \frac{L}{2}\right)$, where L is the latitude of the point.

a. Write this expression in terms of a trigonometric function of L.

b. Find the value of this expression if $L = 60°$.

40. Critical Thinking Determine the tangent of angle α in the figure.



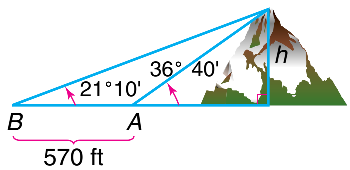
Mixed Review

41. Find the exact value of $\sec \frac{\pi}{12}$. *(Lesson 7-3)*

42. Show that $\sin x^2 + \cos x^2 = 1$ is not an identity. *(Lesson 7-1)*

43. Find the degree measure to the nearest tenth of the central angle of a circle of radius 10 centimeters if the measure of the subtended arc is 17 centimeters. *(Lesson 6-1)*

44. Surveying To find the height of a mountain peak, points A and B were located on a plain in line with the peak, and the angle of elevation was measured from each point. The angle at A was $36°40'$, and the angle at B was $21°10'$. The distance from A to B was 570 feet. How high is the peak above the level of the plain? *(Lesson 5-4)*



45. Write a polynomial equation of least degree with roots −3, 0.5, 6, and 2. *(Lesson 4-1)*

46. Graph $y = 2x + 5$ and its inverse. *(Lesson 3-4)*

47. Solve the system of equations. *(Lesson 2-1)*
$x + 2y = 11$
$3x - 5y = 11$

48. SAT Practice **Grid-In** If $(a - b)^2 = 64$, and $ab = 3$, find $a^2 + b^2$.

Extra Practice See p. A39.