7-5

Solving Trigonometric Equations

OBJECTIVE

- Solve trigonometric equations and inequalities.

Real World Application

ENTERTAINMENT When you ride a Ferris wheel that has a diameter of 40 meters and turns at a rate of 1.5 revolutions per minute, the height above the ground, in meters, of your seat after t minutes can be modeled by the equation $h = 21 - 20 \cos 3\pi t$. How long after the ride starts will your seat first be 31 meters above the ground? *This problem will be solved in Example 4.*

So far in this chapter, we have studied a special type of trigonometric equation called an identity. Trigonometric identities are equations that are true for all values of the variable for which both sides are defined. In this lesson, we will examine another type of trigonometric equation. These equations are true for only certain values of the variable. Solving these equations resembles solving algebraic equations.

Most trigonometric equations have more than one solution. If the variable is not restricted, the periodic nature of trigonometric functions will result in an infinite number of solutions. Also, many trigonometric expressions will take on a given value twice every period.

If the variable is restricted to two adjacent quadrants, a trigonometric equation will have fewer solutions. These solutions are called **principal values.** For $\sin x$ and $\tan x$, the principal values are in Quadrants I and IV. So x is in the interval $-90° \leq x \leq 90°$. For $\cos x$, the principal values are in Quadrants I and II, so x is in the interval $0° \leq x \leq 180°$.

Example 1 **Solve $\sin x \cos x - \frac{1}{2} \cos x = 0$ for principal values of x. Express solutions in degrees.**

$$\sin x \cos x - \frac{1}{2}\cos x = 0$$

$$\cos x\left(\sin x - \frac{1}{2}\right) = 0 \quad \textit{Factor.}$$

$\cos x = 0$ or $\sin x - \frac{1}{2} = 0$ *Set each factor equal to 0.*

$x = 90°$

$\sin x = \frac{1}{2}$

$x = 30°$

The principal values are 30° and 90°.

If an equation cannot be solved easily by factoring, try writing the expressions in terms of only one trigonometric function. Remember to use your knowledge of identities.

Example 2 **Solve $\cos^2 x - \cos x + 1 = \sin^2 x$ for $0 \le x < 2\pi$.**

This equation can be written in terms of cos x only.

$\cos^2 x - \cos x + 1 = \sin^2 x$

$\cos^2 x - \cos x + 1 = 1 - \cos^2 x$ *Pythagorean identity: $\sin^2 x = 1 - \cos^2 x$*

$2\cos^2 x - \cos x = 0$

$\cos x(2\cos x - 1) = 0$ *Factor.*

$\cos x = 0$ or $2\cos x - 1 = 0$

$x = \frac{\pi}{2}$ or $x = \frac{3\pi}{2}$ $\quad$ $\cos x = \frac{1}{2}$

$x = \frac{\pi}{3}$ or $x = \frac{5\pi}{3}$

The solutions are $\frac{\pi}{3}$, $\frac{\pi}{2}$, $\frac{3\pi}{2}$, and $\frac{5\pi}{3}$.

As indicated earlier, most trigonometric equations have infinitely many solutions. When all of the values of x are required, the solution should be represented as $x + 360k°$ or $x + 2\pi k$ for sin x and cos x and $x + 180k°$ or $x + \pi k$ for tan x, where k is any integer.

Example 3 **Solve $2\sec^2 x - \tan^4 x = -1$ for all real values of x.**

A Pythagorean identity can be used to write this equation in terms of tan x only.

$2\sec^2 x - \tan^4 x = -1$

$2(1 + \tan^2 x) - \tan^4 x = -1$ *Pythagorean identity: $\sec^2 x = 1 + \tan^2 x$*

$2 + 2\tan^2 x - \tan^4 x = -1$ *Simplify.*

$\tan^4 x - 2\tan^2 x - 3 = 0$

$(\tan^2 x - 3)(\tan^2 x + 1) = 0$ *Factor.*

$\tan^2 x - 3 = 0$ or $\tan^2 x + 1 = 0$

$\tan^2 x = 3$ $\quad$ $\tan^2 x = -1$ *This part gives no solutions since $\tan^2 x \ge 0$.*

$\tan x = \pm\sqrt{3}$

$x = \frac{\pi}{3} + \pi k$ or $x = -\frac{\pi}{3} + \pi k$, where k is any integer.

When a problem asks for real values of x, use radians.

The solutions are $\frac{\pi}{3} + \pi k$ and $-\frac{\pi}{3} + \pi k$.

There are times when a general expression for all of the solutions is helpful for determining a specific solution.

Example 4 (Real World Application) **ENTERTAINMENT** **Refer to the application at the beginning of the lesson. How long after the Ferris wheel starts will your seat first be 31 meters above the ground?**

$h = 21 - 20 \cos 3\pi t$

$31 = 21 - 20 \cos 3\pi t$ *Replace h with 31.*

$-\frac{1}{2} = \cos 3\pi t$

$\frac{2\pi}{3} + 2\pi k = 3\pi t$ or $\frac{4\pi}{3} + 2\pi k = 3\pi t$ *where k is any integer*

$\frac{2}{9} + \frac{2}{3}k = t$ or $\frac{4}{9} + \frac{2}{3}k = t$

The least positive value for t is obtained by letting $k = 0$ in the first expression. Therefore, $t = \frac{2}{9}$ of a minute or about 13 seconds.

You can solve some trigonometric inequalities using the same techniques as for algebraic inequalities. The unit circle can be useful when deciding which angles to include in the answer.

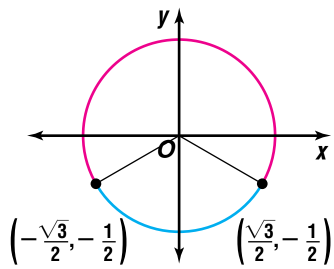
Example 5 **Solve $2 \sin \theta + 1 > 0$ for $0 \leq \theta < 2\pi$.**

$2 \sin \theta + 1 > 0$

$\sin \theta > -\frac{1}{2}$ *Solve for sin θ.*

In terms of the unit circle, we need to find points with y-coordinates greater than $-\frac{1}{2}$.

The values of θ for which $\sin \theta = -\frac{1}{2}$ are $\frac{7\pi}{6}$ and $\frac{11\pi}{6}$. The figure shows that the solution of the inequality is $0 \leq \theta < \frac{7\pi}{6}$ or $\frac{11\pi}{6} < \theta < 2\pi$.



GRAPHING CALCULATOR EXPLORATION

Some trigonometric equations and inequalities are difficult or impossible to solve with only algebraic methods. A graphing calculator is helpful in such cases.

TRY THESE **Graph each side of the equation as a separate function.**

1. $\sin x = 2 \cos x$ for $0 \leq x \leq 2\pi$
2. $\tan 0.5x = \cos x$ for $-2\pi \leq x \leq 2\pi$
3. Use the **CALC** menu to find the intersection point(s) of the graphs in Exercises 1 and 2.

WHAT DO YOU THINK?

4. What do the values in Exercise 3 represent? How could you verify this conjecture?
5. Graph $y = 2 \cos x - \sin x$ for $0 \leq x \leq 2\pi$.
   a. How could you use the graph to solve the equation $\sin x = 2 \cos x$? How does this solution compare with those found in Exercise 3?
   b. What equation would you use to apply this method to $\tan 0.5x = \cos x$?

CHECK FOR UNDERSTANDING

Communicating Mathematics

Read and study the lesson to answer each question.

1. **Explain** the difference between a trigonometric identity and a trigonometric equation that is not an identity.
2. **Explain** why many trigonometric equations have infinitely many solutions.
3. **Write** all the solutions to a trigonometric equation in terms of sin x, given that the solutions between $0°$ and $360°$ are $45°$ and $135°$.
4. *Math Journal* **Compare and contrast** solving trigonometric equations with solving linear and quadratic equations. What techniques are the same? What techniques are different? How many solutions do you expect? Do you use a graphing calculator in a similar manner?

Guided Practice

Solve each equation for principal values of x. Express solutions in degrees.

5. $2 \sin x + 1 = 0$
6. $2 \cos x - \sqrt{3} = 0$

Solve each equation for $0° \le x < 360°$.

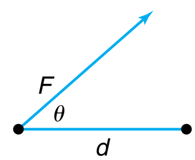
7. $\sin x \cot x = \frac{\sqrt{3}}{2}$
8. $\cos 2x = \sin^2 x - 2$

Solve each equation for $0 \le x < 2\pi$.

9. $3 \tan^2 x - 1 = 0$
10. $2 \sin^2 x = 5 \sin x + 3$

Solve each equation for all real values of x.

11. $\sin^2 2x + \cos^2 x = 0$
12. $\tan^2 x + 2 \tan x + 1 = 0$
13. $\cos^2 x + 3 \cos x = -2$
14. $\sin 2x - \cos x = 0$
15. Solve $2 \cos \theta + 1 < 0$ for $0 \le \theta < 2\pi$.
16. **Physics** The work done in moving an object through a displacement of d meters is given by $W = Fd \cos \theta$, where θ is the angle between the displacement and the force F exerted. If Lisa does 1500 joules of work while exerting a 100-newton force over 20 meters, at what angle was she exerting the force?



EXERCISES

Practice

Solve each equation for principal values of x. Express solutions in degrees.

17. $\sqrt{2} \sin x - 1 = 0$
18. $2 \cos x + 1 = 0$
19. $\sin 2x - 1 = 0$
20. $\tan 2x - \sqrt{3} = 0$
21. $\cos^2 x = \cos x$
22. $\sin x = 1 + \cos^2 x$

Solve each equation for $0° \le x < 360°$.

23. $\sqrt{2} \cos x + 1 = 0$
24. $\cos x \tan x = \frac{1}{2}$
25. $\sin x \tan x - \sin x = 0$
26. $2 \cos^2 x + 3 \cos x - 2 = 0$
27. $\sin 2x = -\sin x$
28. $\cos (x + 45°) + \cos (x - 45°) = \sqrt{2}$

29. Find all solutions to $2 \sin \theta \cos \theta + \sqrt{3} \sin \theta = 0$ in the interval $0° \le \theta < 360°$.

Solve each equation for $0 \le x < 2\pi$.

30. $(2 \sin x - 1)(2 \cos^2 x - 1) = 0$

31. $4 \sin^2 x + 1 = -4 \sin x$

32. $\sqrt{2} \tan x = 2 \sin x$

33. $\sin x = \cos 2x - 1$

34. $\cot^2 x - \csc x = 1$

35. $\sin x + \cos x = 0$

36. Find all values of θ between 0 and 2π that satisfy $-1 - 3 \sin \theta = \cos 2\theta$.

Solve each equation for all real values of x.

37. $\sin x = -\frac{1}{2}$

38. $\cos x \tan x - 2 \cos^2 x = -1$

39. $3 \tan^2 x = \sqrt{3} \tan x$

40. $2 \cos^2 x = 3 \sin x$

41. $\frac{1}{\cos x - \sin x} = \cos x + \sin x$

42. $2 \tan^2 x - 3 \sec x = 0$

43. $\sin x \cos x = \frac{1}{2}$

44. $\cos^2 x - \sin^2 x = \frac{\sqrt{3}}{2}$

45. $\sin^4 x - 1 = 0$

46. $\sec^2 x + 2 \sec x = 0$

47. $\sin x + \cos x = 1$

48. $2 \sin x + \csc x = 3$

Solve each inequality for $0 \le \theta < 2\pi$.

49. $\cos \theta \le -\frac{\sqrt{3}}{2}$

50. $\cos \theta - \frac{1}{2} > 0$

51. $\sqrt{2} \sin \theta - 1 < 0$

Graphing Calculator

Solve each equation graphically on the interval $0 \le x < 2\pi$.

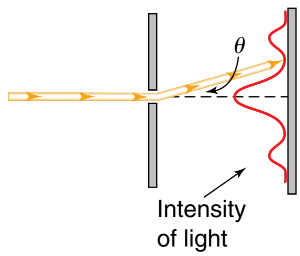
52. $\tan x = 0.5$

53. $\sin x - \frac{x}{2} = 0$

54. $\cos x = 3 \sin x$

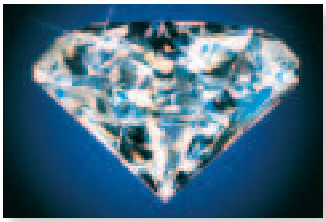
Applications and Problem Solving

55. **Optics** When light passes through a small slit, it is diffracted. The angle θ subtended by the first diffraction minimum is related to the wavelength λ of the light and the width D of the slit by the equation $\sin \theta = \frac{\lambda}{D}$. Consider light of wavelength 550 nanometers (5.5×10^{-7} m). What is the angle subtended by the first diffraction minimum when the light passes through a slit of width 3 millimeters?



56. **Critical Thinking** Solve the inequality $\sin 2x < \sin x$ for $0 \le x < 2\pi$ without a calculator.

57. **Physics** The range of a projectile that is launched with an initial velocity v at an angle of θ with the horizontal is given by $R = \frac{v^2}{g} \sin 2\theta$, where g is the acceleration due to gravity or 9.8 meters per second squared. If a projectile is launched with an initial velocity of 15 meters per second, what angle is required to achieve a range of 20 meters?

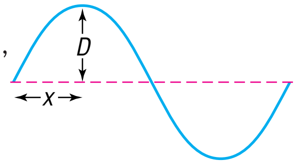


58. Gemology The sparkle of a diamond is created by *refracted* light. Light travels at different speeds in different mediums. When light rays pass from one medium to another in which they travel at a different velocity, the light is bent, or refracted. According to Snell's Law, $n_1 \sin i = n_2 \sin r$, where n_1 is the index of refraction of the medium the light is exiting, n_2 is the index of refraction of the medium the light is entering, i is the angle of incidence, and r is the angle of refraction.

a. The index of refraction of a diamond is 2.42, and the index of refraction of air is 1.00. If a beam of light strikes a diamond at an angle of 35°, what is the angle of refraction?

b. Explain how a gemologist might use Snell's Law to determine if a diamond is genuine.

59. Music A wave traveling in a guitar string can be modeled by the equation $D = 0.5 \sin(6.5x) \sin(2500t)$, where D is the displacement in millimeters at the position x meters from the left end of the string at time t seconds. Find the first positive time when the point 0.5 meter from the left end has a displacement of 0.01 millimeter.



60. Critical Thinking How many solutions in the interval $0° \leq x < 360°$ should you expect for the equation $a \sin(bx + c) + d = d + \frac{a}{2}$, if $a \neq 0$ and b is a positive integer?

61. Geometry The point $P(x, y)$ can be rotated θ degrees counterclockwise about the origin by multiplying the matrix $\begin{bmatrix} x \\ y \end{bmatrix}$ on the left by the rotation matrix $R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$. Determine the angle required to rotate the point $P(3, 4)$ to the point $P'(\sqrt{17}, 2\sqrt{2})$.

Mixed Review

62. Find the exact value of cot 67.5°. *(Lesson 7-4)*

63. Find a numerical value of one trigonometric function of x if $\frac{\tan x}{\sec x} = \frac{\sqrt{2}}{5}$. *(Lesson 7-2)*

64. Graph $y = \frac{2}{3} \cos \theta$. *(Lesson 6-4)*

65. **Transportation** A boat trailer has wheels with a diameter of 14 inches. If the trailer is being pulled by a car going 45 miles per hour, what is the angular velocity of the wheels in revolutions per second? *(Lesson 6-2)*

66. Use the unit circle to find the value of csc 180°. *(Lesson 5-3)*

67. Determine the binomial factors of $x^3 - 3x - 2$. *(Lesson 4-3)*

68. Graph $y = x^3 - 3x + 5$. Find and classify its extrema. *(Lesson 3-6)*

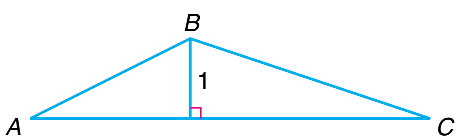
69. Find the values of x and y for which $\begin{bmatrix} 3x + 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 16 \\ 2y \end{bmatrix}$ is true. *(Lesson 2-3)*

70. Solve the system $x - y + z = 1$, $2x + y + 3z = 5$, and $x + y - z = 11$. *(Lesson 2-2)*

71. Graph $g(x) = |x + 3|$. *(Lesson 1-7)*

72. **SAT/ACT Practice** If $AC = 6$, what is the area of triangle ABC?

A 1 B $\sqrt{6}$ C 3
D 6 E 12



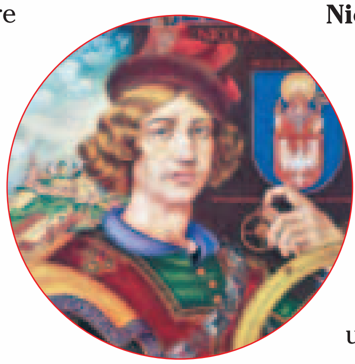
Extra Practice See p. A39.

HISTORY of MATHEMATICS

TRIGONOMETRY

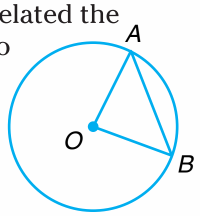
Trigonometry was developed in response to the needs of astronomers. In fact, it was not until the thirteenth century that trigonometry and astronomy were treated as separate disciplines.

Early Evidence The earliest use of trigonometry was to calculate the heights of objects using the lengths of shadows. Egyptian mathematicians produced tables relating the lengths of shadows at particular times of day as early as the thirteenth century B.C.



Copernicus

The Greek mathematician **Hipparchus** (190–120 B.C.), is generally credited with laying the foundations of trigonometry. Hipparchus is believed to have produced a twelve-book treatise on the construction of a table of chords. This table related the lengths of chords of a circle to the angles subtended by those chords. In the diagram, $\angle AOB$ would be compared to the length of chord $\overline{AB}$.



In about 100 A.D., the Greek mathematician **Menelaus**, credited with the first work on spherical trigonometry, also produced a treatise on chords in a circle. **Ptolemy**, a Babylonian mathematician, produced yet another book of chords, believed to have been adapted from Hipparchus' treatise. He used an identity similar to $\sin^2 x + \cos^2 x = 1$, except that it was relative to chords. He also used the formulas $\sin (x + y) = \sin x \cos y + \cos x \sin y$ and $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$ as they related to chords.

In about 500 A.D., **Aryabhata**, a Hindu mathematician, was the first person to use the sine function as we know it today. He produced a table of sines and called the sine *jya.* In 628 A.D., another Hindu mathematician, **Brahmagupta**, also produced a table of sines.

The Renaissance Many mathematicians developed theories and applications of trigonometry during this time period. **Nicolas Copernicus** (1473–1543) published a book highlighting all the trigonometry necessary for astronomy at that time. During this period, the sine and versed sine were the most important trigonometric functions. Today, the versed sine, which is defined as $\text{versin } x = 1 - \cos x$, is rarely used.

Modern Era Mathematicians of the 1700s, 1800s, and 1900s worked on more sophisticated trigonometric ideas such as those relating to complex variables and hyperbolic functions. Renowned mathematicians who made contributions to trigonometry during this era were **Bernoulli**, **Cotes**, **DeMoivre**, **Euler**, and **Lambert**.

Today architects, such as Dennis Holloway of Santa Fe, New Mexico, use trigonometry in their daily work. Mr. Holloway is particularly interested in Native American designs. He uses trigonometry to determine the best angles for the walls of his buildings and for finding the correct slopes for landscaping.

ACTIVITIES

1. Draw a circle of radius 5 centimeters. Make a table of chords for angles of measure 10° through 90°(use 10° intervals). The table headings should be "angle measure" and "length of chord." (In the diagram of circle O, you are using $\angle AOB$ and chord $\overline{AB}$.)

2. Find out more about personalities referenced in this article and others who contributed to the history of trigonometry. Visit **www.amc.glencoe.com**