EXPLORING DATA AND STATISTICS

10.1 The Distance and Midpoint Formulas

What you should learn

GOAL 1 Find the distance between two points and find the midpoint of the line segment joining two points.

GOAL 2 Use the distance and midpoint formulas in **real-life** situations, such as finding the diameter of a broken dish in **Example 5**.

Why you should learn it

▼ To solve **real-life** problems, such as finding the distance a medical helicopter must travel to a hospital in **Exs. 53–56**.

REAL LIFE

GOAL 1 USING THE DISTANCE AND MIDPOINT FORMULAS

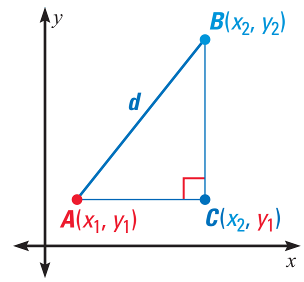
To find the distance d between $A(x_1, y_1)$ and $B(x_2, y_2)$, you can apply the Pythagorean theorem to right triangle ABC.

$$(AB)^2 = (AC)^2 + (BC)^2$$

$$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

The third equation is called the **distance formula**.



THE DISTANCE FORMULA

The distance d between the points (x_1, y_1) and (x_2, y_2) is as follows:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

EXAMPLE 1 *Finding the Distance Between Two Points*

Find the distance between $(-2, 5)$ and $(3, -1)$.

SOLUTION

Let $(x_1, y_1) = (-2, 5)$ and $(x_2, y_2) = (3, -1)$.

$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ **Use distance formula.**

$= \sqrt{(3 - (-2))^2 + (-1 - 5)^2}$ **Substitute.**

$= \sqrt{25 + 36}$ **Simplify.**

$= \sqrt{61} \approx 7.81$ **Use a calculator.**

EXAMPLE 2 *Classifying a Triangle Using the Distance Formula*

Classify $\triangle ABC$ as *scalene*, *isosceles*, or *equilateral*.

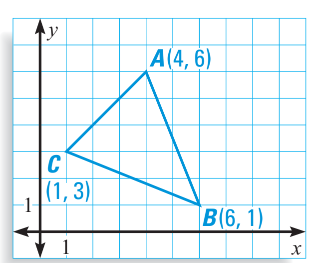
SOLUTION

$AB = \sqrt{(6 - 4)^2 + (1 - 6)^2} = \sqrt{29}$

$BC = \sqrt{(1 - 6)^2 + (3 - 1)^2} = \sqrt{29}$

$AC = \sqrt{(1 - 4)^2 + (3 - 6)^2} = 3\sqrt{2}$

▶ Because $AB = BC$, $\triangle ABC$ is isosceles.



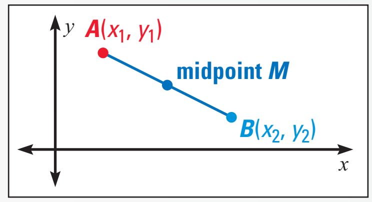
Another formula involving two points in a coordinate plane is the **midpoint formula.** Recall that the midpoint of a segment is the point on the segment that is equidistant from the two endpoints.

THE MIDPOINT FORMULA

The midpoint of the line segment joining $A(x_1, y_1)$ and $B(x_2, y_2)$ is as follows:

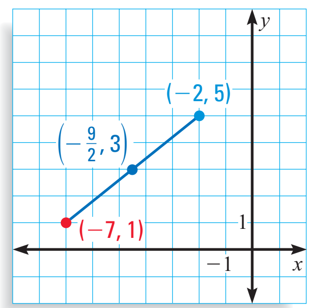
$$M\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

Each coordinate of M is the mean of the corresponding coordinates of A and B.



EXAMPLE 3 *Finding the Midpoint of a Segment*

Find the midpoint of the line segment joining $(-7, 1)$ and $(-2, 5)$.



SOLUTION

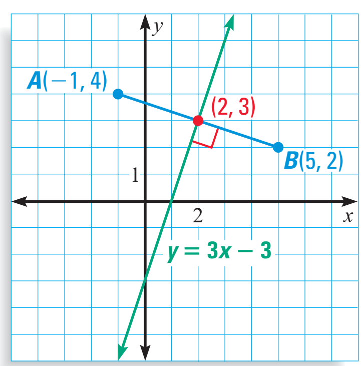
Let $(x_1, y_1) = (-7, 1)$ and $(x_2, y_2) = (-2, 5)$.

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) = \left(\frac{-7 + (-2)}{2}, \frac{1 + 5}{2}\right)$$

$$= \left(-\frac{9}{2}, 3\right)$$

EXAMPLE 4 *Finding a Perpendicular Bisector*

Write an equation for the perpendicular bisector of the line segment joining $A(-1, 4)$ and $B(5, 2)$.



SOLUTION

First find the midpoint of the line segment:

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) = \left(\frac{-1 + 5}{2}, \frac{4 + 2}{2}\right) = (2, 3)$$

Then find the slope of $\overline{AB}$:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{2 - 4}{5 - (-1)} = \frac{-2}{6} = -\frac{1}{3}$$

The slope of the perpendicular bisector is the negative reciprocal of $-\frac{1}{3}$, or $m_{\perp} = 3$.

STUDENT HELP

Look Back
For help with perpendicular lines, see p. 92.

Since you know the slope of the perpendicular bisector and a point that the bisector passes through, you can use the point-slope form to write its equation.

$$y - 3 = 3(x - 2)$$

$$y = 3x - 3$$

▶ An equation for the perpendicular bisector of $\overline{AB}$ is $y = 3x - 3$.

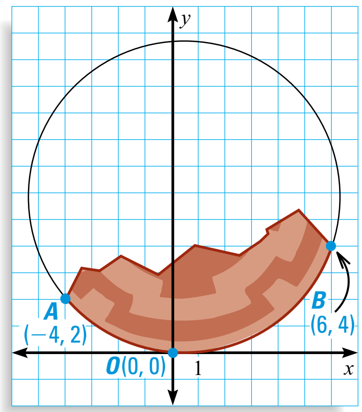
FOCUS ON APPLICATIONS

ARCHEOLOGISTS use grids to systematically explore a site. By labeling the grid squares, they can record where each artifact is found.

GOAL 2 DISTANCE AND MIDPOINT FORMULAS IN REAL LIFE

Recall from geometry that the perpendicular bisector of a chord of a circle passes through the center of the circle. Using this theorem, you can find the center of a circle given three points on the circle.

EXAMPLE 5 *Using the Distance and Midpoint Formulas in Real Life*



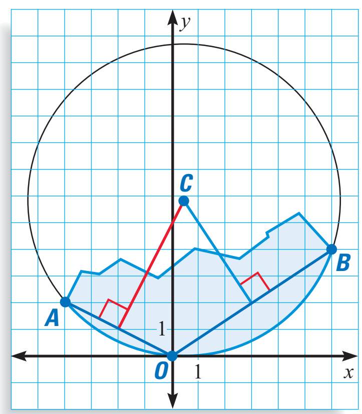
ARCHEOLOGY While on an archeological dig, you discover a piece of a broken dish. To estimate the original diameter of the dish, you lay the piece on a coordinate plane and mark three points on the circular edge, as shown. Use these points to find the diameter of the dish. (Each unit in the coordinate plane represents 1 inch.)

SOLUTION

Use the method illustrated in Example 4 to find the perpendicular bisectors of $\overline{AO}$ and $\overline{OB}$.

$y = 2x + 5$ **Perpendicular bisector of $\overline{AO}$**

$y = -\frac{3}{2}x + \frac{13}{2}$ **Perpendicular bisector of $\overline{OB}$**



Both bisectors pass through the circle's center. Therefore, the center of the circle is the solution of the system formed by these two equations.

$y = 2x + 5$ **Write first equation.**

$-\frac{3}{2}x + \frac{13}{2} = 2x + 5$ **Substitute for *y*.**

$-3x + 13 = 4x + 10$ **Multiply each side by 2.**

$-7x = -3$ **Simplify.**

$x = \frac{3}{7}$ **Divide each side by −7.**

$y = 2\left(\frac{3}{7}\right) + 5$ **Substitute the *x*-value into the first equation.**

$y = \frac{41}{7}$ **Simplify.**

The center of the circle is $C\left(\frac{3}{7}, \frac{41}{7}\right)$. The radius of the circle is the distance between C and any of the three given points.

$$CO = \sqrt{\left(0 - \frac{3}{7}\right)^2 + \left(0 - \frac{41}{7}\right)^2}$$

$$= \sqrt{\frac{1690}{49}}$$

$$\approx 5.87$$

▶ The dish had a diameter of about 2(5.87) = 11.74 inches.

STUDENT HELP

Look Back
For help with solving systems, see p. 148.

Guided Practice

Vocabulary Check ✓

1. State the distance and midpoint formulas.

Concept Check ✓

2. Look back at Example 1. Find the distance between $(-2, 5)$ and $(3, -1)$, but this time letting $(x_1, y_1) = (3, -1)$ and $(x_2, y_2) = (-2, 5)$. How are the calculations different? Do you get the same answer?
3. **a.** Write a formula for the distance between a point (x, y) and the origin.

   b. Write a formula for the midpoint of the segment joining a point (x, y) and the origin.

Skill Check ✓

Find the distance between the two points.

4. $(2, -1), (2, 3)$ **5.** $(-5, -2), (0, -2)$ **6.** $(0, 6), (4, 9)$

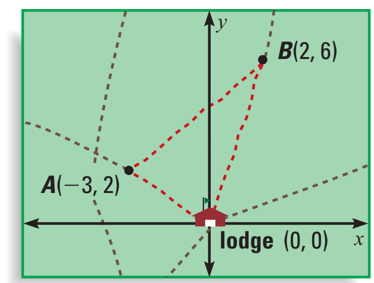
7. $(10, -2), (7, 4)$ **8.** $(-3, 8), (5, 6)$ **9.** $(6, -1), (-9, 8)$

Find the midpoint of the line segment joining the two points.

10. $(0, 0), (-8, 14)$ **11.** $(0, 3), (4, 9)$ **12.** $(1, -2), (1, 6)$

13. $(1, 3), (3, 11)$ **14.** $(-5, 4), (2, -4)$ **15.** $(-1, 5), (-8, -6)$

16. **HIKING** You are going on a two-day hike. The map at the right shows the trails you plan to follow. (Each unit represents 1 mile.)



a. You hike from the lodge to point A and decide that you will hike to the midpoint of $\overline{AB}$ before you camp for the night. At what point in the plane will you be camping?

b. How far will you hike each day?

Practice and Applications

STUDENT HELP
Extra Practice to help you master skills is on p. 953.

USING THE FORMULAS **Find the distance between the two points. Then find the midpoint of the line segment joining the two points.**

17. $(0, 0), (3, 4)$ **18.** $(0, 0), (4, 12)$ **19.** $(0, 4), (8, -3)$

20. $(-2, 8), (6, 0)$ **21.** $(-3, -1), (7, 4)$ **22.** $(9, -2), (3, 6)$

23. $(-5, -8), (1, 6)$ **24.** $(-2, 10), (10, -2)$ **25.** $(8, 3), (2, -1)$

26. $(-10, -15), (12, 18)$ **27.** $(-3.5, 1.2), (6, -3.8)$ **28.** $(6.3, -9), (1.3, -8.5)$

29. $(-7, 2), \left(-\frac{11}{2}, 4\right)$ **30.** $\left(\frac{2}{3}, -\frac{11}{4}\right), \left(-\frac{7}{2}, -\frac{11}{2}\right)$ **31.** $\left(-\frac{3}{4}, 2\right), \left(5, -\frac{7}{4}\right)$

STUDENT HELP
HOMEWORK HELP
Example 1: Exs. 17–31, 47–50
Example 2: Exs. 32–40
Example 3: Exs. 17–31
Example 4: Exs. 41–46
Example 5: Exs. 51–58

GEOMETRY CONNECTION **The vertices of a triangle are given. Classify the triangle as *scalene*, *isosceles*, or *equilateral*.**

32. $(2, 0), (0, 8), (-2, 0)$ **33.** $(4, 1), (1, -2), (6, -4)$ **34.** $(1, 9), (-4, 2), (4, 2)$

35. $(2, 5), (8, 2), (4, -1)$ **36.** $(5, -1), (-4, 0), (3, 5)$ **37.** $(4, 4), (8, 1), (6, -5)$

38. $(0, -3), (3, 5), (-5, 2)$ **39.** $(1, 1), (-4, 0), (-2, 5)$ **40.** $(2, 4), (3, -2), (-1, 1)$

FINDING EQUATIONS **Write an equation for the perpendicular bisector of the line segment joining the two points.**

41. $(2, 2), (6, 14)$ **42.** $(0, 0), (-8, -10)$ **43.** $(0, -6), (-4, 9)$

44. $(3, -7), (-3, 1)$ **45.** $(-3, -7.2), (-4.2, 1.8)$ **46.** $\left(\frac{3}{2}, -6\right), (-3, 1)$

FINDING A COORDINATE **Use the given distance *d* between the two points to solve for *x*.**

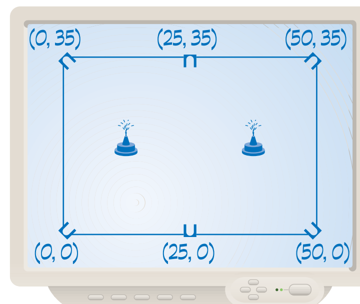
47. $(0, 1), (x, 4); d = \sqrt{34}$ **48.** $(1, 3), (-6, x); d = \sqrt{74}$

49. $(x, -10), (-8, 4); d = 7\sqrt{5}$ **50.** $(0.5, x), (7, 2); d = 8.5$

STUDENT HELP

HOMEWORK HELP Visit our Web site www.mcdougallittell.com for help with problem solving in Exs. 51 and 52.

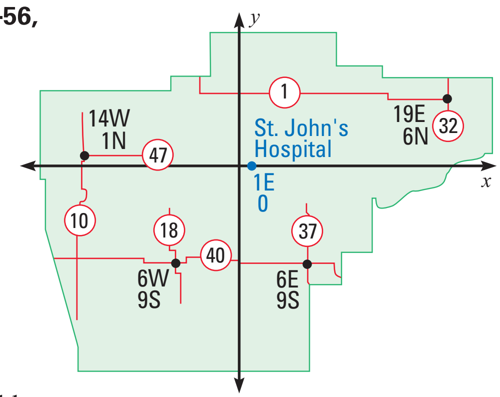
URBAN PLANNING **In Exercises 51 and 52, use the following information.**
You are designing a city park like the one shown at the right. You want the park to have two fountains so that each fountain is equidistant from four of the six park entrances. The labeled points shown in the coordinate plane represent the park entrances.



51. Where should the fountains be placed?

52. How far apart should the fountains be placed?

HELICOPTER RESCUE **In Exercises 53–56, use the following information to find the distance a medical helicopter would have to travel to St. John's Hospital from each highway intersection.**
The Highway Department of Sangamon County in Illinois uses a map with a coordinate plane whose origin represents downtown Springfield. Each unit represents one mile and the letters N, S, E, and W are used to indicate the direction. For example, 3E 5S corresponds to $(3, -5)$, a point 3 miles east and 5 miles south of downtown Springfield. St. John's Hospital is located at 1E 0, or $(1, 0)$.



53. Rt. 1–Rt. 32 intersection at 19E 6N **54.** Rt. 37–Rt. 40 intersection at 6E 9S

55. Rt. 18–Rt. 40 intersection at 6W 9S **56.** Rt. 10–Rt. 47 intersection at 14W 1N

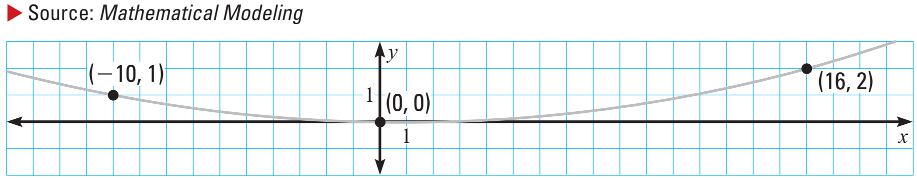
FOCUS ON CAREERS

REAL LIFE **ACCIDENT RECONSTRUCTIONIST** An accident reconstructionist uses physical evidence, such as skid marks, to determine how accidents occurred.

CAREER LINK www.mcdougallittell.com

57. **ACCIDENT RECONSTRUCTION** When a car makes a fast, sharp turn, an accident reconstructionist can use the car's skid mark to determine its speed. The equation $v = \sqrt{ar}$ gives the car's speed v (in meters per second) as a function of the radius r of the circle (in meters) along which the car was traveling. The constant a (measured in meters per second squared) varies depending on road conditions. Find the radius of the skid mark shown below. Then use the given equation and 6.86 for a to find how fast the car was going.

▶ Source: *Mathematical Modeling*



58. **STATISTICS CONNECTION** A physician uses many tests to evaluate a patient's condition. Some of these tests yield numerical results. In these cases, the physician can treat two test results as an ordered pair and use the distance formula to determine how close to average the patient is. In the table below the serum creatinine (C) and systolic blood pressure (P) for several patients are given. Tell how far from normal each patient is, where normal is represented by the ordered pair $(C, P) = (1, 127)$.

C	2	5	1	7	3	4	1
P	120	127	140	115	112	125	130

Test Preparation

QUANTITATIVE COMPARISON In Exercises 59–62, choose the statement that is true about the given quantities.

Ⓐ The quantity in column A is greater.

Ⓑ The quantity in column B is greater.

Ⓒ The two quantities are equal.

Ⓓ The relationship cannot be determined from the given information.

	Column A	Column B
59.	Distance between (0, 7) and (1, −1)	Distance between (9, 2) and (3, 8)
60.	Distance between (−5, −2) and (5, 2)	Distance between (−5, 5) and (2, −2)
61.	Distance between (−3, 0) and (2, −4)	Distance between (7, 6) and (1, 5)
62.	Distance between (2, −5) and (1, 6)	Distance between (0, 8) and (6, 0)

★ **Challenge**

63. **FINDING A FORMULA** Find formulas for the distance between a point (x, y) and each of the following: **(a)** a horizontal line $y = k$ and **(b)** a vertical line $x = h$.

MIXED REVIEW

GRAPHING FUNCTIONS Graph the quadratic function. (Review 5.1 for 10.2)

64. $y = 4x^2$ **65.** $y = 3x^2$ **66.** $y = -3x^2$ **67.** $y = -2x^2$

68. $y = \frac{1}{3}x^2$ **69.** $y = -\frac{2}{3}x^2$ **70.** $y = -\frac{3}{4}x^2$ **71.** $y = \frac{5}{6}x^2$

SOLVING EQUATIONS Solve the equation. Check for extraneous solutions. (Review 7.6)

72. $x^{2/3} + 13 = 17$ **73.** $\sqrt{x + 100} = 25$ **74.** $\sqrt{2x} = x - 4$

75. $\sqrt{x + 2} = \sqrt{3x}$ **76.** $2\sqrt[3]{3x} = 6$ **77.** $-2x^{3/2} = -8$

OPERATIONS WITH RATIONAL EXPRESSIONS Perform the indicated operation and simplify. (Review 9.5)

78. $\frac{2}{x + 1} - \frac{x}{x^2 - 1}$ **79.** $\frac{4}{2x^2} + \frac{1}{3x}$ **80.** $\frac{11}{4(x - 5)} - \frac{x + 1}{4x}$

81. $\frac{3x}{x^2} - \frac{x - 1}{x + 3}$ **82.** $\frac{2}{3x + 2} + \frac{5x^2}{x - 4}$ **83.** $\frac{1 - 3x}{x - 6} + \frac{2}{2x + 1}$