5.4 Complex Numbers

GOAL 1 OPERATIONS WITH COMPLEX NUMBERS

What you should learn

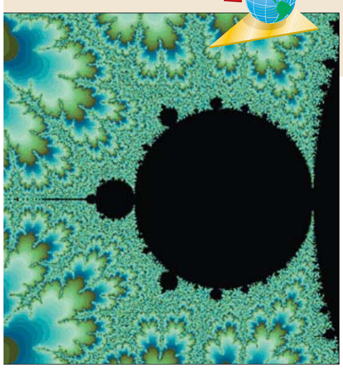
GOAL 1 Solve quadratic equations with complex solutions and perform operations with complex numbers.

GOAL 2 Apply complex numbers to fractal geometry.

Why you should learn it

▼ To solve problems, such as determining whether a complex number belongs to the Mandelbrot set in **Example 7**.

REAL LIFE



Not all quadratic equations have real-number solutions. For instance, $x^2 = -1$ has no real-number solutions because the square of any real number x is never negative. To overcome this problem, mathematicians created an expanded system of numbers using the **imaginary unit** i, defined as $i = \sqrt{-1}$. Note that $i^2 = -1$. The imaginary unit i can be used to write the square root of *any* negative number.

THE SQUARE ROOT OF A NEGATIVE NUMBER

PROPERTY	EXAMPLE
1. If r is a positive real number, then $\sqrt{-r} = i\sqrt{r}$.	$\sqrt{-5} = i\sqrt{5}$
2. By Property (1), it follows that $(i\sqrt{r})^2 = -r$.	$(i\sqrt{5})^2 = i^2 \cdot 5 = -5$

EXAMPLE 1 *Solving a Quadratic Equation*

Solve $3x^2 + 10 = -26$.

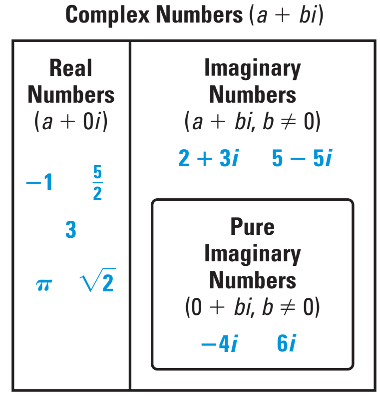
SOLUTION

$3x^2 + 10 = -26$	Write original equation.
$3x^2 = -36$	Subtract 10 from each side.
$x^2 = -12$	Divide each side by 3.
$x = \pm\sqrt{-12}$	Take square roots of each side.
$x = \pm i\sqrt{12}$	Write in terms of i.
$x = \pm 2i\sqrt{3}$	Simplify the radical.

▶ The solutions are $2i\sqrt{3}$ and $-2i\sqrt{3}$.

A **complex number** written in **standard form** is a number $a + bi$ where a and b are real numbers. The number a is the *real part* of the complex number, and the number bi is the *imaginary part*. If $b \neq 0$, then $a + bi$ is an **imaginary number**. If $a = 0$ and $b \neq 0$, then $a + bi$ is a **pure imaginary number**. The diagram shows how different types of complex numbers are related.

Complex Numbers $(a + bi)$

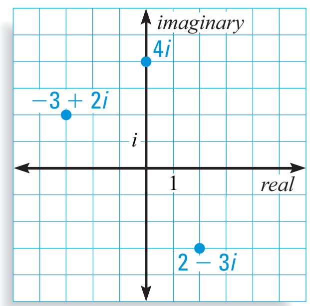


Just as every real number corresponds to a point on the real number line, every complex number corresponds to a point in the **complex plane**. As shown in the next example, the complex plane has a horizontal axis called the *real axis* and a vertical axis called the *imaginary axis*.

EXAMPLE 2 *Plotting Complex Numbers*

Plot the complex numbers in the complex plane.

a. $2 - 3i$ **b.** $-3 + 2i$ **c.** $4i$



SOLUTION

a. To plot $2 - 3i$, start at the origin, move 2 units to the right, and then move 3 units down.

b. To plot $-3 + 2i$, start at the origin, move 3 units to the left, and then move 2 units up.

c. To plot $4i$, start at the origin and move 4 units up.

Two complex numbers $a + bi$ and $c + di$ are equal if and only if $a = c$ and $b = d$. For instance, if $x + yi = 8 - i$, then $x = 8$ and $y = -1$.

To add (or subtract) two complex numbers, add (or subtract) their real parts and their imaginary parts separately.

Sum of complex numbers: $(a + bi) + (c + di) = (a + c) + (b + d)i$

Difference of complex numbers: $(a + bi) - (c + di) = (a - c) + (b - d)i$

EXAMPLE 3 *Adding and Subtracting Complex Numbers*

Write the expression as a complex number in standard form.

a. $(4 - i) + (3 + 2i)$ **b.** $(7 - 5i) - (1 - 5i)$ **c.** $6 - (-2 + 9i) + (-8 + 4i)$

SOLUTION

a. $(4 - i) + (3 + 2i) = (4 + 3) + (-1 + 2)i$ **Definition of complex addition**

$= 7 + i$ **Standard form**

b. $(7 - 5i) - (1 - 5i) = (7 - 1) + (-5 + 5)i$ **Definition of complex subtraction**

$= 6 + 0i$ **Simplify.**

$= 6$ **Standard form**

c. $6 - (-2 + 9i) + (-8 + 4i) = [(6 + 2) - 9i] + (-8 + 4i)$ **Subtract.**

$= (8 - 9i) + (-8 + 4i)$ **Simplify.**

$= (8 - 8) + (-9 + 4)i$ **Add.**

$= 0 - 5i$ **Simplify.**

$= -5i$ **Standard form**

To multiply two complex numbers, use the distributive property or the FOIL method just as you do when multiplying real numbers or algebraic expressions. Other properties of real numbers that also apply to complex numbers include the associative and commutative properties of addition and multiplication.

EXAMPLE 4 *Multiplying Complex Numbers*

Write the expression as a complex number in standard form.

a. $5i(-2 + i)$ **b.** $(7 - 4i)(-1 + 2i)$ **c.** $(6 + 3i)(6 - 3i)$

SOLUTION

a. $5i(-2 + i) = -10i + 5i^2$ — **Distributive property**

$= -10i + 5(-1)$ — **Use $i^2 = -1$.**

$= -5 - 10i$ — **Standard form**

b. $(7 - 4i)(-1 + 2i) = -7 + 14i + 4i - 8i^2$ — **Use FOIL.**

$= -7 + 18i - 8(-1)$ — **Simplify and use $i^2 = -1$.**

$= 1 + 18i$ — **Standard form**

c. $(6 + 3i)(6 - 3i) = 36 - 18i + 18i - 9i^2$ — **Use FOIL.**

$= 36 - 9(-1)$ — **Simplify and use $i^2 = -1$.**

$= 45$ — **Standard form**

....

In part (c) of Example 4, notice that the two factors $6 + 3i$ and $6 - 3i$ have the form $a + bi$ and $a - bi$. Such numbers are called **complex conjugates**. The product of complex conjugates is always a real number. You can use complex conjugates to write the quotient of two complex numbers in standard form.

EXAMPLE 5 *Dividing Complex Numbers*

Write the quotient $\frac{5 + 3i}{1 - 2i}$ in standard form.

SOLUTION

The key step here is to multiply the numerator and the denominator by the complex conjugate of the denominator.

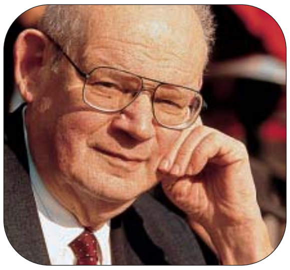
$\frac{5 + 3i}{1 - 2i} = \frac{5 + 3i}{1 - 2i} \cdot \frac{1 + 2i}{1 + 2i}$ — **Multiply by $1 + 2i$, the conjugate of $1 - 2i$.**

$= \frac{5 + 10i + 3i + 6i^2}{1 + 2i - 2i - 4i^2}$ — **Use FOIL.**

$= \frac{-1 + 13i}{5}$ — **Simplify.**

$= -\frac{1}{5} + \frac{13}{5}i$ — **Standard form**

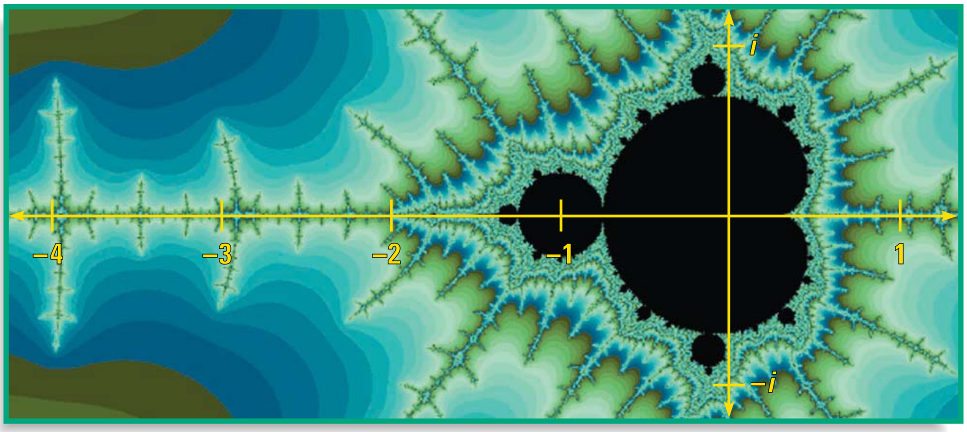
FOCUS ON PEOPLE



BENOIT MANDELBROT was born in Poland in 1924, came to the United States in 1958, and is now a professor at Yale University. He pioneered the study of fractal geometry in the 1970s.

GOAL 2 USING COMPLEX NUMBERS IN FRACTAL GEOMETRY

In the hands of a person who understands *fractal geometry*, the complex plane can become an easel on which stunning pictures called *fractals* are drawn. One very famous fractal is the *Mandelbrot set,* named after mathematician Benoit Mandelbrot. The Mandelbrot set is the black region in the complex plane below. (The points in the colored regions are *not* part of the Mandelbrot set.)



To understand how the Mandelbrot set is constructed, you need to know how the *absolute value* of a complex number is defined.

ABSOLUTE VALUE OF A COMPLEX NUMBER

The **absolute value** of a complex number $z = a + bi$, denoted $|z|$, is a nonnegative *real* number defined as follows:

$$|z| = \sqrt{a^2 + b^2}$$

Geometrically, the absolute value of a complex number is the number's distance from the origin in the complex plane.

EXAMPLE 6 *Finding Absolute Values of Complex Numbers*

Find the absolute value of each complex number. Which number is farthest from the origin in the complex plane?

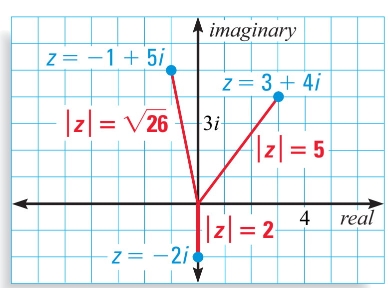
a. $3 + 4i$ **b.** $-2i$ **c.** $-1 + 5i$

SOLUTION

a. $|3 + 4i| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$

b. $|-2i| = |0 + (-2i)| = \sqrt{0^2 + (-2)^2} = 2$

c. $|-1 + 5i| = \sqrt{(-1)^2 + 5^2} = \sqrt{26} \approx 5.10$



Since $-1 + 5i$ has the greatest absolute value, it is farthest from the origin in the complex plane.

The following result shows how absolute value can be used to tell whether a given complex number belongs to the Mandelbrot set.

COMPLEX NUMBERS IN THE MANDELBROT SET

To determine whether a complex number c belongs to the Mandelbrot set, consider the function $f(z) = z^2 + c$ and this infinite list of complex numbers:

$$z_0 = 0,\ z_1 = f(z_0),\ z_2 = f(z_1),\ z_3 = f(z_2),\ \ldots$$

- If the absolute values $|z_0|, |z_1|, |z_2|, |z_3|, \ldots$ are all less than some fixed number N, then c belongs to the Mandelbrot set.
- If the absolute values $|z_0|, |z_1|, |z_2|, |z_3|, \ldots$ become infinitely large, then c does not belong to the Mandelbrot set.

EXAMPLE 7 *Determining if a Complex Number Is in the Mandelbrot Set*

Tell whether the complex number c belongs to the Mandelbrot set.

a. $c = i$ **b.** $c = 1 + i$ **c.** $c = -2$

SOLUTION

a. Let $f(z) = z^2 + i$.

$z_0 = 0$ $|z_0| = 0$

$z_1 = f(0) = 0^2 + i = i$ $|z_1| = 1$

$z_2 = f(i) = i^2 + i = -1 + i$ $|z_2| = \sqrt{2} \approx 1.41$

$z_3 = f(-1 + i) = (-1 + i)^2 + i = -i$ $|z_3| = 1$

$z_4 = f(-i) = (-i)^2 + i = -1 + i$ $|z_4| = \sqrt{2} \approx 1.41$

At this point the absolute values alternate between 1 and $\sqrt{2}$, and so all the absolute values are less than $N = 2$. Therefore, $c = i$ belongs to the Mandelbrot set.

b. Let $f(z) = z^2 + (1 + i)$.

$z_0 = 0$ $|z_0| = 0$

$z_1 = f(0) = 0^2 + (1 + i) = 1 + i$ $|z_1| \approx 1.41$

$z_2 = f(1 + i) = (1 + i)^2 + (1 + i) = 1 + 3i$ $|z_2| \approx 3.16$

$z_3 = f(1 + 3i) = (1 + 3i)^2 + (1 + i) = -7 + 7i$ $|z_3| \approx 9.90$

$z_4 = f(-7 + 7i) = (-7 + 7i)^2 + (1 + i) = 1 - 97i$ $|z_4| \approx 97.0$

The next few absolute values in the list are (approximately) 9409, 8.85×10^7, and 7.84×10^{15}. Since the absolute values are becoming infinitely large, $c = 1 + i$ does not belong to the Mandelbrot set.

c. Let $f(z) = z^2 + (-2)$, or $f(z) = z^2 - 2$. You can show that $z_0 = 0$, $z_1 = -2$, and $z_n = 2$ for $n > 1$. Therefore, the absolute values of $z_0, z_1, z_2, z_3, \ldots$ are all less than $N = 3$, and so $c = -2$ belongs to the Mandelbrot set.

Guided Practice

Vocabulary Check ✓

1. Complete this statement: For the complex number $3 - 7i$, the real part is __?__ and the imaginary part is __?__.

Concept Check ✓

2. **Error Analysis** A student thinks that the complex conjugate of $-5 + 2i$ is $5 - 2i$. Explain the student's mistake, and give the correct complex conjugate of $-5 + 2i$.

3. Geometrically, what does the absolute value of a complex number represent?

Skill Check ✓

Solve the equation.

4. $x^2 = -9$
5. $2x^2 + 3 = -13$
6. $(x - 1)^2 = -7$

Write the expression as a complex number in standard form.

7. $(1 + 5i) + (6 - 2i)$
8. $(4 + 3i) - (-2 + 4i)$
9. $(1 - i)(7 + 2i)$
10. $\frac{3 - 4i}{1 + i}$

Find the absolute value of the complex number.

11. $1 + i$
12. $3i$
13. $-2 + 3i$
14. $5 - 5i$

15. Plot the numbers in Exercises 11–14 in the same complex plane.

16. **Fractal Geometry** Tell whether $c = 1 - i$ belongs to the Mandelbrot set. Use absolute value to justify your answer.

Practice and Applications

Student Help

Extra Practice to help you master skills is on p. 946.

Solving Quadratic Equations Solve the equation.

17. $x^2 = -4$
18. $x^2 = -11$
19. $3x^2 = -81$
20. $2x^2 + 9 = -41$
21. $5x^2 + 18 = 3$
22. $-x^2 - 4 = 14$
23. $8r^2 + 7 = 5r^2 + 4$
24. $3s^2 - 1 = 7s^2$
25. $(t - 2)^2 = -16$
26. $-6(u + 5)^2 = 120$
27. $-\frac{1}{8}(v + 3)^2 = 7$
28. $9(w - 4)^2 + 1 = 0$

Plotting Complex Numbers Plot the numbers in the same complex plane.

29. $4 + 2i$
30. $-1 + i$
31. $-4i$
32. 3
33. $-2 - i$
34. $1 + 5i$
35. $6 - 3i$
36. $-5 + 4i$

Student Help

Homework Help
Example 1: Exs. 17–28
Example 2: Exs. 29–36
Example 3: Exs. 37–46
Example 4: Exs. 47–55
Example 5: Exs. 56–63
Example 6: Exs. 64–71
Example 7: Exs. 72–79

Adding and Subtracting Write the expression as a complex number in standard form.

37. $(2 + 3i) + (7 + i)$
38. $(6 + 2i) + (5 - i)$
39. $(-4 + 7i) + (-4 - 7i)$
40. $(-1 - i) + (9 - 3i)$
41. $(8 + 5i) - (1 + 2i)$
42. $(2 - 6i) - (-10 + 4i)$
43. $(-0.4 + 0.9i) - (-0.6 + i)$
44. $(25 + 15i) - (25 - 6i)$
45. $-i + (8 - 2i) - (5 - 9i)$
46. $(30 - i) - (18 + 6i) + 30i$

MULTIPLYING **Write the expression as a complex number in standard form.**

47. $i(3 + i)$ **48.** $4i(6 - i)$ **49.** $-10i(4 + 7i)$

50. $(5 + i)(8 + i)$ **51.** $(-1 + 2i)(11 - i)$ **52.** $(2 - 9i)(9 - 6i)$

53. $(7 + 5i)(7 - 5i)$ **54.** $(3 + 10i)^2$ **55.** $(15 - 8i)^2$

DIVIDING **Write the expression as a complex number in standard form.**

56. $\frac{8}{1 + i}$ **57.** $\frac{2i}{1 - i}$ **58.** $\frac{-5 - 3i}{4i}$ **59.** $\frac{3 + i}{3 - i}$

60. $\frac{2 + 5i}{5 + 2i}$ **61.** $\frac{-7 + 6i}{9 - 4i}$ **62.** $\frac{\sqrt{10}}{\sqrt{10} - i}$ **63.** $\frac{6 - i\sqrt{2}}{6 + i\sqrt{2}}$

ABSOLUTE VALUE **Find the absolute value of the complex number.**

64. $3 - 4i$ **65.** $5 + 12i$ **66.** $-2 - i$ **67.** $-7 + i$

68. $2 + 5i$ **69.** $4 - 8i$ **70.** $-9 + 6i$ **71.** $\sqrt{11} + i\sqrt{5}$

MANDELBROT SET **Tell whether the complex number *c* belongs to the Mandelbrot set. Use absolute value to justify your answer.**

72. $c = 1$ **73.** $c = -1$ **74.** $c = -i$ **75.** $c = -1 - i$

76. $c = 2$ **77.** $c = -1 + i$ **78.** $c = -0.5$ **79.** $c = 0.5i$

STUDENT HELP

Skills Review
For help with disproving statements by counterexample, see p. 927.

LOGICAL REASONING **In Exercises 80–85, tell whether the statement is *true* or *false*. If the statement is false, give a counterexample.**

80. Every complex number is an imaginary number.

81. Every irrational number is a complex number.

82. All real numbers lie on a single line in the complex plane.

83. The sum of two imaginary numbers is always an imaginary number.

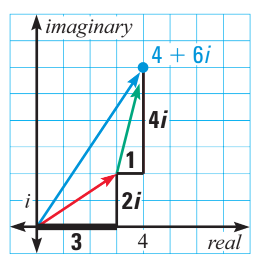
84. Every real number equals its complex conjugate.

85. The absolute values of a complex number and its complex conjugate are always equal.

86. **VISUAL THINKING** The graph shows how you can geometrically add two complex numbers (in this case, $3 + 2i$ and $1 + 4i$) to find their sum (in this case, $4 + 6i$). Find each of the following sums by drawing a graph.

a. $(2 + i) + (3 + 5i)$

b. $(-1 + 6i) + (7 - 4i)$



COMPARING REAL AND COMPLEX NUMBERS **Tell whether the property is true for (a) the set of real numbers and (b) the set of complex numbers.**

87. If r, s, and t are numbers in the set, then $(r + s) + t = r + (s + t)$.

88. If r is a number in the set and $|r| = k$, then $r = k$ or $r = -k$.

89. If r and s are numbers in the set, then $r - s = s - r$.

90. If r, s, and t are numbers in the set, then $r(s + t) = rs + rt$.

91. If r and s are numbers in the set, then $|r + s| = |r| + |s|$.

FOCUS ON CAREERS

REAL LIFE ELECTRICIAN An electrician installs, maintains, and repairs electrical systems. This often involves working with the types of circuits described in Exs. 95 and 96.

INTERNET CAREER LINK www.mcdougallittell.com

92. CRITICAL THINKING Evaluate $\sqrt{-4} \cdot \sqrt{-9}$ and $\sqrt{36}$. Does the rule $\sqrt{a} \cdot \sqrt{b} = \sqrt{ab}$ on page 264 hold when a and b are negative numbers?

93. *Writing* Give both an algebraic argument and a geometric argument explaining why the definitions of absolute value on pages 50 and 275 are consistent when applied to real numbers.

94. EXTENSION: ADDITIVE AND MULTIPLICATIVE INVERSES The *additive inverse* of a complex number z is a complex number z_a such that $z + z_a = 0$. The *multiplicative inverse* of z is a complex number z_m such that $z \cdot z_m = 1$. Find the additive and multiplicative inverses of each complex number.

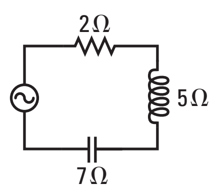
a. $z = 1 + i$ **b.** $z = 3 - i$ **c.** $z = -2 + 8i$

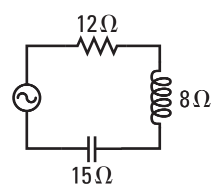
ELECTRICITY In Exercises 95 and 96, use the following information.

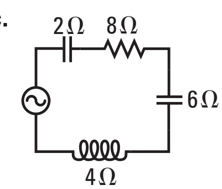
Electrical circuits may contain several types of components such as resistors, inductors, and capacitors. The resistance of each component to the flow of electrical current is the component's *impedance*, denoted by Z. The value of Z is a real number R for a resistor of R ohms (Ω), a pure imaginary number Li for an inductor of L ohms, and a pure imaginary number $-Ci$ for a capacitor of C ohms. Examples are given in the table.

Component	Symbol	Z
Resistor	3Ω	3
Inductor	5Ω	$5i$
Capacitor	6Ω	$-6i$

95. SERIES CIRCUITS A *series circuit* is a type of circuit found in switches, fuses, and circuit breakers. In a series circuit, there is only one pathway through which current can flow. To find the total impedance of a series circuit, add the impedances of the components in the circuit. What is the impedance of each series circuit shown below? (*Note:* The symbol ~ denotes an alternating current source and does not affect the calculation of impedance.)

a.


b.


c.


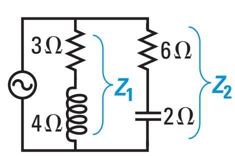
STUDENT HELP

INTERNET HOMEWORK HELP Visit our Web site www.mcdougallittell.com for help with problem solving in Exs. 95 and 96.

96. PARALLEL CIRCUITS *Parallel circuits* are used in household lighting and appliances. In a parallel circuit, there is more than one pathway through which current can flow. To find the impedance Z of a parallel circuit with two pathways, first calculate the impedances Z_1 and Z_2 of the pathways separately by treating each pathway as a series circuit. Then apply this formula:

$$Z = \frac{Z_1 Z_2}{Z_1 + Z_2}$$

What is the impedance of each parallel circuit shown below?

a.


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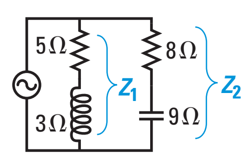
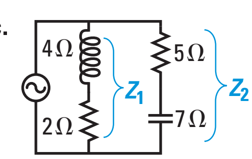


c.


Test Preparation

QUANTITATIVE COMPARISON **In Exercises 97–99, choose the statement that is true about the given quantities.**

(A) The quantity in column A is greater.

(B) The quantity in column B is greater.

(C) The two quantities are equal.

(D) The relationship cannot be determined from the given information.

	Column A	Column B
97.	$\lvert 5 + 4i \rvert$	$\lvert 3 - 6i \rvert$
98.	$\lvert -6 + 8i \rvert$	$\lvert -10i \rvert$
99.	$\lvert 2 + bi \rvert$ where $b < -1$	$\lvert \sqrt{3} + ci \rvert$ where $0 < c < 1$

★ **Challenge**

100. **POWERS OF *i*** In this exercise you will investigate a pattern that appears when the imaginary unit i is raised to successively higher powers.

a. Copy and complete the table.

Power of *i*	i^1	i^2	i^3	i^4	i^5	i^6	i^7	i^8
Simplified form	i	-1	$-i$	?	?	?	?	?

b. *Writing* Describe the pattern you observe in the table. Verify that the pattern continues by evaluating the next four powers of i.

c. Use the pattern you described in part (b) to evaluate i^{26} and i^{83}.

EXTRA CHALLENGE
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MIXED REVIEW

EVALUATING FUNCTIONS **Evaluate $f(x)$ for the given value of x.** (Review 2.1)

101. $f(x) = 4x - 1$ when $x = 3$

102. $f(x) = x^2 - 5x + 8$ when $x = -4$

103. $f(x) = \lvert -x + 6 \rvert$ when $x = 9$

104. $f(x) = 2$ when $x = -30$

SOLVING SYSTEMS **Use an inverse matrix to solve the system.** (Review 4.5)

105. $3x + y = 5$
$5x + 2y = 9$

106. $x + y = 2$
$7x + 8y = 21$

107. $x - 2y = 10$
$3x + 4y = 0$

SOLVING QUADRATIC EQUATIONS **Solve the equation.** (Review 5.3 for 5.5)

108. $(x + 4)^2 = 1$

109. $(x + 2)^2 = 36$

110. $(x - 11)^2 = 25$

111. $-(x - 5)^2 = -10$

112. $2(x + 7)^2 = 24$

113. $3(x - 6)^2 - 8 = 13$

114. **STATISTICS CONNECTION** The table shows the cumulative number N (in thousands) of DVD players sold in the United States from the end of February, 1997, to time t (in months). Make a scatter plot of the data. Approximate the equation of the best-fitting line. (Review 2.5)

t	1	2	3	4	5	6	7	8	9	10	11	12
N	34	69	96	125	144	178	213	269	307	347	383	416

INTERNET DATA UPDATE of *DVD Insider* data at www.mcdougallittell.com