5.5 Completing the Square

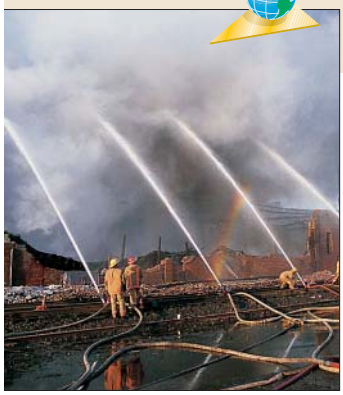
What you should learn

GOAL 1 Solve quadratic equations by completing the square.

GOAL 2 Use completing the square to write quadratic functions in vertex form, as applied in **Example 7**.

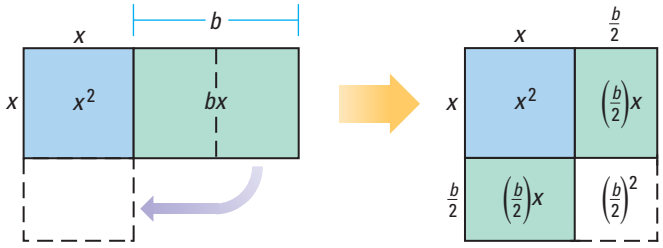
Why you should learn it

▼ To solve **real-life** problems, such as finding where to position a fire hose in **Ex. 91**.



GOAL 1 Solving Quadratic Equations by Completing the Square

Completing the square is a process that allows you to write an expression of the form $x^2 + bx$ as the square of a binomial. This process can be illustrated using an area model, as shown below.



You can see that to complete the square for $x^2 + bx$, you need to add $\left(\frac{b}{2}\right)^2$, the area of the incomplete corner of the square in the second diagram. This diagram models the following rule:

$$x^2 + bx + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2$$

EXAMPLE 1 Completing the Square

Find the value of c that makes $x^2 - 7x + c$ a perfect square trinomial. Then write the expression as the square of a binomial.

SOLUTION

In the expression $x^2 - 7x + c$, note that $b = -7$. Therefore:

$$c = \left(\frac{b}{2}\right)^2 = \left(\frac{-7}{2}\right)^2 = \frac{49}{4}$$

Use this value of c to write $x^2 - 7x + c$ as a perfect square trinomial, and then as the square of a binomial.

$$x^2 - 7x + \mathbf{c} = x^2 - 7x + \frac{49}{4} \qquad \text{Perfect square trinomial}$$

$$= \left(x - \frac{7}{2}\right)^2 \qquad \text{Square of a binomial: } \left(x + \frac{b}{2}\right)^2$$

In Lesson 5.2 you learned how to solve quadratic equations by factoring. However, many quadratic equations, such as $x^2 + 10x - 3 = 0$, contain expressions that cannot be factored. Completing the square is a method that lets you solve *any* quadratic equation, as the next example illustrates.

EXAMPLE 2 Solving a Quadratic Equation if the Coefficient of x^2 Is 1

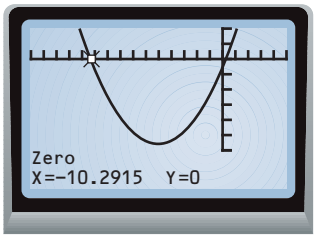
Solve $x^2 + 10x - 3 = 0$ by completing the square.

SOLUTION

$x^2 + 10x - 3 = 0$	Write original equation.
$x^2 + 10x = 3$	Write the left side in the form $x^2 + bx$.
$x^2 + 10x + \mathbf{5^2} = 3 + \mathbf{25}$	Add $\left(\frac{10}{2}\right)^2 = 5^2 = 25$ to each side.
$(x + 5)^2 = 28$	Write the left side as a binomial squared.
$x + 5 = \pm\sqrt{28}$	Take square roots of each side.
$x = -5 \pm \sqrt{28}$	Solve for x.
$x = -5 \pm 2\sqrt{7}$	Simplify.

▶ The solutions are $-5 + 2\sqrt{7}$ and $-5 - 2\sqrt{7}$.

✓ **CHECK** You can check the solutions by substituting them back into the original equation. Alternatively, you can graph $y = x^2 + 10x - 3$ and observe that the x-intercepts are about $0.29 \approx -5 + 2\sqrt{7}$ and $-10.29 \approx -5 - 2\sqrt{7}$.



STUDENT HELP

Study Tip

In Example 2 note that you must add 25 to *both sides* of the equation $x^2 + 10x = 3$ when completing the square.

If the coefficient of x^2 in a quadratic equation is not 1, you should divide each side of the equation by this coefficient before completing the square.

EXAMPLE 3 Solving a Quadratic Equation if the Coefficient of x^2 Is Not 1

Solve $3x^2 - 6x + 12 = 0$ by completing the square.

SOLUTION

$3x^2 - 6x + 12 = 0$	Write original equation.
$x^2 - 2x + 4 = 0$	Divide each side by the coefficient of x^2.
$x^2 - 2x = -4$	Write the left side in the form $x^2 + bx$.
$x^2 - 2x + \mathbf{(-1)^2} = -4 + \mathbf{1}$	Add $\left(\frac{-2}{2}\right)^2 = (-1)^2 = 1$ to each side.
$(x - 1)^2 = -3$	Write the left side as a binomial squared.
$x - 1 = \pm\sqrt{-3}$	Take square roots of each side.
$x = 1 \pm \sqrt{-3}$	Solve for x.
$x = 1 \pm i\sqrt{3}$	Write in terms of the imaginary unit i.

▶ The solutions are $1 + i\sqrt{3}$ and $1 - i\sqrt{3}$.

✓ **CHECK** Because the solutions are imaginary, you cannot check them graphically. However, you can check the solutions algebraically by substituting them back into the original equation.

REAL LIFE
Traffic Engineering

EXAMPLE 4 Using a Quadratic Equation to Model Distance

On dry asphalt the distance d (in feet) needed for a car to stop is given by

$$d = 0.05s^2 + 1.1s$$

where s is the car's speed (in miles per hour). What speed limit should be posted on a road where drivers round a corner and have 80 feet to come to a stop?

SOLUTION

$d = 0.05s^2 + 1.1s$	Write original equation.
$80 = 0.05s^2 + 1.1s$	Substitute 80 for d.
$1600 = s^2 + 22s$	Divide each side by the coefficient of s^2.
$1600 + 121 = s^2 + 22s + 11^2$	Add $\left(\frac{22}{2}\right)^2 = 11^2 = 121$ to each side.
$1721 = (s + 11)^2$	Write the right side as a binomial squared.
$\pm\sqrt{1721} = s + 11$	Take square roots of each side.
$-11 \pm \sqrt{1721} = s$	Solve for s.
$s \approx 30$ or $s \approx -52$	Use a calculator.

▶ Reject the solution -52 because a car's speed cannot be negative. The posted speed limit should be at most 30 miles per hour.

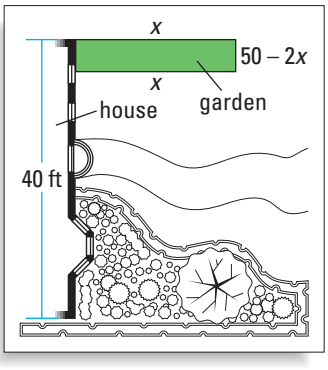
REAL LIFE
Landscape Design

EXAMPLE 5 Using a Quadratic Equation to Model Area

You want to plant a rectangular garden along part of a 40 foot side of your house. To keep out animals, you will enclose the garden with wire mesh along its three open sides. You will also cover the garden with mulch. If you have 50 feet of mesh and enough mulch to cover 100 square feet, what should the garden's dimensions be?

SOLUTION

Draw a diagram. Let x be the length of the sides of the garden perpendicular to the house. Then $50 - 2x$ is the length of the third fenced side of the garden.



$x(50 - 2x) = 100$	Length × Width = Area
$50x - 2x^2 = 100$	Distributive property
$-2x^2 + 50x = 100$	Write the x^2-term first.
$x^2 - 25x = -50$	Divide each side by -2.
$x^2 - 25x + (-12.5)^2 = -50 + 156.25$	Complete the square.
$(x - 12.5)^2 = 106.25$	Write as a binomial squared.
$x - 12.5 = \pm\sqrt{106.25}$	Take square roots of each side.
$x = 12.5 \pm \sqrt{106.25}$	Solve for x.
$x \approx 22.8$ or $x \approx 2.2$	Use a calculator.

▶ Reject $x = 2.2$ since $50 - 2x = 45.6$ is greater than the house's length. If $x = 22.8$, then $50 - 2x = 4.4$. The garden should be about 22.8 feet by 4.4 feet.

GOAL 2 WRITING QUADRATIC FUNCTIONS IN VERTEX FORM

Given a quadratic function in standard form, $y = ax^2 + bx + c$, you can use completing the square to write the function in vertex form, $y = a(x - h)^2 + k$.

EXAMPLE 6 *Writing a Quadratic Function in Vertex Form*

STUDENT HELP

INTERNET HOMEWORK HELP

Visit our Web site www.mcdougallittell.com for extra examples.

Write the quadratic function $y = x^2 - 8x + 11$ in vertex form. What is the vertex of the function's graph?

SOLUTION

$$y = x^2 - 8x + 11$$ **Write original function.**

$$y + \underline{?} = (x^2 - 8x + \underline{?}) + 11$$ **Prepare to complete the square for $x^2 - 8x$.**

$$y + \mathbf{16} = (x^2 - 8x + \mathbf{16}) + 11$$ **Add $\left(\frac{-8}{2}\right)^2 = (-4)^2 = 16$ to each side.**

$$y + 16 = (x - 4)^2 + 11$$ **Write $x^2 - 8x + 16$ as a binomial squared.**

$$y = (x - 4)^2 - 5$$ **Solve for y.**

▶ The vertex form of the function is $y = (x - 4)^2 - 5$. The vertex is $(4, -5)$.

REAL LIFE Agriculture

EXAMPLE 7 *Finding the Maximum Value of a Quadratic Function*

The amount s (in pounds per acre) of sugar produced from sugarbeets can be modeled by the function

$$s = -0.0655n^2 + 7.855n + 5562$$

where n is the amount (in pounds per acre) of nitrogen fertilizer used. How much fertilizer should you use to maximize sugar production? What is the maximum amount of sugar you can produce?

▶ Source: Sugarbeet Research and Education Board of Minnesota and North Dakota

SOLUTION

The optimal amount of fertilizer and the maximum amount of sugar are the coordinates of the vertex of the function's graph. One way to find the vertex is to write the function in vertex form.

$$s = -0.0655n^2 + 7.855n + 5562$$

$$s = -0.0655(n^2 - 120n) + 5562$$

$$s - 0.0655(\underline{?}) = -0.0655(n^2 - 120n + \underline{?}) + 5562$$

$$s - 0.0655(\mathbf{3600}) = -0.0655(n^2 - 120n + \mathbf{3600}) + 5562$$

$$s - 236 = -0.0655(n - 60)^2 + 5562$$

$$s = -0.0655(n - 60)^2 + 5798$$

▶ The vertex is approximately $(60, 5798)$. To maximize sugar production, you should use about 60 pounds per acre of nitrogen fertilizer. The maximum amount of sugar you can produce is about 5800 pounds per acre.

GUIDED PRACTICE

Vocabulary Check ✓

1. Describe what it means to "complete the square" for an expression of the form $x^2 + bx$.

Concept Check ✓

2. Which method for solving quadratic equations—factoring or completing the square—is more general? Explain.

3. **ERROR ANALYSIS** A student tried to write $y = -x^2 - 6x + 4$ in vertex form as shown. Explain the student's mistake. Then write the correct vertex form of the function.

$$y = -x^2 - 6x + 4$$
$$y = -(x^2 + 6x) + 4$$
$$y + 9 = -(x^2 + 6x + 9) + 4$$
$$y + 9 = -(x + 3)^2 + 4$$
$$y = -(x + 3)^2 - 5$$

Skill Check ✓

Find the value of *c* that makes the expression a perfect square trinomial. Then write the expression as the square of a binomial.

4. $x^2 + 2x + c$
5. $x^2 + 14x + c$
6. $x^2 - 6x + c$
7. $x^2 - 10x + c$
8. $x^2 + 5x + c$
9. $x^2 - 13x + c$

Solve the equation by completing the square.

10. $x^2 + 4x = -1$
11. $x^2 - 2x = 4$
12. $x^2 - 16x + 76 = 0$
13. $x^2 + 8x + 9 = 0$
14. $2x^2 + 12x = 4$
15. $3x^2 - 12x + 93 = 0$

Write the quadratic function in vertex form and identify the vertex.

16. $y = x^2 + 12x$
17. $y = x^2 - 4x + 7$
18. $y = x^2 - 8x + 31$
19. $y = x^2 + 10x + 17$
20. $y = -x^2 + 14x - 45$
21. $y = 2x^2 + 4x - 4$

22. **LANDSCAPE DESIGN** Suppose the homeowner in Example 5 has 60 feet of wire mesh to put around the garden and enough mulch to cover an area of 140 square feet. What should the dimensions of the garden be?

PRACTICE AND APPLICATIONS

STUDENT HELP

Extra Practice to help you master skills is on p. 946.

REWRITING EXPRESSIONS **Write the expression as the square of a binomial.**

23. $x^2 + 16x + 64$
24. $x^2 + 20x + 100$
25. $x^2 - 24x + 144$
26. $x^2 - 38x + 361$
27. $x^2 + x + 0.25$
28. $x^2 - 1.4x + 0.49$
29. $x^2 - 3x + \frac{9}{4}$
30. $x^2 + \frac{1}{6}x + \frac{1}{144}$
31. $x^2 - \frac{4}{9}x + \frac{4}{81}$

COMPLETING THE SQUARE **Find the value of *c* that makes the expression a perfect square trinomial. Then write the expression as the square of a binomial.**

32. $x^2 - 12x + c$
33. $x^2 + 18x + c$
34. $x^2 + 26x + c$
35. $x^2 - 44x + c$
36. $x^2 + 9x + c$
37. $x^2 - 11x + c$
38. $x^2 - 23x + c$
39. $x^2 + 15x + c$
40. $x^2 - 0.2x + c$
41. $x^2 - 5.8x + c$
42. $x^2 + 1.6x + c$
43. $x^2 + 9.4x + c$
44. $x^2 - \frac{2}{7}x + c$
45. $x^2 + \frac{10}{3}x + c$
46. $x^2 + \frac{17}{8}x + c$

STUDENT HELP

HOMEWORK HELP

Example 1: Exs. 23–46
Example 2: Exs. 47–54, 63–64
Example 3: Exs. 55–72
Example 4: Exs. 89–91
Example 5: Exs. 92, 93
Example 6: Exs. 73–84
Example 7: Exs. 94, 95

COEFFICIENT OF x^2 IS 1 **Solve the equation by completing the square.**

47. $x^2 + 2x = 9$

48. $x^2 - 12x = -28$

49. $x^2 + 20x + 104 = 0$

50. $x^2 + 3x - 1 = 0$

51. $u^2 - 4u = 2u + 35$

52. $v^2 - 17v + 200 = 13v - 43$

53. $m^2 + 1.8m - 1.5 = 0$

54. $n^2 - \frac{4}{3}n - \frac{14}{9} = 0$

COEFFICIENT OF x^2 IS NOT 1 **Solve the equation by completing the square.**

55. $2x^2 - 12x = -14$

56. $-3x^2 + 24x = 27$

57. $6x^2 + 84x + 300 = 0$

58. $4x^2 + 40x + 280 = 0$

59. $-4r^2 + 21r = r + 13$

60. $3s^2 - 26s + 2 = 5s^2 + 1$

61. $0.4t^2 + 0.7t = 0.3t - 0.2$

62. $\frac{w^2}{24} - \frac{w}{2} + \frac{13}{6} = 0$

SOLVING BY ANY METHOD **Solve the equation by factoring, by finding square roots, or by completing the square.**

63. $x^2 + 4x - 12 = 0$

64. $x^2 - 6x - 15 = 0$

65. $9x^2 - 23 = 0$

66. $2x^2 + 9x + 7 = 0$

67. $3x^2 + x = 2x - 6$

68. $4(x + 8)^2 = 144$

69. $7k^2 + 10k - 100 = 2k^2 + 55$

70. $14b^2 - 19b + 4 = -11b^2 + 11b - 5$

71. $0.01p^2 - 0.22p + 2.9 = 0$

72. $\frac{q^2}{4} - \frac{9q^2}{20} = 18$

WRITING IN VERTEX FORM **Write the quadratic function in vertex form and identify the vertex.**

73. $y = x^2 - 6x + 11$

74. $y = x^2 - 2x - 9$

75. $y = x^2 + 16x + 14$

76. $y = x^2 + 26x + 68$

77. $y = x^2 - 3x - 2$

78. $y = x^2 + 7x - 1$

79. $y = -x^2 + 20x - 80$

80. $y = -x^2 - 14x - 47$

81. $y = 3x^2 - 12x + 1$

82. $y = -2x^2 - 2x - 7$

83. $y = 1.4x^2 + 5.6x + 3$

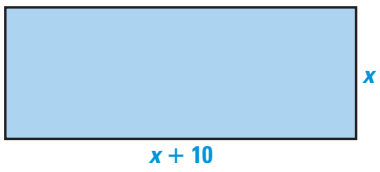
84. $y = \frac{2}{3}x^2 - \frac{4}{5}x$

STUDENT HELP

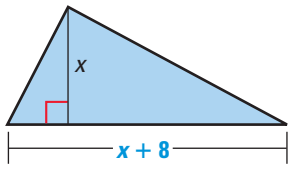
Skills Review
For help with areas of geometric figures, see p. 914.

GEOMETRY CONNECTION **Find the value of x.**

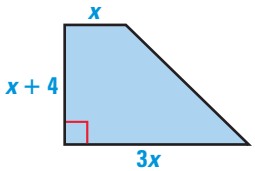
85. Area of rectangle = 100



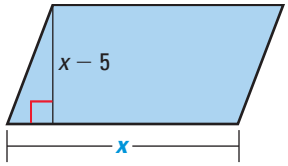
86. Area of triangle = 40



87. Area of trapezoid = 70



88. Area of parallelogram = 54



FOCUS ON PEOPLE

JACKIE JOYNER-KERSEE became one of the greatest female athletes in history despite having severe asthma as a child and as an adult. She has won six Olympic medals: three gold, one silver, and two bronze.

89. **TRAFFIC ENGINEERING** For a road covered with dry, packed snow, the formula for a car's stopping distance given in Example 4 becomes:

$$d = 0.08s^2 + 1.1s$$

Show that, in snowy conditions, a driver cannot safely round the corner in Example 4 when traveling at the calculated speed limit of 30 miles per hour. What is a safe speed limit if the road is covered with snow?

90. **SPORTS** Jackie Joyner-Kersee won the women's heptathlon during the 1992 Olympics in Barcelona, Spain. Her throw in the shot put, one of the seven events in the heptathlon, can be modeled by

$$y = -0.0241x^2 + x + 5.5$$

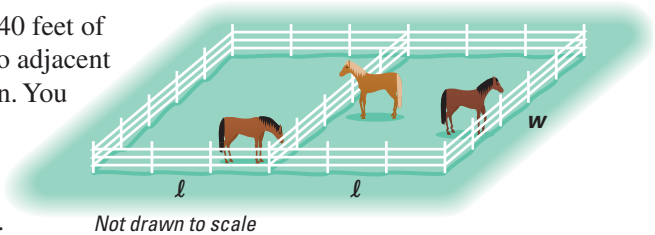
where x is the shot put's horizontal distance traveled (in feet) and y is its corresponding height (in feet). How long was Joyner-Kersee's throw?

91. **FIREFIGHTING** In firefighting, a good water stream can be modeled by

$$y = -0.003x^2 + 0.62x + 3$$

where x is the water's horizontal distance traveled (in feet) and y is its corresponding height (in feet). If a firefighter is aiming a good water stream at a building's window 25 feet above the ground, at what two distances can the firefighter stand from the building?

92. **CORRALS** You have 240 feet of wooden fencing to form two adjacent rectangular corrals as shown. You want each corral to have an area of 1000 square feet.



Not drawn to scale

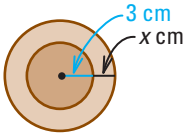
a. Show that $w = 80 - \frac{4}{3}\ell$.

b. Use your answer from part (a) to find the possible dimensions of each corral.

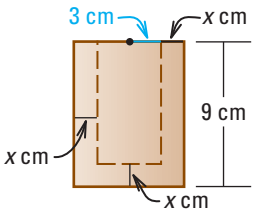
93. **POTTERY** You are taking a pottery class. As an assignment, you are given a lump of clay whose volume is 200 cubic centimeters and asked to make a cylindrical pencil holder. The pencil holder should be 9 centimeters high and have an inner radius of 3 centimeters. What thickness x should your pencil holder have if you want to use all the clay? (*Hint:* The volume of clay equals the difference of the volumes of two cylinders.)


Pencil holder

Not drawn to scale



Top view



Side view

94. **BIOLOGY CONNECTION** When a gray kangaroo jumps, its path through the air can be modeled by

$$y = -0.0267x^2 + 0.8x$$

where x is the kangaroo's horizontal distance traveled (in feet) and y is its corresponding height (in feet). How high can a gray kangaroo jump? How far can it jump?

95. **SCIENCE CONNECTION** In a fireplace, the heat loss q (in Btu/ft^3) resulting from hot gases escaping through the chimney can be modeled by

$$q = -0.00002T^2 + 0.0203T - 1.24$$

where T is the temperature (in degrees Fahrenheit) of the gases. (This model assumes an indoor temperature of 65°F.) For what gas temperature is heat loss maximized? What is the maximum heat loss? ▶ Source: *Workshop Math*

Test Preparation

96. **MULTIPLE CHOICE** If $x^2 - 28x + c$ is a perfect square trinomial, what is the value of c?

Ⓐ -14 Ⓑ 28 Ⓒ 196 Ⓓ 784

97. **MULTIPLE CHOICE** What are the solutions of $x^2 + 12x + 61 = 0$?

Ⓐ $-1, -11$ Ⓑ $-6 \pm 5i$ Ⓒ $-6 \pm \sqrt{97}$ Ⓓ $-6 \pm i\sqrt{61}$

98. **MULTIPLE CHOICE** What is the vertex form of $y = 2x^2 - 8x + 3$?

Ⓐ $y = 2(x - 2)^2 - 5$ Ⓑ $y = 2(x - 2)^2 + 3$

Ⓒ $y = 2(x - 4)^2 - 29$ Ⓓ $y = 2(x - 4)^2 + 3$

★ Challenge

CRITICAL THINKING **Exercises 99 and 100 should be done together.**

99. Graph the two functions in the same coordinate plane.

a. $y = x^2 + 2x$, $y = (x + 1)^2$ **b.** $y = x^2 + 4x$, $y = (x + 2)^2$ **c.** $y = x^2 - 6x$, $y = (x - 3)^2$

EXTRA CHALLENGE
www.mcdougallittell.com

100. Compare the graphs of $y = x^2 + bx$ and $y = \left(x + \frac{b}{2}\right)^2$. What happens to the graph of $y = x^2 + bx$ when you complete the square for $x^2 + bx$?

MIXED REVIEW

EVALUATING EXPRESSIONS **Evaluate $b^2 - 4ac$ for the given values of a, b, and c.** (Review 1.2 for 5.6)

101. $a = 1, b = 5, c = 2$ **102.** $a = 3, b = -8, c = 7$

103. $a = -5, b = 0, c = 2.6$ **104.** $a = 11, b = 4, c = -1$

105. $a = 16, b = -24, c = 9$ **106.** $a = -1.4, b = 2, c = -0.5$

EQUATIONS OF LINES **Write an equation in slope-intercept form of the line through the given point and having the given slope.** (Review 2.4)

107. $(3, 1), m = 2$ **108.** $(2, -4), m = 1$ **109.** $(-7, 10), m = -5$

110. $(-8, -8), m = -3$ **111.** $(6, 9), m = \frac{1}{3}$ **112.** $(11, -2), m = -\frac{5}{4}$

SYSTEMS OF LINEAR INEQUALITIES **Graph the system of inequalities.** (Review 3.3)

113. $x \geq 2$, $y \leq 3$ **114.** $y > -1$, $y < 2$ **115.** $x \geq 0$, $x + y < 4$

116. $y < x - 2$, $x - 3y \leq 6$ **117.** $3x - 2y < 8$, $2x + y > 0$ **118.** $y \leq 2x + 3$, $y \geq 2x - 3$