

5.6

The Quadratic Formula and the Discriminant

What you should learn

GOAL 1 Solve quadratic equations using the quadratic formula.

GOAL 2 Use the quadratic formula in **real-life** situations, such as baton twirling in **Example 5**.

Why you should learn it

▼ To solve **real-life** problems, such as finding the speed and duration of a thrill ride in **Ex. 84**.

**GOAL 1 SOLVING EQUATIONS WITH THE QUADRATIC FORMULA**

In Lesson 5.5 you solved quadratic equations by completing the square for *each equation separately*. By completing the square *once* for the general equation $ax^2 + bx + c = 0$, you can develop a formula that gives the solutions of *any* quadratic equation. The formula for the solutions is called the **quadratic formula**. A derivation of the quadratic formula appears on page 895.

THE QUADRATIC FORMULA

Let a , b , and c be real numbers such that $a \neq 0$. The solutions of the quadratic equation $ax^2 + bx + c = 0$ are:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Remember that *before* you apply the quadratic formula to a quadratic equation, you must write the equation in standard form, $ax^2 + bx + c = 0$.

EXAMPLE 1 Solving a Quadratic Equation with Two Real Solutions

Solve $2x^2 + x = 5$.

SOLUTION

$$2x^2 + x = 5$$

$$2x^2 + x - 5 = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-1 \pm \sqrt{1^2 - 4(2)(-5)}}{2(2)}$$

$$x = \frac{-1 \pm \sqrt{41}}{4}$$

Write original equation.

Write in standard form.

Quadratic formula

$$a = 2, b = 1, c = -5$$

Simplify.

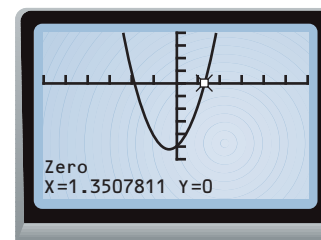
▶ The solutions are

$$x = \frac{-1 + \sqrt{41}}{4} \approx 1.35$$

and

$$x = \frac{-1 - \sqrt{41}}{4} \approx -1.85.$$

✓ **CHECK** Graph $y = 2x^2 + x - 5$ and note that the x -intercepts are about 1.35 and about -1.85 .



**EXAMPLE 2** Solving a Quadratic Equation with One Real SolutionSolve $x^2 - x = 5x - 9$.**SOLUTION**

$$x^2 - x = 5x - 9$$

$$x^2 - 6x + 9 = 0$$

$$x = \frac{6 \pm \sqrt{(-6)^2 - 4(1)(9)}}{2(1)}$$

$$x = \frac{6 \pm \sqrt{0}}{2}$$

$$x = 3$$

Write original equation.

$$a = 1, b = -6, c = 9$$

Quadratic formula

Simplify.

Simplify.

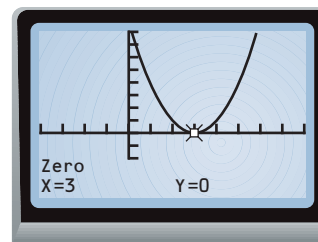
▶ The solution is 3.

✓ **CHECK** Graph $y = x^2 - 6x + 9$ and note that the only x -intercept is 3. Alternatively, substitute 3 for x in the original equation.

$$3^2 - 3 \stackrel{?}{=} 5(3) - 9$$

$$9 - 3 \stackrel{?}{=} 15 - 9$$

$$6 = 6 \checkmark$$

**EXAMPLE 3** Solving a Quadratic Equation with Two Imaginary SolutionsSolve $-x^2 + 2x = 2$.**SOLUTION**

$$-x^2 + 2x = 2$$

$$-x^2 + 2x - 2 = 0$$

$$x = \frac{-2 \pm \sqrt{2^2 - 4(-1)(-2)}}{2(-1)}$$

$$x = \frac{-2 \pm \sqrt{-4}}{-2}$$

$$x = \frac{-2 \pm 2i}{-2}$$

$$x = 1 \pm i$$

Write original equation.

$$a = -1, b = 2, c = -2$$

Quadratic formula

Simplify.

Write using the imaginary unit i .

Simplify.

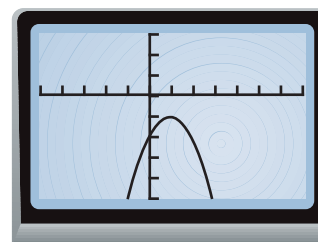
▶ The solutions are $1 + i$ and $1 - i$.

✓ **CHECK** Graph $y = -x^2 + 2x - 2$ and note that there are no x -intercepts. So, the original equation has no real solutions. To check the imaginary solutions $1 + i$ and $1 - i$, substitute them into the original equation. The check for $1 + i$ is shown.

$$-(1 + i)^2 + 2(1 + i) \stackrel{?}{=} 2$$

$$-2i + 2 + 2i \stackrel{?}{=} 2$$

$$2 = 2 \checkmark$$

**STUDENT HELP****HOMEWORK HELP**

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for extra examples.



In the quadratic formula, the expression $b^2 - 4ac$ under the radical sign is called the **discriminant** of the associated equation $ax^2 + bx + c = 0$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \leftarrow \text{discriminant}$$

You can use the discriminant of a quadratic equation to determine the equation's number and type of solutions.

NUMBER AND TYPE OF SOLUTIONS OF A QUADRATIC EQUATION

Consider the quadratic equation $ax^2 + bx + c = 0$.

- If $b^2 - 4ac > 0$, then the equation has two real solutions.
- If $b^2 - 4ac = 0$, then the equation has one real solution.
- If $b^2 - 4ac < 0$, then the equation has two imaginary solutions.

EXAMPLE 4 Using the Discriminant

Find the discriminant of the quadratic equation and give the number and type of solutions of the equation.

a. $x^2 - 6x + 10 = 0$

b. $x^2 - 6x + 9 = 0$

c. $x^2 - 6x + 8 = 0$

SOLUTION

EQUATION	DISCRIMINANT	SOLUTION(S)
$ax^2 + bx + c = 0$	$b^2 - 4ac$	$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
a. $x^2 - 6x + 10 = 0$	$(-6)^2 - 4(1)(10) = -4$	Two imaginary: $3 \pm i$
b. $x^2 - 6x + 9 = 0$	$(-6)^2 - 4(1)(9) = 0$	One real: 3
c. $x^2 - 6x + 8 = 0$	$(-6)^2 - 4(1)(8) = 4$	Two real: 2, 4

In Example 4 notice that the number of real solutions of $x^2 - 6x + c = 0$ can be changed just by changing the value of c . A graph can help you see why this occurs. By changing c , you can move the graph of

$$y = x^2 - 6x + c$$

up or down in the coordinate plane. If the graph is moved too high, it won't have an x -intercept and the equation $x^2 - 6x + c = 0$ won't have a real-number solution.

$$y = x^2 - 6x + 10$$

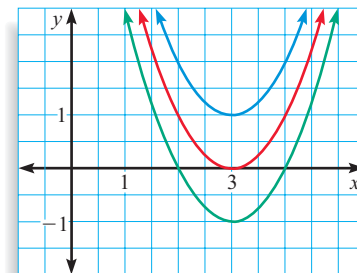
Graph is above x -axis (no x -intercept).

$$y = x^2 - 6x + 9$$

Graph touches x -axis (one x -intercept).

$$y = x^2 - 6x + 8$$

Graph crosses x -axis (two x -intercepts).

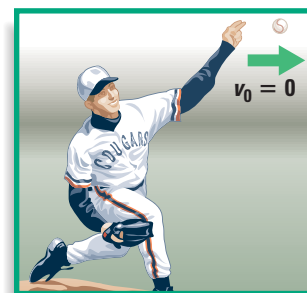
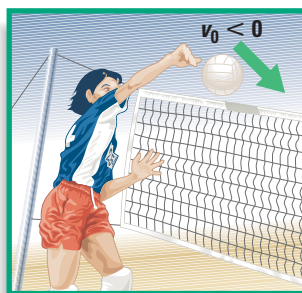
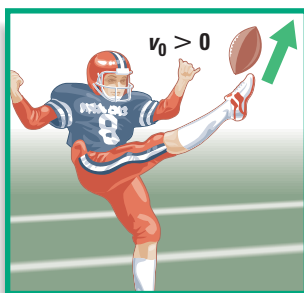


**GOAL 2 USING THE QUADRATIC FORMULA IN REAL LIFE**

In Lesson 5.3 you studied the model $h = -16t^2 + h_0$ for the height of an object that is *dropped*. For an object that is *launched or thrown*, an extra term v_0t must be added to the model to account for the object's initial vertical velocity v_0 .

Models	$h = -16t^2 + h_0$	Object is dropped.
	$h = -16t^2 + v_0t + h_0$	Object is launched or thrown.
Labels	$h =$ height	(feet)
	$t =$ time in motion	(seconds)
	$h_0 =$ initial height	(feet)
	$v_0 =$ initial vertical velocity	(feet per second)

The initial vertical velocity of a launched object can be positive, negative, or zero. If the object is launched upward, its initial vertical velocity is positive ($v_0 > 0$). If the object is launched downward, its initial vertical velocity is negative ($v_0 < 0$). If the object is launched parallel to the ground, its initial vertical velocity is zero ($v_0 = 0$).

**EXAMPLE 5 Solving a Vertical Motion Problem**

A baton twirler tosses a baton into the air. The baton leaves the twirler's hand 6 feet above the ground and has an initial vertical velocity of 45 feet per second. The twirler catches the baton when it falls back to a height of 5 feet. For how long is the baton in the air?

SOLUTION

Since the baton is thrown (not dropped), use the model $h = -16t^2 + v_0t + h_0$ with $v_0 = 45$ and $h_0 = 6$. To determine how long the baton is in the air, find the value of t for which $h = 5$.

$$h = -16t^2 + v_0t + h_0 \quad \text{Write height model.}$$

$$5 = -16t^2 + 45t + 6 \quad h = 5, v_0 = 45, h_0 = 6$$

$$0 = -16t^2 + 45t + 1 \quad a = -16, b = 45, c = 1$$

$$t = \frac{-45 \pm \sqrt{2089}}{-32} \quad \text{Quadratic formula}$$

$$t \approx -0.022 \text{ or } t \approx 2.8 \quad \text{Use a calculator.}$$

- ▶ Reject the solution -0.022 since the baton's time in the air cannot be negative. The baton is in the air for about 2.8 seconds.

GUIDED PRACTICE

Vocabulary Check ✓

Concept Check ✓


Skill Check ✓

- In the quadratic formula, what is the expression $b^2 - 4ac$ called?
- How many solutions does a quadratic equation have if its discriminant is positive? if its discriminant is zero? if its discriminant is negative?
- Describe a real-life situation in which you can use the model $h = -16t^2 + v_0t + h_0$ but not the model $h = -16t^2 + h_0$.

Use the quadratic formula to solve the equation.

- | | |
|------------------------|--------------------------------|
| 4. $x^2 - 4x + 3 = 0$ | 5. $x^2 + x - 1 = 0$ |
| 6. $2x^2 + 3x + 5 = 0$ | 7. $9x^2 + 6x - 1 = 0$ |
| 8. $-x^2 + 8x = 1$ | 9. $5x^2 - 2x + 37 = x^2 + 2x$ |

Find the discriminant of the quadratic equation and give the number and type of solutions of the equation.

- | | | |
|--------------------------|--------------------------|-------------------------|
| 10. $x^2 + 5x + 2 = 0$ | 11. $x^2 + 2x + 5 = 0$ | 12. $4x^2 - 4x + 1 = 0$ |
| 13. $-2x^2 + 3x - 7 = 0$ | 14. $9x^2 + 12x + 4 = 0$ | 15. $5x^2 - x - 13 = 0$ |
16.  **BASKETBALL** A basketball player passes the ball to a teammate who catches it 11 ft above the court, just above the rim of the basket, and slam-dunks it through the hoop. (This play is called an “alley-oop.”) The first player releases the ball 5 ft above the court with an initial vertical velocity of 21 ft/sec. How long is the ball in the air before being caught, assuming it is caught as it rises?

PRACTICE AND APPLICATIONS

STUDENT HELP

→ **Extra Practice**
to help you master
skills is on p. 946.

EQUATIONS IN STANDARD FORM Use the quadratic formula to solve the equation.

- | | | |
|--------------------------|---------------------------|--------------------------|
| 17. $x^2 - 5x - 14 = 0$ | 18. $x^2 + 3x - 2 = 0$ | 19. $x^2 - 2x - 4 = 0$ |
| 20. $x^2 + 10x + 22 = 0$ | 21. $x^2 + 6x + 58 = 0$ | 22. $-x^2 + 7x - 19 = 0$ |
| 23. $5x^2 + 3x - 1 = 0$ | 24. $3x^2 - 11x - 4 = 0$ | 25. $2x^2 + x + 1 = 0$ |
| 26. $6p^2 - 8p + 3 = 0$ | 27. $-7q^2 + 2q + 9 = 0$ | 28. $8r^2 + 4r + 5 = 0$ |
| 29. $-4t^2 - 9t - 3 = 0$ | 30. $9u^2 - 12u + 85 = 0$ | 31. $10v^2 + 8v - 1 = 0$ |

EQUATIONS NOT IN STANDARD FORM Use the quadratic formula to solve the equation.

- | | |
|---------------------------------|--|
| 32. $x^2 + 4x = -20$ | 33. $x^2 - 2x = 99$ |
| 34. $x^2 + 14 = 10x$ | 35. $x^2 = 8x - 35$ |
| 36. $-x^2 - 3x = -7$ | 37. $-x^2 = 16x + 46$ |
| 38. $3x^2 + 6x = -2$ | 39. $8x^2 - 8x = 1$ |
| 40. $5x^2 + 9x = -x^2 + 5x + 1$ | 41. $40x - 7x^2 = 101 - 3x^2$ |
| 42. $-16k^2 = 20k^2 + 24k + 5$ | 43. $13n^2 + 11n - 9 = 4n^2 - n - 4$ |
| 44. $3(d - 1)^2 = 4d + 2$ | 45. $3.5y^2 + 2.6y - 8.2 = -0.4y^2 - 6.9y$ |

STUDENT HELP

→ **HOMEWORK HELP**
Examples 1–3: Exs. 17–55
Example 4: Exs. 56–64
Example 5: Exs. 74–80

SOLVING BY ANY METHOD Solve the equation by factoring, by finding square roots, or by using the quadratic formula.

46. $6x^2 - 12 = 0$

47. $x^2 - 3x - 15 = 0$

48. $x^2 + 4x + 29 = 0$

49. $x^2 - 18x + 32 = 0$

50. $4x^2 + 28x = -49$

51. $3(x + 4)^2 = -27$

52. $-2u^2 + 5 = 3u^2 - 10u$

53. $11m^2 - 1 = 7m^2 + 2$

54. $-9v^2 + 35v - 30 = 1 - v$

55. $20p^2 + 6p = 6p^2 - 13p + 3$

USING THE DISCRIMINANT Find the discriminant of the quadratic equation and give the number and type of solutions of the equation.

56. $x^2 - 4x + 10 = 0$

57. $x^2 + 3x - 6 = 0$

58. $x^2 + 14x + 49 = 0$

59. $3x^2 - 10x - 5 = 0$

60. $64x^2 - 16x + 1 = 0$

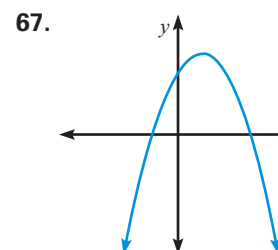
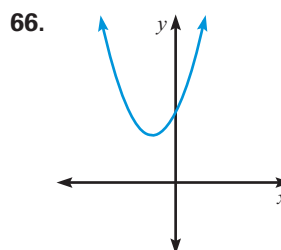
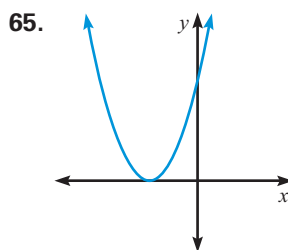
61. $-2x^2 - 5x - 4 = 0$

62. $7r^2 - 3 = 0$

63. $s^2\sqrt{5} + s + \sqrt{5} = 0$

64. $-4t^2 + 20t - 25 = 0$

VISUAL THINKING In Exercises 65–67, the graph of a quadratic function $y = ax^2 + bx + c$ is shown. Tell whether the discriminant of $ax^2 + bx + c = 0$ is **positive, negative, or zero**.



THE CONSTANT TERM Find all values of c for which the equation has (a) two real solutions, (b) one real solution, and (c) two imaginary solutions.

68. $x^2 - 2x + c = 0$

69. $x^2 + 4x + c = 0$

70. $x^2 + 10x + c = 0$

71. $x^2 - 8x + c = 0$

72. $x^2 + 6x + c = 0$

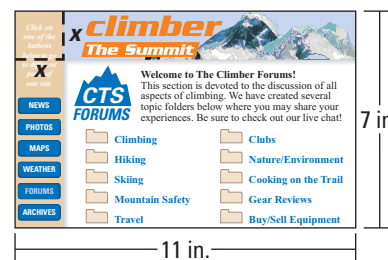
73. $x^2 - 12x + c = 0$

74. **CRITICAL THINKING** Explain why the height model $h = -16t^2 + v_0t + h_0$ applies not only to launched or thrown objects, but to dropped objects as well. (*Hint:* What is the initial vertical velocity of a dropped object?)

75. **DIVING** In July of 1997, the first Cliff Diving World Championships were held in Brontallo, Switzerland. Participants performed acrobatic dives from heights of up to 92 feet. Suppose a cliff diver jumps from this height with an initial upward velocity of 5 feet per second. How much time does the diver have to perform acrobatic maneuvers before hitting the water?

▶ Source: World High Diving Federation

76. **WORLD WIDE WEB** A Web developer is creating a Web site devoted to mountain climbing. Each page on the Web site will have frames along its top and left sides showing the name of the site and links to different parts of the site. These frames will take up one third of the computer screen. What will the width x of the frames be on the screen shown?



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WEB DEVELOPER


Web developers use *hypertext markup language* (HTML) to create electronic pages for the World Wide Web. A *Web browser* translates HTML into pages that can be viewed on a computer screen.

CAREER LINK

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77.  **VOLLEYBALL** In a volleyball game, a player on one team spikes the ball over the net when the ball is 10 feet above the court. The spike drives the ball downward with an initial vertical velocity of -55 feet per second. Players on the opposing team must hit the ball back over the net before the ball touches the court. How much time do the opposing players have to hit the spiked ball?

78.  **AVIATION** The length l (in feet) of runway needed for a small airplane to land is given by $l = 0.1s^2 - 3s + 22$ where s is the airplane's speed (in feet per second). If a pilot is landing a small airplane on a runway 2000 feet long, what is the maximum speed at which the pilot can land?

79.  **TELECOMMUNICATIONS** For the years 1989–1996, the amount A (in billions of dollars) spent on long distance telephone calls in the United States can be modeled by $A = 0.560t^2 + 0.488t + 51$ where t is the number of years since 1989. In what year did the amount spent reach \$60 billion?



DATA UPDATE of *Statistical Abstract of the United States* data at www.mcdougallittell.com

80.  **EARTH SCIENCE** The volcanic cinder cone Puu Puai in Hawaii was formed in 1959 when a massive “lava fountain” erupted at Kilauea Iki Crater, shooting lava hundreds of feet into the air. When the eruption was most intense, the height h (in feet) of the lava t seconds after being ejected from the ground could be modeled by $h = -16t^2 + 350t$.  Source: Volcano World

a. What was the initial vertical velocity of the lava? What was the lava's maximum height above the ground?

b. **CHOOSING A METHOD** For how long was the lava in the air? Solve the problem either by factoring or by using the quadratic formula.



Test Preparation

QUANTITATIVE COMPARISON In Exercises 81–83, choose the statement that is true about the given quantities.

- (A) The quantity in column A is greater.
- (B) The quantity in column B is greater.
- (C) The two quantities are equal.
- (D) The relationship cannot be determined from the given information.

	Column A	Column B
81.	Discriminant of $x^2 - 6x - 1 = 0$	Discriminant of $x^2 + 5x - 4 = 0$
82.	Discriminant of $x^2 + 2kx + 1 = 0$	Discriminant of $kx^2 + 3x - k = 0$
83.	Least zero of $f(x) = x^2 - 10x + 23$	Greatest zero of $f(x) = x^2 - 2x - 2$

★ Challenge

84.  **THRILL RIDES** The Stratosphere Tower in Las Vegas is 921 feet tall and has a “needle” at its top that extends even higher into the air. A thrill ride called the Big Shot catapults riders 160 feet up the needle and then lets them fall back to the launching pad.  Source: Stratosphere Tower

a. The height h (in feet) of a rider on the Big Shot can be modeled by $h = -16t^2 + v_0t + 921$ where t is the elapsed time (in seconds) after launch and v_0 is the initial vertical velocity (in feet per second). Find v_0 using the fact that the maximum value of h is $921 + 160 = 1081$ feet.

b. A brochure for the Big Shot states that the ride up the needle takes 2 seconds. Compare this time with the time given by the model $h = -16t^2 + v_0t + 921$ where v_0 is the value you found in part (a). Discuss the model's accuracy.

EXTRA CHALLENGE

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MIXED REVIEW

SOLVING LINEAR INEQUALITIES Solve the inequality. Then graph your solution. (Review 1.6 for 5.7)

85. $3x + 6 > 12$

86. $16 - 7x \geq -5$

87. $-2(x + 9) \leq 8$

88. $10x + 3 < 6x - 1$

89. $4 \leq 5x - 11 \leq 29$

90. $\frac{3}{2}x + 20 \leq 14$ or $1 > 8 - x$

GRAPHING LINEAR INEQUALITIES Graph the inequality. (Review 2.6 for 5.7)

91. $y > x$

92. $y \leq -2x$

93. $y < 3x - 2$

94. $x + y > 5$

95. $2x - 3y \geq 12$

96. $7x + 4y \leq -28$

ABSOLUTE VALUE FUNCTIONS Graph the function. (Review 2.8)

97. $y = |x - 3|$

98. $y = |x| + 2$

99. $y = -2|x| - 1$

100. $y = 3|x + 4|$

101. $y = |x + 2| + 3$

102. $y = \frac{1}{2}|x - 5| - 4$

QUIZ 2

Self-Test for Lessons 5.4–5.6

Write the expression as a complex number in standard form. (Lesson 5.4)

1. $(7 + 5i) + (-2 + 11i)$

2. $(-1 + 8i) - (3 - 2i)$

3. $(4 - i)(6 + 7i)$

4. $\frac{1 - 3i}{5 + i}$

Plot the numbers in the same complex plane and find their absolute values. (Lesson 5.4)

5. $2 + 4i$

6. $-5i$

7. $-3 + i$

8. $4 + 3i$

9. -4

10. $-\frac{3}{2} - \frac{7}{2}i$

Solve the quadratic equation by completing the square. (Lesson 5.5)

11. $x^2 + 8x = -14$

12. $x^2 - 2x + 17 = 0$

13. $4p^2 - 40p - 8 = 0$

14. $3q^2 + 20q = -2q^2 - 19$

Write the quadratic function in vertex form. (Lesson 5.5)

15. $y = x^2 + 6x + 1$

16. $y = x^2 - 18x + 50$

17. $y = -2x^2 + 8x - 7$


Use the quadratic formula to solve the equation. (Lesson 5.6)

18. $x^2 + 2x - 10 = 0$

19. $x^2 - 16x + 73 = 0$

20. $3w^2 + 3w = 4w^2 + 4$

21. $14 + 2y - 25y^2 = 42y + 6$

22.  **ENTERTAINMENT** A juggler throws a ball into the air, releasing it 5 feet above the ground with an initial vertical velocity of 15 ft/sec. She catches the ball with her other hand when the ball is 4 feet above the ground. Using the model $h = -16t^2 + v_0t + h_0$, find how long the ball is in the air. (Lesson 5.6)