

6.2

Evaluating and Graphing Polynomial Functions

What you should learn

GOAL 1 Evaluate a polynomial function.

GOAL 2 Graph a polynomial function, as applied in Example 5.

Why you should learn it

▼ To find values of **real-life** functions, such as the amount of prize money awarded at the U.S. Open Tennis Tournament in Ex. 86.

**GOAL 1** EVALUATING POLYNOMIAL FUNCTIONS

A **polynomial function** is a function of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

where $a_n \neq 0$, the exponents are all whole numbers, and the coefficients are all real numbers. For this polynomial function, a_n is the **leading coefficient**, a_0 is the **constant term**, and n is the **degree**. A polynomial function is in **standard form** if its terms are written in descending order of exponents from left to right.

You are already familiar with some types of polynomial functions. For instance, the linear function $f(x) = 3x + 2$ is a polynomial function of degree 1. The quadratic function $f(x) = x^2 + 3x + 2$ is a polynomial function of degree 2. Here is a summary of common types of polynomial functions.

Degree	Type	Standard form
0	Constant	$f(x) = a_0$
1	Linear	$f(x) = a_1 x + a_0$
2	Quadratic	$f(x) = a_2 x^2 + a_1 x + a_0$
3	Cubic	$f(x) = a_3 x^3 + a_2 x^2 + a_1 x + a_0$
4	Quartic	$f(x) = a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0$

EXAMPLE 1 Identifying Polynomial Functions

Decide whether the function is a polynomial function. If it is, write the function in standard form and state its degree, type, and leading coefficient.

a. $f(x) = \frac{1}{2}x^2 - 3x^4 - 7$

b. $f(x) = x^3 + 3^x$

c. $f(x) = 6x^2 + 2x^{-1} + x$

d. $f(x) = -0.5x + \pi x^2 - \sqrt{2}$

SOLUTION

a. The function is a polynomial function. Its standard form is $f(x) = -3x^4 + \frac{1}{2}x^2 - 7$. It has degree 4, so it is a quartic function. The leading coefficient is -3 .

b. The function is not a polynomial function because the term 3^x does not have a variable base and an exponent that is a whole number.

c. The function is not a polynomial function because the term $2x^{-1}$ has an exponent that is not a whole number.

d. The function is a polynomial function. Its standard form is $f(x) = \pi x^2 - 0.5x - \sqrt{2}$. It has degree 2, so it is a quadratic function. The leading coefficient is π .

One way to evaluate a polynomial function is to use direct substitution. For instance, $f(x) = 2x^4 - 8x^2 + 5x - 7$ can be evaluated when $x = 3$ as follows.

$$\begin{aligned} f(3) &= 2(3)^4 - 8(3)^2 + 5(3) - 7 \\ &= 162 - 72 + 15 - 7 \\ &= 98 \end{aligned}$$

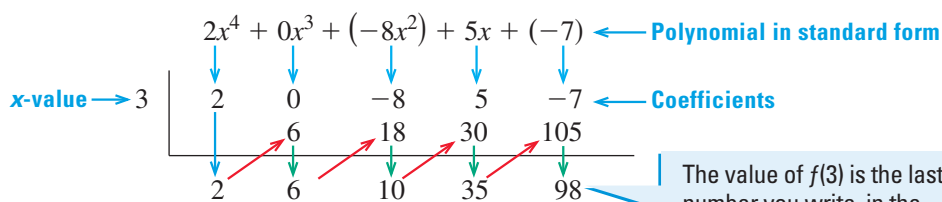
Another way to evaluate a polynomial function is to use **synthetic substitution**.

EXAMPLE 2 Using Synthetic Substitution

Use synthetic substitution to evaluate $f(x) = 2x^4 - 8x^2 + 5x - 7$ when $x = 3$.

SOLUTION

Write the value of x and the coefficients of $f(x)$ as shown. Bring down the leading coefficient. **Multiply by 3** and write the result in the next column. **Add** the numbers in that column and write the sum below the line. Continue to multiply and add, as shown.



The value of $f(3)$ is the last number you write, in the bottom right-hand corner.

▶ $f(3) = 98$

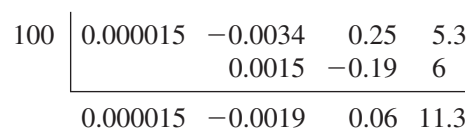
Using synthetic substitution is equivalent to evaluating the polynomial in *nested form*.

$f(x) = 2x^4 + 0x^3 - 8x^2 + 5x - 7$	Write original function.
$= (2x^3 + 0x^2 - 8x + 5)x - 7$	Factor x out of first 4 terms.
$= ((2x^2 + 0x - 8)x + 5)x - 7$	Factor x out of first 3 terms.
$= (((2x + 0)x - 8)x + 5)x - 7$	Factor x out of first 2 terms.

EXAMPLE 3 Evaluating a Polynomial Function in Real Life

PHOTOGRAPHY The time t (in seconds) it takes a camera battery to recharge after flashing n times can be modeled by $t = 0.000015n^3 - 0.0034n^2 + 0.25n + 5.3$. Find the recharge time after 100 flashes. ▶ Source: *Popular Photography*

SOLUTION



▶ The recharge time is about 11 seconds.

STUDENT HELP

Study Tip

In Example 2, note that the row of coefficients for $f(x)$ must include a coefficient of 0 for the “missing” x^3 -term.

FOCUS ON CAREERS



PHOTOGRAPHER

Some photographers work in advertising, some work for newspapers, and some are self-employed. Others specialize in aerial, police, medical, or scientific photography.

CAREER LINK

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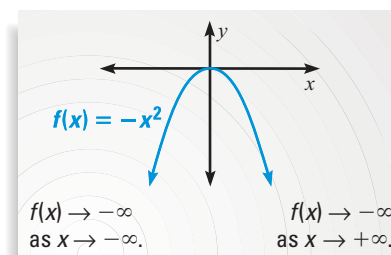
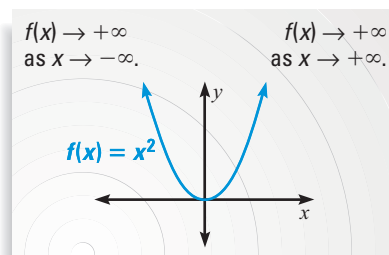
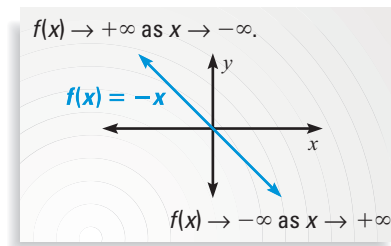
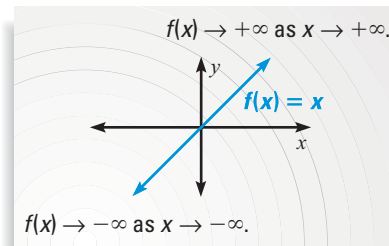
GOAL 2 GRAPHING POLYNOMIAL FUNCTIONS

STUDENT HELP

Look Back

For help with graphing functions, see pp. 69 and 250.

The **end behavior** of a polynomial function's graph is the behavior of the graph as x approaches positive infinity ($+\infty$) or negative infinity ($-\infty$). The expression $x \rightarrow +\infty$ is read as "x approaches positive infinity."



ACTIVITY

Developing Concepts



Investigating End Behavior

- Use a graphing calculator to graph each function. Then complete these statements: $f(x) \rightarrow ?$ as $x \rightarrow -\infty$ and $f(x) \rightarrow ?$ as $x \rightarrow +\infty$.

a. $f(x) = x^3$	b. $f(x) = x^4$	c. $f(x) = x^5$	d. $f(x) = x^6$
e. $f(x) = -x^3$	f. $f(x) = -x^4$	g. $f(x) = -x^5$	h. $f(x) = -x^6$
- How does the sign of the leading coefficient affect the behavior of a polynomial function's graph as $x \rightarrow +\infty$?
- How is the behavior of a polynomial function's graph as $x \rightarrow +\infty$ related to its behavior as $x \rightarrow -\infty$ when the function's degree is odd? when it is even?

In the activity you may have discovered that the end behavior of a polynomial function's graph is determined by the function's degree and leading coefficient.

CONCEPT SUMMARY

END BEHAVIOR FOR POLYNOMIAL FUNCTIONS

The graph of $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$ has this end behavior:

- For $a_n > 0$ and n even, $f(x) \rightarrow +\infty$ as $x \rightarrow -\infty$ and $f(x) \rightarrow +\infty$ as $x \rightarrow +\infty$.
- For $a_n > 0$ and n odd, $f(x) \rightarrow -\infty$ as $x \rightarrow -\infty$ and $f(x) \rightarrow +\infty$ as $x \rightarrow +\infty$.
- For $a_n < 0$ and n even, $f(x) \rightarrow -\infty$ as $x \rightarrow -\infty$ and $f(x) \rightarrow -\infty$ as $x \rightarrow +\infty$.
- For $a_n < 0$ and n odd, $f(x) \rightarrow +\infty$ as $x \rightarrow -\infty$ and $f(x) \rightarrow -\infty$ as $x \rightarrow +\infty$.

STUDENT HELP



HOMEWORK HELP

Visit our Web site
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for extra examples.

EXAMPLE 4 Graphing Polynomial Functions

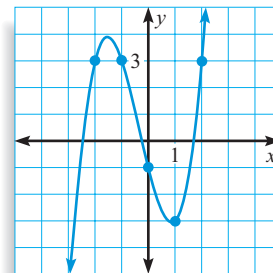
Graph (a) $f(x) = x^3 + x^2 - 4x - 1$ and (b) $f(x) = -x^4 - 2x^3 + 2x^2 + 4x$.

SOLUTION

- a. To graph the function, make a table of values and plot the corresponding points. Connect the points with a smooth curve and check the end behavior.

x	-3	-2	-1	0	1	2	3
$f(x)$	-7	3	3	-1	-3	3	23

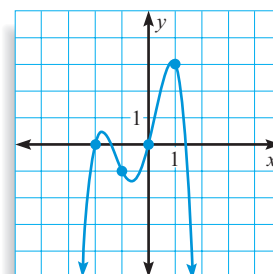
The degree is odd and the leading coefficient is positive, so $f(x) \rightarrow -\infty$ as $x \rightarrow -\infty$ and $f(x) \rightarrow +\infty$ as $x \rightarrow +\infty$.



- b. To graph the function, make a table of values and plot the corresponding points. Connect the points with a smooth curve and check the end behavior.

x	-3	-2	-1	0	1	2	3
$f(x)$	-21	0	-1	0	3	-16	-105

The degree is even and the leading coefficient is negative, so $f(x) \rightarrow -\infty$ as $x \rightarrow -\infty$ and $f(x) \rightarrow -\infty$ as $x \rightarrow +\infty$.



EXAMPLE 5 Graphing a Polynomial Model

A rainbow trout can grow up to 40 inches in length. The weight y (in pounds) of a rainbow trout is related to its length x (in inches) according to the model $y = 0.0005x^3$. Graph the model. Use your graph to estimate the length of a 10 pound rainbow trout.

SOLUTION

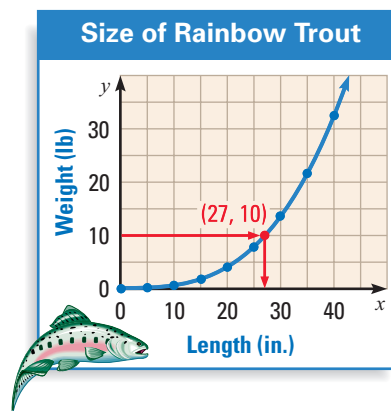
Make a table of values. The model makes sense only for positive values of x .

x	0	5	10	15	20	25	30	35	40
y	0	0.0625	0.5	1.69	4	7.81	13.5	21.4	32

Plot the points and connect them with a smooth curve, as shown at the right. Notice that the leading coefficient of the model is positive and the degree is odd, so the graph rises to the right.

Read the graph backwards to see that $x \approx 27$ when $y = 10$.

- ▶ A 10 pound trout is approximately 27 inches long.



GUIDED PRACTICE

Vocabulary Check ✓

1. Identify the degree, type, leading coefficient, and constant term of the polynomial function $f(x) = 5x - 2x^3$.

Concept Check ✓

2. Complete the synthetic substitution shown at the right. Describe each step of the process.

$$\begin{array}{r|rrrr}
 -2 & 3 & 1 & -9 & 2 \\
 & & ? & ? & ? \\
 \hline
 & 3 & ? & ? & 0
 \end{array}$$

3. Describe the graph of a constant function.

Skill Check ✓

Decide whether each function is a polynomial function. If it is, use synthetic substitution to evaluate the function when $x = -1$.

4. $f(x) = x^4\sqrt{5} - x$

5. $f(x) = x^3 + x^2 - x^{-3} + 3$

6. $f(x) = 6^{2x} - 12x$

7. $f(x) = 14 - 21x^2 + 5x^4$

Describe the end behavior of the graph of the polynomial function by completing the statements $f(x) \rightarrow ?$ as $x \rightarrow -\infty$ and $f(x) \rightarrow ?$ as $x \rightarrow +\infty$.

8. $f(x) = x^3 - 5x$


9. $f(x) = -x^5 - 3x^3 + 2$

10. $f(x) = x^4 - 4x^2 + x$


11. $f(x) = x + 12$

12. $f(x) = -x^2 + 3x + 1$

13. $f(x) = -x^8 + 9x^5 - 2x^4$


14.  **VIDEO RENTALS** The total revenue (actual and projected) from home video rentals in the United States from 1985 to 2005 can be modeled by

$$R = 1.8t^3 - 76t^2 + 1099t + 2600$$

where R is the revenue (in millions of dollars) and t is the number of years since 1985. Graph the function.  Source: *The Wall Street Journal Almanac*

PRACTICE AND APPLICATIONS

STUDENT HELP

 **Extra Practice** to help you master skills is on p. 947.

CLASSIFYING POLYNOMIALS Decide whether the function is a polynomial function. If it is, write the function in standard form and state the degree, type, and leading coefficient.

15. $f(x) = 12 - 5x$

16. $f(x) = 2x + \frac{3}{5}x^4 + 9$

17. $f(x) = x + \pi$

18. $f(x) = x^2\sqrt{2} + x - 5$

19. $f(x) = x - 3x^{-2} - 2x^3$

20. $f(x) = -2$

21. $f(x) = x^2 - x + 1$

22. $f(x) = 22 - 19x + 2^x$

23. $f(x) = 36x^2 - x^3 + x^4$

24. $f(x) = 3x^2 - 2x^{-x}$

25. $f(x) = 3x^3$

26. $f(x) = -6x^2 + x - \frac{3}{x}$

DIRECT SUBSTITUTION Use direct substitution to evaluate the polynomial function for the given value of x .

27. $f(x) = 2x^3 + 5x^2 + 4x + 8, x = -2$

28. $f(x) = 2x^3 - x^4 + 5x^2 - x, x = 3$

29. $f(x) = x + \frac{1}{2}x^3, x = 4$

30. $f(x) = x^2 - x^5 + 1, x = -1$

31. $f(x) = 5x^4 - 8x^3 + 7x^2, x = 1$

32. $f(x) = x^3 + 3x^2 - 2x + 5, x = -3$

33. $f(x) = 11x^3 - 6x^2 + 2, x = 0$

34. $f(x) = x^4 - 2x + 7, x = 2$

35. $f(x) = 7x^3 + 9x^2 + 3x, x = 10$

36. $f(x) = -x^5 - 4x^3 + 6x^2 - x, x = -2$

STUDENT HELP

HOMework HELP

Example 1: Exs. 15–26

Example 2: Exs. 37–46

Example 3: Exs. 81, 82

Example 4: Exs. 47–79

Example 5: Exs. 83–86

SYNTHETIC SUBSTITUTION Use synthetic substitution to evaluate the polynomial function for the given value of x .

37. $f(x) = 5x^3 + 4x^2 + 8x + 1, x = 2$

38. $f(x) = -3x^3 + 7x^2 - 4x + 8, x = 3$

39. $f(x) = x^3 + 3x^2 + 6x - 11, x = -5$

40. $f(x) = x^3 - x^2 + 12x + 15, x = -1$

41. $f(x) = -4x^3 + 3x - 5, x = 2$

42. $f(x) = -x^4 + x^3 - x + 1, x = -3$

43. $f(x) = 2x^4 + x^3 - 3x^2 + 5x, x = -1$

44. $f(x) = 3x^5 - 2x^2 + x, x = 2$

45. $f(x) = 2x^3 - x^2 + 6x, x = 5$

46. $f(x) = -x^4 + 8x^3 + 13x - 4, x = -2$

END BEHAVIOR PATTERNS Graph each polynomial function in the table. Then copy and complete the table to describe the end behavior of the graph of each function.

Function	As $x \rightarrow -\infty$	As $x \rightarrow +\infty$
$f(x) = -5x^3$?	?
$f(x) = -x^3 + 1$?	?
$f(x) = 2x - 3x^3$?	?
$f(x) = 2x^2 - x^3$?	?

Function	As $x \rightarrow -\infty$	As $x \rightarrow +\infty$
$f(x) = x^4 + 3x^3$?	?
$f(x) = x^4 + 2$?	?
$f(x) = x^4 - 2x - 1$?	?
$f(x) = 3x^4 - 5x^2$?	?

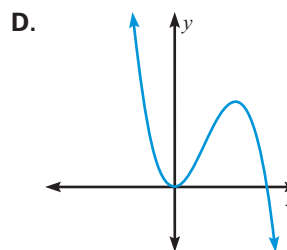
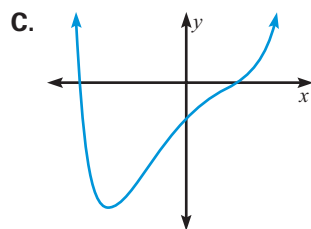
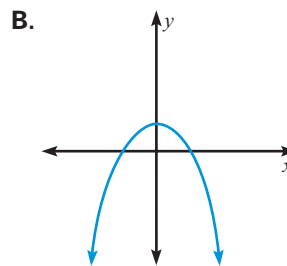
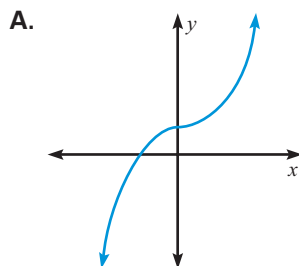
MATCHING Use what you know about end behavior to match the polynomial function with its graph.

49. $f(x) = 4x^6 - 3x^2 + 5x - 2$

50. $f(x) = -2x^3 + 5x^2$

51. $f(x) = -x^4 + 1$

52. $f(x) = 6x^3 + 1$



DESCRIBING END BEHAVIOR Describe the end behavior of the graph of the polynomial function by completing these statements: $f(x) \rightarrow ?$ as $x \rightarrow -\infty$ and $f(x) \rightarrow ?$ as $x \rightarrow +\infty$.

53. $f(x) = -5x^4$

54. $f(x) = -x^2 + 1$

55. $f(x) = 2x$

56. $f(x) = -10x^3$

57. $f(x) = -x^6 + 2x^3 - x$

58. $f(x) = x^5 + 2x^2$

59. $f(x) = -3x^5 - 4x^2 + 3$

60. $f(x) = x^7 - 3x^3 + 2x$

61. $f(x) = 3x^6 - x - 4$

62. $f(x) = 3x^8 - 4x^3$

63. $f(x) = -6x^3 + 10x$

64. $f(x) = x^4 - 5x^3 + x - 1$

GRAPHING POLYNOMIALS Graph the polynomial function.

65. $f(x) = -x^3$

66. $f(x) = -x^4$

67. $f(x) = x^5 + 2$

68. $f(x) = x^4 - 4$

69. $f(x) = x^4 + 6x^2 - 5$

70. $f(x) = 2 - x^3$

71. $f(x) = x^5 - 2$

72. $f(x) = -x^4 + 3$

73. $f(x) = -x^3 + 3x$

74. $f(x) = -x^3 + 2x^2 - 4$

75. $f(x) = -x^5 + x^2 + 1$

76. $f(x) = x^3 - 3x - 1$

77. $f(x) = x^5 + 3x^3 - x$

78. $f(x) = x^4 - 2x - 3$

79. $f(x) = -x^4 + 2x - 1$

80. **CRITICAL THINKING** Give an example of a polynomial function f such that $f(x) \rightarrow -\infty$ as $x \rightarrow -\infty$ and $f(x) \rightarrow +\infty$ as $x \rightarrow +\infty$.

81. **SHOPPING** The retail space in shopping centers in the United States from 1972 to 1996 can be modeled by

$$S = -0.0068t^3 - 0.27t^2 + 150t + 1700$$

where S is the amount of retail space (in millions of square feet) and t is the number of years since 1972. How much retail space was there in 1990?

82. **CABLE TELEVISION** The average monthly cable TV rate from 1980 to 1997 can be modeled by

$$R = -0.0036t^3 + 0.13t^2 - 0.073t + 7.7$$

where R is the monthly rate (in dollars) and t is the number of years since 1980. What was the monthly rate in 1983?

NURSING In Exercises 83 and 84, use the following information.

From 1985 to 1995, the number of graduates from nursing schools in the United States can be modeled by

$$y = -0.036t^4 + 0.605t^3 - 1.87t^2 - 4.67t + 82.5$$

where y is the number of graduates (in thousands) and t is the number of years since 1985. ▶ Source: *Statistical Abstract of the United States*

83. Describe the end behavior of the graph of the function. From the end behavior, would you expect the number of nursing graduates in the year 2010 to be more than or less than the number of nursing graduates in 1995? Explain.

84. Graph the function for $0 \leq t \leq 10$. Use the graph to find the first year in which there were over 82,500 nursing graduates.

TENNIS In Exercises 85 and 86, use the following information.

The amount of prize money for the women's U.S. Open Tennis Tournament from 1970 to 1997 can be modeled by

$$P = 1.141t^2 - 5.837t + 14.31$$

where P is the prize money (in thousands of dollars) and t is the number of years since 1970. ▶ Source: U.S. Open

85. Describe the end behavior of the graph of the function. From the end behavior, would you expect the amount of prize money in the year 2005 to be more than or less than the amount in 1995? Explain.

86. Graph the function for $0 \leq t \leq 40$. Use the graph to estimate the amount of prize money in the year 2005.

FOCUS ON CAREERS**REAL LIFE NURSE**

Although the majority of nurses work in hospitals, nurses also work in doctors' offices, in private homes, at nursing homes, and in other community settings.

CAREER LINK

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Test Preparation

- 87. MULTI-STEP PROBLEM** To determine whether a Holstein heifer's height is normal, a veterinarian can use the cubic functions

$$L = 0.0007t^3 - 0.061t^2 + 2.02t + 30$$

$$H = 0.001t^3 - 0.08t^2 + 2.3t + 31$$

where L is the minimum normal height (in inches), H is the maximum normal height (in inches), and t is the age (in months).

► Source: Journal of Dairy Science

- What is the normal height range for an 18-month-old Holstein heifer?
- Describe the end behavior of each function's graph.
- Graph the two height functions.
- Writing* Suppose a veterinarian examines a Holstein heifer that is 43 inches tall. About how old do you think the cow is? How did you get your answer?



A heifer is a young cow that has not yet had calves.

★ Challenge

 **EXAMINING END BEHAVIOR** Use a spreadsheet or a graphing calculator to evaluate the polynomial functions $f(x) = x^3$ and $g(x) = x^3 - 2x^2 + 4x + 5$ for the given values of x .

- 88.** Copy and complete the table.
- 89.** Use the results of Exercise 88 to complete this statement:

$$\text{As } x \rightarrow +\infty, \frac{f(x)}{g(x)} \rightarrow \underline{\quad}.$$

Explain how this statement shows that the functions f and g have the same end behavior as $x \rightarrow +\infty$.

x	$f(x)$	$g(x)$	$\frac{f(x)}{g(x)}$
50	?	?	?
100	?	?	?
500	?	?	?
1000	?	?	?
5000	?	?	?

EXTRA CHALLENGE

► www.mcdougallittell.com

MIXED REVIEW

SIMPLIFYING EXPRESSIONS Simplify the expression. (Review 1.2 for 6.3)

- 90.** $x + 3 - 2x - x + 2$ **91.** $-2x^2 + 3x + 4x + 2x^2$ **92.** $-3x^2 + 1 - (x^2 + 2)$
93. $x^2 + x + 1 + 3(x - 4)$ **94.** $4x - 2x^2 + 3 - x^2 - 4$ **95.** $x^2 - 1 - (2x^2 + x - 3)$

STANDARD FORM Write the quadratic function in standard form. (Review 5.1 for 6.3)

- 96.** $y = -4(x - 2)^2 + 5$ **97.** $y = -2(x + 6)(x - 5)$ **98.** $y = 2(x - 7)(x + 4)$
99. $y = 4(x - 3)^2 - 24$ **100.** $y = -(x + 5)^2 + 12$ **101.** $y = -3(x - 5)^2 + 3$

SOLVING QUADRATIC EQUATIONS Solve the equation. (Review 5.4)

- 102.** $x^2 = -9$ **103.** $x^2 = -5$ **104.** $-3x^2 + 1 = 7$
105. $4x^2 + 15 = 3$ **106.** $6x^2 + 5 = 2x^2 + 1$ **107.** $x^2 = 7x^2 + 1$
108. $x^2 - 4 = -3x^2 - 24$ **109.** $3x^2 + 5 = 5x^2 + 10$ **110.** $5x^2 + 2 = -2x^2 + 1$