6.4 Factoring and Solving Polynomial Equations

What *you should learn*

GOAL 1 Factor polynomial expressions.

GOAL 2 Use factoring to solve polynomial equations, as applied in **Ex. 87**.

Why *you should learn it*

▼ To solve **real-life** problems, such as finding the dimensions of a block discovered at an underwater archeological site in **Example 5**.

REAL LIFE

GOAL 1 FACTORING POLYNOMIAL EXPRESSIONS

In Chapter 5 you learned how to factor the following types of quadratic expressions.

TYPE	EXAMPLE
General trinomial	$2x^2 - 5x - 12 = (2x + 3)(x - 4)$
Perfect square trinomial	$x^2 + 10x + 25 = (x + 5)^2$
Difference of two squares	$4x^2 - 9 = (2x + 3)(2x - 3)$
Common monomial factor	$6x^2 + 15x = 3x(2x + 5)$

In this lesson you will learn how to factor other types of polynomials.

ACTIVITY Developing Concepts

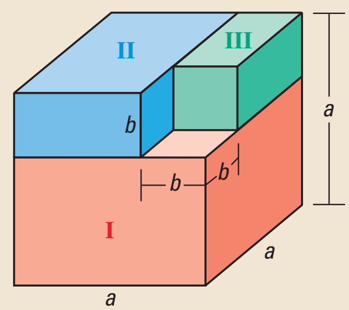
The Difference of Two Cubes

Use the diagram to answer the questions.

1. Explain why $a^3 - b^3 =$ [Volume of solid I] + [Volume of solid II] + [Volume of solid III].

2. For each of solid I, solid II, and solid III, write an algebraic expression for the solid's volume. Leave your expressions in factored form.

3. Substitute your expressions from **Step 2** into the equation from **Step 1**. Use the resulting equation to factor $a^3 - b^3$ completely.



In the activity you may have discovered how to factor the difference of two cubes. This factorization and the factorization of the sum of two cubes are given below.

SPECIAL FACTORING PATTERNS

SUM OF TWO CUBES	**Example**
$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$	$x^3 + 8 = (x + 2)(x^2 - 2x + 4)$
DIFFERENCE OF TWO CUBES	
$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$	$8x^3 - 1 = (2x - 1)(4x^2 + 2x + 1)$

EXAMPLE 1 Factoring the Sum or Difference of Cubes

Factor each polynomial.

a. $x^3 + 27$ **b.** $16u^5 - 250u^2$

SOLUTION

a. $x^3 + 27 = x^3 + 3^3$ **Sum of two cubes**

$= (x + 3)(x^2 - 3x + 9)$

b. $16u^5 - 250u^2 = 2u^2(8u^3 - 125)$ **Factor common monomial.**

$= 2u^2\left[(2u)^3 - 5^3\right]$ **Difference of two cubes**

$= 2u^2(2u - 5)(4u^2 + 10u + 25)$

. . . .

For some polynomials, you can **factor by grouping** pairs of terms that have a common monomial factor. The pattern for this is as follows.

$$ra + rb + sa + sb = r(a + b) + s(a + b)$$
$$= (r + s)(a + b)$$

EXAMPLE 2 Factoring by Grouping

Factor the polynomial $x^3 - 2x^2 - 9x + 18$.

SOLUTION

$x^3 - 2x^2 - 9x + 18 = x^2(x - 2) - 9(x - 2)$ **Factor by grouping.**

$= (x^2 - 9)(x - 2)$

$= (x + 3)(x - 3)(x - 2)$ **Difference of squares**

. . . .

An expression of the form $au^2 + bu + c$ where u is any expression in x is said to be in **quadratic form**. The factoring techniques you studied in Chapter 5 can sometimes be used to factor such expressions.

EXAMPLE 3 Factoring Polynomials in Quadratic Form

Factor each polynomial.

a. $81x^4 - 16$ **b.** $4x^6 - 20x^4 + 24x^2$

SOLUTION

a. $81x^4 - 16 = (9x^2)^2 - 4^2$

$= (9x^2 + 4)(9x^2 - 4)$

$= (9x^2 + 4)(3x + 2)(3x - 2)$

b. $4x^6 - 20x^4 + 24x^2 = 4x^2(x^4 - 5x^2 + 6)$

$= 4x^2(x^2 - 2)(x^2 - 3)$

GOAL 2 SOLVING POLYNOMIAL EQUATIONS BY FACTORING

In Chapter 5 you learned how to use the zero product property to solve factorable quadratic equations. You can extend this technique to solve some higher-degree polynomial equations.

EXAMPLE 4 *Solving a Polynomial Equation*

Solve $2x^5 + 24x = 14x^3$.

STUDENT HELP

Study Tip
In the solution of Example 4, do not divide both sides of the equation by a variable or a variable expression. Doing so will result in the loss of solutions.

SOLUTION

$2x^5 + 24x = 14x^3$ Write original equation.

$2x^5 - 14x^3 + 24x = 0$ Rewrite in standard form.

$2x(x^4 - 7x^2 + 12) = 0$ Factor common monomial.

$2x(x^2 - 3)(x^2 - 4) = 0$ Factor trinomial.

$2x(x^2 - 3)(x + 2)(x - 2) = 0$ Factor difference of squares.

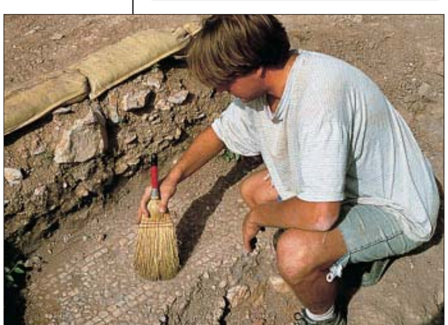
$x = 0, x = \sqrt{3}, x = -\sqrt{3}, x = -2, \text{ or } x = 2$ Zero product property

▶ The solutions are 0, $\sqrt{3}$, $-\sqrt{3}$, −2, and 2. Check these in the original equation.

EXAMPLE 5 *Solving a Polynomial Equation in Real Life*

ARCHEOLOGY In 1980 archeologists at the ruins of Caesara discovered a huge hydraulic concrete block with a volume of 330 cubic yards. The block's dimensions are x yards high by $13x - 11$ yards long by $13x - 15$ yards wide. What is the height?

FOCUS ON CAREERS



ARCHEOLOGIST
Archeologists excavate, classify, and date items used by ancient people. They may specialize in a particular geographical region and/or time period.

CAREER LINK
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SOLUTION

VERBAL MODEL Volume = Height · Length · Width

LABELS

Volume = 330 (cubic yards)

Height = x (yards)

Length = $13x - 11$ (yards)

Width = $13x - 15$ (yards)

ALGEBRAIC MODEL

$330 = x(13x - 11)(13x - 15)$

$0 = 169x^3 - 338x^2 + 165x - 330$ Write in standard form.

$0 = 169x^2(x - 2) + 165(x - 2)$ Factor by grouping.

$0 = (169x^2 + 165)(x - 2)$

▶ The only real solution is $x = 2$, so $13x - 11 = 15$ and $13x - 15 = 11$. The block is 2 yards high. The dimensions are 2 yards by 15 yards by 11 yards.

GUIDED PRACTICE

Vocabulary Check ✓

1. Give an example of a polynomial in quadratic form that contains an x^3-term.

Concept Check ✓

2. State which factoring method you would use to factor each of the following.

   a. $6x^3 - 2x^2 + 9x - 3$ **b.** $8x^3 - 125$ **c.** $16x^4 - 9$

3. **ERROR ANALYSIS** What is wrong with the solution at the right?

4. **a.** Factor the polynomial $x^3 + 1$ into the product of a linear binomial and a quadratic trinomial.

   b. Show that you can't factor the quadratic trinomial from part (a).

$$2x^4 - 18x^2 = 0$$
$$2x^2(x^2 - 9) = 0$$
$$x^2 - 9 = 0$$
$$(x + 3)(x - 3) = 0$$
$$x = -3 \text{ or } x = 3$$

Ex. 3

Skill Check ✓

Factor the polynomial using any method.

5. $x^6 + 125$
6. $4x^3 + 16x^2 + x + 4$
7. $x^4 - 1$
8. $2x^3 - 3x^2 - 10x + 15$
9. $5x^3 - 320$
10. $x^4 + 7x^2 + 10$

Find the real-number solutions of the equation.

11. $x^3 - 27 = 0$
12. $3x^3 + 7x^2 - 12x = 28$
13. $x^3 + 2x^2 - 9x = 18$
14. $54x^3 = -2$
15. $9x^4 - 12x^2 + 4 = 0$
16. $16x^8 = 81$
17. **BUSINESS** The revenue R (in thousands of dollars) for a small business can be modeled by

$$R = t^3 - 8t^2 + t + 82$$

where t is the number of years since 1990. In what year did the revenue reach $90,000?

PRACTICE AND APPLICATIONS

STUDENT HELP

Extra Practice to help you master skills is on p. 948.

MONOMIAL FACTORS **Find the greatest common factor of the terms in the polynomial.**

18. $14x^2 + 8x + 72$
19. $3x^4 - 12x^3$
20. $7x + 28x^2 - 35x^3$
21. $24x^4 - 6x$
22. $39x^5 + 13x^3 - 78x^2$
23. $145x^9 - 17$
24. $6x^6 - 3x^4 - 9x^2$
25. $72x^9 + 15x^6 + 9x^3$
26. $6x^4 - 18x^3 + 15x^2$

MATCHING **Match the polynomial with its factorization.**

27. $3x^2 + 11x + 6$
28. $x^3 - 4x^2 + 4x - 16$
29. $125x^3 - 216$
30. $2x^7 - 32x^3$
31. $2x^5 + 4x^4 - 4x^3 - 8x^2$
32. $2x^3 - 32x$

A. $2x^3(x + 2)(x - 2)(x^2 + 4)$
B. $2x(x + 4)(x - 4)$
C. $(3x + 2)(x + 3)$
D. $(x^2 + 4)(x - 4)$
E. $2x^2(x^2 - 2)(x + 2)$
F. $(5x - 6)(25x^2 + 30x + 36)$

STUDENT HELP

HOMEWORK HELP
Example 1: Exs. 18–40, 59–67
Example 2: Exs. 18–32, 41–49, 59–67
Example 3: Exs. 18–32, 50–67
Example 4: Exs. 68–85
Example 5: Exs. 87–92

SUM OR DIFFERENCE OF CUBES **Factor the polynomial.**

33. $x^3 - 8$ **34.** $x^3 + 64$ **35.** $216x^3 + 1$ **36.** $125x^3 - 8$

37. $1000x^3 + 27$ **38.** $27x^3 + 216$ **39.** $32x^3 - 4$ **40.** $2x^3 + 54$

GROUPING **Factor the polynomial by grouping.**

41. $x^3 + x^2 + x + 1$ **42.** $10x^3 + 20x^2 + x + 2$ **43.** $x^3 + 3x^2 + 10x + 30$

44. $x^3 - 2x^2 + 4x - 8$ **45.** $2x^3 - 5x^2 + 18x - 45$ **46.** $-2x^3 - 4x^2 - 3x - 6$

47. $3x^3 - 6x^2 + x - 2$ **48.** $2x^3 - x^2 + 2x - 1$ **49.** $3x^3 - 2x^2 - 9x + 6$

QUADRATIC FORM **Factor the polynomial.**

50. $16x^4 - 1$ **51.** $x^4 + 3x^2 + 2$ **52.** $x^4 - 81$

53. $81x^4 - 256$ **54.** $4x^4 - 5x^2 - 9$ **55.** $x^4 + 10x^2 + 16$

56. $81 - 16x^4$ **57.** $32x^6 - 2x^2$ **58.** $6x^5 - 51x^3 - 27x$

CHOOSING A METHOD **Factor using any method.**

59. $18x^3 - 2x^2 + 27x - 3$ **60.** $6x^3 + 21x^2 + 15x$ **61.** $4x^4 + 39x^2 - 10$

62. $8x^3 - 12x^2 - 2x + 3$ **63.** $8x^3 - 64$ **64.** $3x^4 - 300x^2$

65. $3x^4 - 24x$ **66.** $5x^4 + 31x^2 + 6$ **67.** $3x^4 + 9x^3 + x^2 + 3x$

SOLVING EQUATIONS **Find the real-number solutions of the equation.**

68. $x^3 - 3x^2 = 0$ **69.** $2x^3 - 6x^2 = 0$ **70.** $3x^4 + 15x^2 - 72 = 0$

71. $x^3 + 27 = 0$ **72.** $x^3 + 2x^2 - x = 2$ **73.** $x^4 + 7x^3 - 8x - 56 = 0$

74. $2x^4 - 26x^2 + 72 = 0$ **75.** $3x^7 - 243x^3 = 0$ **76.** $x^3 + 3x^2 - 2x - 6 = 0$

77. $8x^3 - 1 = 0$ **78.** $x^3 + 8x^2 = -16x$ **79.** $x^3 - 5x^2 + 5x - 25 = 0$

80. $3x^4 + 3x^3 = 6x^2 + 6x$ **81.** $x^4 + x^3 - x = 1$ **82.** $4x^4 + 20x^2 = -25$

83. $-2x^6 = 16$ **84.** $3x^7 = 81x^4$ **85.** $2x^5 - 12x^3 = -16x$

86. *Writing* You have now factored several different types of polynomials. Explain which factoring techniques or patterns are useful for factoring binomials, trinomials, and polynomials with more than three terms.

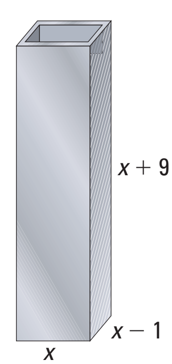
87. **PACKAGING** A candy factory needs a box that has a volume of 30 cubic inches. The width should be 2 inches less than the height and the length should be 5 inches greater than the height. What should the dimensions of the box be?

STUDENT HELP

INTERNET HOMEWORK HELP
Visit our Web site www.mcdougallittell.com for help with problem solving in Ex. 88.

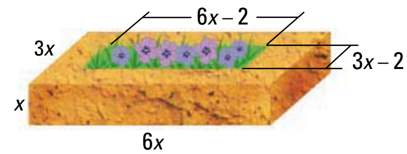
88. **MANUFACTURING** A manufacturer wants to build a rectangular stainless steel tank with a holding capacity of 500 gallons, or about 66.85 cubic feet. If steel that is one half inch thick is used for the walls of the tank, then about 5.15 cubic feet of steel is needed. The manufacturer wants the outside dimensions of the tank to be related as follows:

- The width should be one foot less than the length.
- The height should be nine feet more than the length.



What should the outside dimensions of the tank be?

89. **CITY PARK** For the city park commission, you are designing a marble planter in which to plant flowers. You want the length of the planter to be six times the height and the width to be three times the height. The sides should be one foot thick. Since the planter will be on the sidewalk, it does not need a bottom. What should the outer dimensions of the planter be if it is to hold 4 cubic feet of dirt?



SCULPTURE In Exercises 90 and 91, refer to the sculpture shown in the picture.

90. The "cube" portion of the sculpture is actually a rectangular prism with dimensions x feet by $5x - 10$ feet by $2x - 1$ feet. The volume of the prism is 25 cubic feet. What are the dimensions of the prism?

"Charred Sphere, Cube, and Pyramid" by David Nash

91. Suppose a pyramid like the one in the sculpture is $3x$ feet high and has a square base measuring $x - 5$ feet on each side. If the volume is 250 cubic feet, what are the dimensions of the pyramid? (Use the formula $V = \frac{1}{3}Bh$.)

92. **CRAFTS** Suppose you have 250 cubic inches of clay with which to make a rectangular prism for a sculpture. If you want the height and width each to be 5 inches less than the length, what should the dimensions of the prism be?

Test Preparation

93. **MULTIPLE CHOICE** The expression $(3x - 4)(9x^2 + 12x + 16)$ is the factorization of which of the following?

(A) $27x^3 - 8$ (B) $27x^3 + 36x^2$ (C) $27x^3 - 64$ (D) $27x^3 + 64$

94. **MULTIPLE CHOICE** Which of the following is the factorization of $x^3 - 8$?

(A) $(x - 2)(x^2 + 4x + 4)$ (B) $(x + 2)(x^2 - 2x + 4)$

(C) $(x + 2)(x^2 - 4x + 4)$ (D) $(x - 2)(x^2 + 2x + 4)$

95. **MULTIPLE CHOICE** What are the real solutions of the equation $x^5 = 81x$?

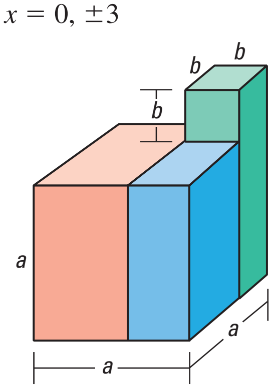
(A) $x = \pm 3, \pm 3i$ (B) $x = 0, \pm 9$

(C) $x = 0, \pm 3, \pm 3i$ (D) $x = 0, \pm 3$

★ Challenge

96. **GEOMETRY CONNECTION** Explain how the figure shown at the right can be used as a geometric factoring model for the sum of two cubes.

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$



Ex. 96

Factor the polynomial.

97. $30x^2y + 36x^2 - 20xy - 24x$

98. $2x^7 - 127x$

EXTRA CHALLENGE
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MIXED REVIEW

SIMPLIFYING EXPRESSIONS **Simplify the expression.** **(Review 6.1 for 6.5)**

99. $\dfrac{6x^3y^9}{36x^3y^{-2}}$ **100.** $\dfrac{5^{-2}x^2y^{-1}}{5^2xy^3}$ **101.** $\dfrac{7^2x^{-3}y^2}{49x^{-3}y^{-2}}$

SYNTHETIC SUBSTITUTION **Use synthetic substitution to evaluate the polynomial function for the given value of *x*.** **(Review 6.2 for 6.5)**

102. $f(x) = 3x^4 + 2x^3 - x^2 - 12x + 1, x = 3$

103. $f(x) = 2x^5 - x^3 + 7x + 1, x = 3$

104. **SEWING** At the fabric store you are buying solid fabric at \$4 per yard, print fabric at \$6 per yard, and a pattern for \$8. Write an equation for the amount you spend as a function of the amount of solid and print fabric you buy. **(Review 3.5)**

MATH & History — Solving Polynomial Equations

APPLICATION LINK www.mcdougallittell.com

THEN

IN 2000 B.C. the Babylonians solved polynomial equations by referring to tables of values. One such table gave the values of $y^3 + y^2$. To be able to use this table, the Babylonians sometimes had to manipulate the equation, as shown below.

$$ax^3 + bx^2 = c \qquad \text{Write original equation.}$$

$$\frac{a^3x^3}{b^3} + \frac{a^2x^2}{b^2} = \frac{a^2c}{b^3} \qquad \text{Multiply by } \frac{a^2}{b^3}.$$

$$\left(\frac{ax}{b}\right)^3 + \left(\frac{ax}{b}\right)^2 = \frac{a^2c}{b^3} \qquad \text{Re-express cubes and squares.}$$

Then they would find $\frac{a^2c}{b^3}$ in the $y^3 + y^2$ column of the table.

Because they knew that the corresponding y-value was equal to $\frac{ax}{b}$, they could conclude that $x = \frac{by}{a}$.

1. Calculate $y^3 + y^2$ for $y = 1, 2, 3, \ldots, 10$. Record the values in a table.

Use your table and the method discussed above to solve the equation.

2. $x^3 + x^2 = 252$ **3.** $x^3 + 2x^2 = 288$ **4.** $3x^3 + x^2 = 90$

5. $2x^3 + 5x^2 = 2500$ **6.** $7x^3 + 6x^2 = 1728$ **7.** $10x^3 + 3x^2 = 297$

NOW

TODAY computers use polynomial equations to accomplish many things, such as making robots move.