7.1 *n*th Roots and Rational Exponents

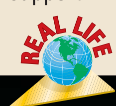
What you should learn

GOAL 1 Evaluate *n*th roots of real numbers using both radical notation and rational exponent notation.

GOAL 2 Use *n*th roots to solve **real-life** problems, such as finding the total mass of a spacecraft that can be sent to Mars in **Example 5**.

Why you should learn it

▼ To solve **real-life** problems, such as finding the number of reptile and amphibian species that Puerto Rico can support in **Ex. 67**.



GOAL 1 EVALUATING *N*TH ROOTS

You can extend the concept of a square root to other types of roots. For instance, 2 is a cube root of 8 because $2^3 = 8$, and 3 is a fourth root of 81 because $3^4 = 81$. In general, for an integer n greater than 1, if $b^n = a$, then b is an **nth root of a**. An nth root of a is written as $\sqrt[n]{a}$, where n is the **index** of the radical.

You can also write an nth root of a as a power of a. For the particular case of a square root, suppose that $\sqrt{a} = a^k$. Then you can determine a value for k as follows:

$$\sqrt{a} \cdot \sqrt{a} = a$$ **Definition of square root**

$$a^k \cdot a^k = a$$ **Substitute a^k for $\sqrt{a}$.**

$$a^{2k} = a^1$$ **Product of powers property**

$$2k = 1$$ **Set exponents equal when bases are equal.**

$$k = \frac{1}{2}$$ **Solve for k.**

Therefore, you can see that $\sqrt{a} = a^{1/2}$. In a similar way you can show that $\sqrt[3]{a} = a^{1/3}$ and $\sqrt[4]{a} = a^{1/4}$. In general, $\sqrt[n]{a} = a^{1/n}$ for any integer n greater than 1.

REAL *N*TH ROOTS

Let n be an integer greater than 1 and let a be a real number.

- If n is odd, then a has one real nth root: $\sqrt[n]{a} = a^{1/n}$
- If n is even and $a > 0$, then a has two real nth roots: $\pm\sqrt[n]{a} = \pm a^{1/n}$
- If n is even and $a = 0$, then a has one nth root: $\sqrt[n]{0} = 0^{1/n} = 0$
- If n is even and $a < 0$, then a has no real nth roots.

EXAMPLE 1 *Finding nth Roots*

Find the indicated real nth root(s) of a.

a. $n = 3, a = -125$ **b.** $n = 4, a = 16$

SOLUTION

a. Because $n = 3$ is odd, $a = -125$ has one real cube root. Because $(-5)^3 = -125$, you can write:

$$\sqrt[3]{-125} = -5 \quad \text{or} \quad (-125)^{1/3} = -5$$

b. Because $n = 4$ is even and $a = 16 > 0$, 16 has two real fourth roots. Because $2^4 = 16$ and $(-2)^4 = 16$, you can write:

$$\pm\sqrt[4]{16} = \pm 2 \quad \text{or} \quad \pm 16^{1/4} = \pm 2$$

A rational exponent does not have to be of the form $\frac{1}{n}$ where n is an integer greater than 1. Other rational numbers such as $\frac{3}{2}$ and $-\frac{1}{2}$ can also be used as exponents.

RATIONAL EXPONENTS

Let $a^{1/n}$ be an nth root of a, and let m be a positive integer.

- $a^{m/n} = (a^{1/n})^m = (\sqrt[n]{a})^m$
- $a^{-m/n} = \frac{1}{a^{m/n}} = \frac{1}{(a^{1/n})^m} = \frac{1}{(\sqrt[n]{a})^m}, \ a \neq 0$

EXAMPLE 2 *Evaluating Expressions with Rational Exponents*

a. $9^{3/2} = (\sqrt{9})^3 = 3^3 = 27$ — **Using radical notation**

$9^{3/2} = (9^{1/2})^3 = 3^3 = 27$ — **Using rational exponent notation**

b. $32^{-2/5} = \frac{1}{32^{2/5}} = \frac{1}{(\sqrt[5]{32})^2} = \frac{1}{2^2} = \frac{1}{4}$ — **Using radical notation**

$32^{-2/5} = \frac{1}{32^{2/5}} = \frac{1}{(32^{1/5})^2} = \frac{1}{2^2} = \frac{1}{4}$ — **Using rational exponent notation**

When using a graphing calculator to approximate an nth root, you may have to rewrite the nth root using a rational exponent. Then use the calculator's power key.

EXAMPLE 3 *Approximating a Root with a Calculator*

Use a graphing calculator to approximate $(\sqrt[4]{5})^3$.

SOLUTION First rewrite $(\sqrt[4]{5})^3$ as $5^{3/4}$. Then enter the following:

Keystrokes: 5 [^] [(] 3 [÷] 4 [)] [ENTER] **Display:** 3.343701525

▶ $(\sqrt[4]{5})^3 \approx 3.34$

STUDENT HELP

Study Tip

To use a scientific calculator in Example 3, replace [^] with [y^x] and replace [ENTER] with [=].

To solve simple equations involving x^n, isolate the power and then take the nth root of *each* side.

EXAMPLE 4 *Solving Equations Using nth Roots*

a.
$$2x^4 = 162$$
$$x^4 = 81$$
$$x = \pm\sqrt[4]{81}$$
$$x = \pm 3$$

b.
$$(x-2)^3 = 10$$
$$x - 2 = \sqrt[3]{10}$$
$$x = \sqrt[3]{10} + 2$$
$$x \approx 4.15$$

GOAL 2 USING *N*TH ROOTS IN REAL LIFE

REAL LIFE
Space Science

EXAMPLE 5 *Evaluating a Model with nth Roots*

The total mass M (in kilograms) of a spacecraft that can be propelled by a magnetic sail is, in theory, given by

$$M = \frac{0.015m^2}{fd^{4/3}}$$

where m is the mass (in kilograms) of the magnetic sail, f is the drag force (in newtons) of the spacecraft, and d is the distance (in astronomical units) to the sun. Find the total mass of a spacecraft that can be sent to Mars using $m = 5000$ kg, $f = 4.52$ N, and $d = 1.52$ AU. ▶ Source: *Journal of Spacecraft and Rockets*

Artist's rendition of a magnetic sail

SOLUTION

$M = \dfrac{0.015m^2}{fd^{4/3}}$ — **Write model for total mass.**

$= \dfrac{0.015(5000)^2}{4.52(1.52)^{4/3}}$ — **Substitute for *m*, *f*, and *d*.**

$\approx 47{,}500$ — **Use a calculator.**

▶ The spacecraft can have a total mass of about 47,500 kilograms. (For comparison, the liftoff weight for a space shuttle is usually about 2,040,000 kilograms.)

EXAMPLE 6 *Solving an Equation Using an nth Root*

NAUTICAL SCIENCE The *Olympias* is a reconstruction of a trireme, a type of Greek galley ship used over 2000 years ago. The power P (in kilowatts) needed to propel the *Olympias* at a desired speed s (in knots) can be modeled by this equation:

$$P = 0.0289s^3$$

A volunteer crew of the *Olympias* was able to generate a maximum power of about 10.5 kilowatts. What was their greatest speed? ▶ Source: *Scientific American*

SOLUTION

$P = 0.0289s^3$ — **Write model for power.**

$10.5 = 0.0289s^3$ — **Substitute 10.5 for *P*.**

$363 \approx s^3$ — **Divide each side by 0.0289.**

$\sqrt[3]{363} \approx s$ — **Take cube root of each side.**

$7 \approx s$ — **Use a calculator.**

▶ The greatest speed attained by the *Olympias* was approximately 7 knots (about 8 miles per hour).

FOCUS ON APPLICATIONS

REAL LIFE **NAUTICAL SCIENCE** The *Olympias* was completed and first launched in 1987. A crew of 170 rowers is needed to run the ship.

APPLICATION LINK
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GUIDED PRACTICE

Vocabulary Check ✓

1. What is the index of a radical?

Concept Check ✓

2. **LOGICAL REASONING** Let n be an integer greater than 1. Tell whether the given statement is *always true*, *sometimes true*, or *never true*. Explain.

   a. If $x^n = a$, then $x = \sqrt[n]{a}$. **b.** $a^{1/n} = \frac{1}{a^n}$

3. Try to evaluate the expressions $-\sqrt[4]{625}$ and $\sqrt[4]{-625}$. Explain the difference in your results.

Skill Check ✓

Evaluate the expression.

4. $\sqrt[4]{81}$ 5. $-(49^{1/2})$ 6. $(\sqrt[3]{-8})^5$ 7. $3125^{2/5}$

Solve the equation.

8. $x^3 = 125$ 9. $3x^5 = -3$ 10. $(x + 4)^2 = 0$ 11. $x^4 - 7 = 9993$

12. **SHOT PUT** The shot (a metal sphere) used in men's shot put has a volume of about 905 cubic centimeters. Find the radius of the shot. (*Hint:* Use the formula $V = \frac{4}{3}\pi r^3$ for the volume of a sphere.)

PRACTICE AND APPLICATIONS

STUDENT HELP

Extra Practice to help you master skills is on p. 949.

USING RATIONAL EXPONENT NOTATION **Rewrite the expression using rational exponent notation.**

13. $\sqrt[4]{14}$ 14. $\sqrt[3]{11}$ 15. $(\sqrt[7]{5})^2$ 16. $(\sqrt[9]{16})^5$ 17. $(\sqrt[8]{2})^{11}$

USING RADICAL NOTATION **Rewrite the expression using radical notation.**

18. $6^{1/3}$ 19. $7^{1/4}$ 20. $10^{3/7}$ 21. $5^{2/5}$ 22. $8^{7/4}$

FINDING *n*TH ROOTS **Find the indicated real *n*th root(s) of *a*.**

23. $n = 2, a = 100$ 24. $n = 4, a = 0$ 25. $n = 3, a = -8$

26. $n = 7, a = 128$ 27. $n = 6, a = -1$ 28. $n = 5, a = 0$

EVALUATING EXPRESSIONS **Evaluate the expression without using a calculator.**

29. $\sqrt[3]{64}$ 30. $\sqrt[3]{-1000}$ 31. $-\sqrt[6]{64}$

32. $4^{-1/2}$ 33. $1^{1/3}$ 34. $-(256^{1/4})$

35. $(\sqrt[4]{16})^2$ 36. $(\sqrt[3]{-27})^{-4}$ 37. $(\sqrt[6]{0})^3$

38. $-(25^{-3/2})$ 39. $32^{4/5}$ 40. $(-125)^{-2/3}$

STUDENT HELP

HOMEWORK HELP
Example 1: Exs. 13–28
Example 2: Exs. 29–40
Example 3: Exs. 41–52
Example 4: Exs. 53–61
Example 5: Exs. 62–64
Example 6: Exs. 65–67

APPROXIMATING ROOTS **Evaluate the expression using a calculator. Round the result to two decimal places when appropriate.**

41. $\sqrt[5]{-16{,}807}$ 42. $\sqrt[9]{1124}$ 43. $\sqrt[8]{65{,}536}$

44. $4^{1/10}$ 45. $10^{-1/4}$ 46. $-(1331^{1/3})$

47. $(\sqrt[3]{112})^{-4}$ 48. $(\sqrt[7]{-280})^3$ 49. $(\sqrt[6]{6})^2$

50. $(-190)^{-4/5}$ 51. $26^{-3/4}$ 52. $522^{2/7}$

SOLVING EQUATIONS **Solve the equation. Round your answer to two decimal places when appropriate.**

53. $x^5 = 243$ **54.** $6x^3 = -1296$ **55.** $x^6 + 10 = 10$

56. $(x - 4)^4 = 81$ **57.** $-x^7 = 40$ **58.** $-12x^4 = -48$

59. $(x + 12)^3 = 21$ **60.** $x^3 - 14 = 22$ **61.** $x^8 - 25 = -10$

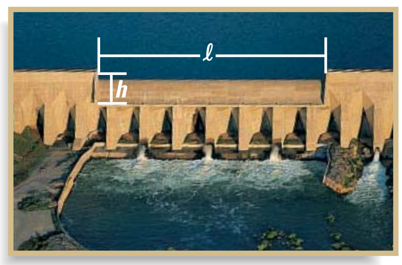
62. **BIOLOGY CONNECTION** For mammals, the lung volume V (in milliliters) can be modeled by $V = 170m^{4/5}$ where m is the body mass (in kilograms). Find the lung volume of each mammal in the table shown.

▶ Source: *Respiration Physiology*

Mammal	Body mass (kg)
Banded mongoose	1.14
Camel	229
Horse	510
Swiss cow	700

63. **SPILLWAY OF A DAM** A dam's spillway capacity is an indication of how the dam will perform under certain flood conditions. The spillway capacity q (in cubic feet per second) of a dam can be calculated using the formula $q = c\ell h^{3/2}$ where c is the discharge coefficient, ℓ is the length (in feet) of the spillway, and h is the height (in feet) of the water on the spillway. A dam with a spillway 40 feet long, 5 feet deep, and 5 feet wide has a discharge coefficient of 2.79. What is the dam's maximum spillway capacity?

▶ Source: *Standard Handbook for Civil Engineers*



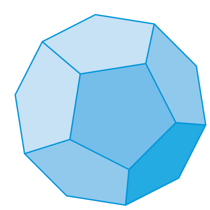
STUDENT HELP

INTERNET **HOMEWORK HELP** Visit our Web site www.mcdougallittell.com for help with problem solving in Ex. 64.

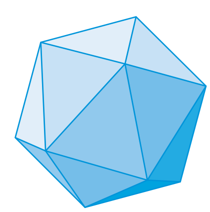
64. **INFLATION** If the price of an item increases from p_1 to p_2 over a period of n years, the annual rate of inflation i (expressed as a decimal) can be modeled by $i = \left(\frac{p_2}{p_1}\right)^{1/n} - 1$. In 1940 the average value of a home was $2900. In 1990 the average value was $79,100. What was the rate of inflation for a home?

▶ Source: Bureau of the Census

65. **GEOMETRY CONNECTION** The formula for the volume V of a regular dodecahedron (a solid with 12 regular pentagons as faces) is $V \approx 7.66a^3$ where a is the length of an edge of the dodecahedron. Find the length of an edge of a regular dodecahedron that has a volume of 30 cubic feet. Round your answer to two decimal places.



66. **GEOMETRY CONNECTION** The formula for the volume V of a regular icosahedron (a solid with 20 congruent equilateral triangles as faces) is $V \approx 2.18a^3$ where a is the length of an edge of the icosahedron. Find the length of an edge of a regular icosahedron that has a volume of 21 cubic centimeters. Round your answer to two decimal places.

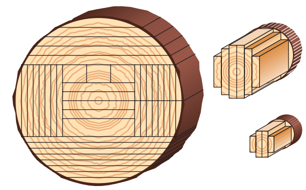


67. **ISLAND SPECIES** Philip Darlington discovered a rule of thumb that relates an island's land area A (in square miles) to the number s of reptile and amphibian species the island can support by the model $A = 0.0779s^3$. The area of Puerto Rico is roughly 4000 square miles. About how many reptile and amphibian species can it support?

▶ Source: *The Song of the Dodo: Island Biogeography in an Age of Extinctions*

Test Preparation

68. MULTI-STEP PROBLEM A board foot is a unit for measuring wood. One board foot has a volume of 144 cubic inches. The Doyle log rule, given by $b = l\left(\frac{r-2}{2}\right)^2$, is a formula for approximating the number b of board feet in a log with length l (in feet) and radius r (in inches). The total volume V (in cubic inches) of wood in the main trunk of a Douglas fir can be modeled by $V = 250r^3$ where r is the radius of the trunk at the base of the tree. Suppose you need 5000 board feet from a 20 foot Douglas fir log.



Log-sawing patterns for maximum board feet

a. What volume of wood do you need?

b. What is the radius of a log that will meet your needs?

c. What is the total volume of wood in the main trunk of a Douglas fir tree that will meet your needs?

d. If you find a suitable tree, what fraction of the tree would you actually use?

e. *Writing* How does your answer to part (d) change if you instead need only 2500 board feet?

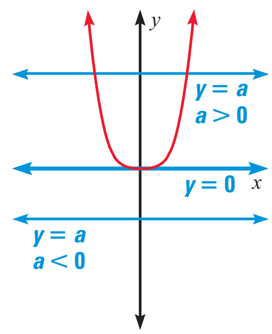
★ Challenge

69. VISUAL THINKING Copy the table. Give the number of nth roots of a for each category.

	$a < 0$	$a = 0$	$a > 0$
n is even	?	?	?
n is odd	?	?	?

70. The graph of $y = x^n$ where n is even is shown in red. Explain how the graph justifies the table for n even.

71. Draw a similar graph to justify the table for n odd.



Ex. 70

EXTRA CHALLENGE
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MIXED REVIEW

SOLVING SYSTEMS **Use Cramer's rule to solve the linear system.** **(Review 4.3)**

72. $x + 4y = 12$
$2x + 5y = 18$

73. $x - 2y = 11$
$2x + 5y = -14$

74. $2x - 4y = 7$
$-x + y = 1$

75. $-3x + 2y = -9$
$x - 4y = 2$

76. $-x - 8y = 10$
$10x + y = 1$

77. $-x - y = 0$
$5x - 6y = 13$

SIMPLIFYING EXPRESSIONS **Simplify the expression. Tell which properties of exponents you used.** **(Review 6.1 for 7.2)**

78. $x^4 \cdot x^{-2}$

79. $(x^{-3})^5$

80. $(2xy^3)^{-2}$

81. $5x^{-2}y^0$

82. $\frac{x^3}{x^{-4}}$

83. $\left(\frac{x^{-2}}{y}\right)^2$

84. $\frac{7x^3y^8}{14xy^{-2}}$

85. $\frac{16xy}{9x^5} \cdot \frac{9x^6y}{4y}$

FINDING ZEROS **Find all the zeros of the polynomial function.** **(Review 6.7)**

86. $f(x) = x^4 + 9x^3 - 5x^2 - 153x - 140$

87. $f(x) = x^4 + x^3 - 19x^2 + 11x + 30$

88. $f(x) = x^3 - 5x^2 + 16x - 80$

89. $f(x) = x^3 - x^2 + 9x - 9$