8.2 Exponential Decay

What you should learn

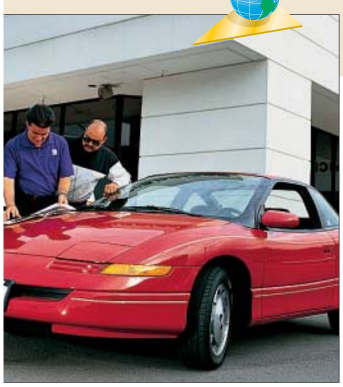
GOAL 1 Graph exponential decay functions.

GOAL 2 Use exponential decay functions to model **real-life** situations, such as the decline of record sales in **Exs. 47–49**.

Why you should learn it

▼ To solve **real-life** problems, such as finding the depreciated value of a car in **Example 4**.

REAL LIFE



GOAL 1 GRAPHING EXPONENTIAL DECAY FUNCTIONS

In Lesson 8.1 you studied exponential growth functions. In this lesson you will study **exponential decay functions**, which have the form $f(x) = ab^x$ where $a > 0$ and $0 < b < 1$.

EXAMPLE 1 *Recognizing Exponential Growth and Decay*

State whether $f(x)$ is an exponential growth or exponential decay function.

a. $f(x) = 5\left(\frac{2}{3}\right)^x$ **b.** $f(x) = 8\left(\frac{3}{2}\right)^x$ **c.** $f(x) = 10(3)^{-x}$

SOLUTION

a. Because $0 < b < 1$, f is an exponential decay function.

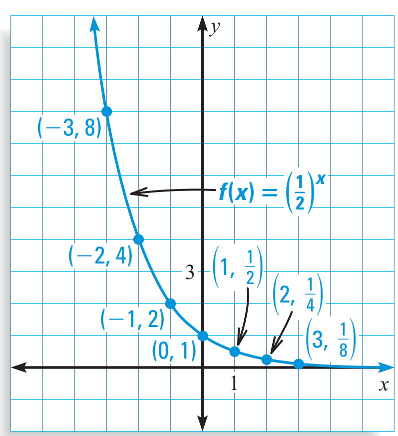
b. Because $b > 1$, f is an exponential growth function.

c. Rewrite the function as $f(x) = 10\left(\frac{1}{3}\right)^x$. Because $0 < b < 1$, f is an exponential decay function.

. . . .

To see the basic shape of the graph of an exponential decay function, you can make a table of values and plot points, as shown below.

x	$f(x) = \left(\frac{1}{2}\right)^x$
-3	$\left(\frac{1}{2}\right)^{-3} = 8$
-2	$\left(\frac{1}{2}\right)^{-2} = 4$
-1	$\left(\frac{1}{2}\right)^{-1} = 2$
0	$\left(\frac{1}{2}\right)^{0} = 1$
1	$\left(\frac{1}{2}\right)^{1} = \frac{1}{2}$
2	$\left(\frac{1}{2}\right)^{2} = \frac{1}{4}$
3	$\left(\frac{1}{2}\right)^{3} = \frac{1}{8}$



Notice the end behavior of the graph. As $x \to -\infty$, $f(x) \to +\infty$, which means that the graph moves up to the left. As $x \to +\infty$, $f(x) \to 0$, which means that the graph has the line $y = 0$ as an asymptote.

Recall that in general the graph of an exponential function $y = ab^x$ passes through the point $(0, a)$ and has the x-axis as an asymptote. The domain is all real numbers, and the range is $y > 0$ if $a > 0$ and $y < 0$ if $a < 0$.

EXAMPLE 2 *Graphing Exponential Functions of the Form $y = ab^x$*

Graph the function.

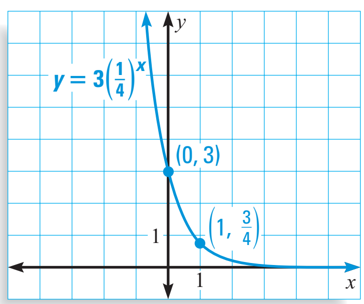
a. $y = 3\left(\frac{1}{4}\right)^x$

b. $y = -5\left(\frac{2}{3}\right)^x$

SOLUTION

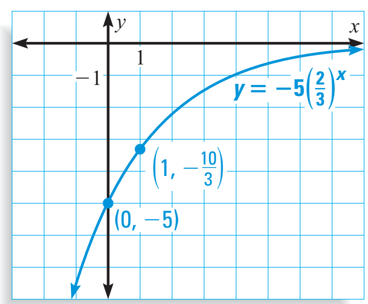
a. Plot $(0, 3)$ and $\left(1, \frac{3}{4}\right)$.

Then, from *right* to *left*, draw a curve that begins just above the x-axis, passes through the two points, and moves up to the left.



b. Plot $(0, -5)$ and $\left(1, -\frac{10}{3}\right)$.

Then, from *right* to *left*, draw a curve that begins just below the x-axis, passes through the two points, and moves down to the left.



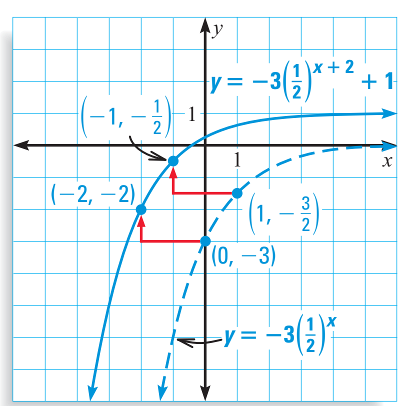
Remember that to graph a general exponential function, $y = ab^{x-h} + k$, begin by sketching the graph of $y = ab^x$. Then translate the graph horizontally by h units and vertically by k units.

EXAMPLE 3 *Graphing a General Exponential Function*

Graph $y = -3\left(\frac{1}{2}\right)^{x+2} + 1$. State the domain and range.

SOLUTION

Begin by lightly sketching the graph of $y = -3\left(\frac{1}{2}\right)^x$, which passes through $(0, -3)$ and $\left(1, -\frac{3}{2}\right)$. Then translate the graph 2 units to the left and 1 unit up. Notice that the graph passes through $(-2, -2)$ and $\left(-1, -\frac{1}{2}\right)$. The graph's asymptote is the line $y = 1$. The domain is all real numbers, and the range is $y < 1$.



GOAL 2 USING EXPONENTIAL DECAY MODELS

When a real-life quantity decreases by a fixed percent each year (or other time period), the amount y of the quantity after t years can be modeled by the equation

$$y = a(1 - r)^t$$

where a is the initial amount and r is the percent decrease expressed as a decimal. The quantity $1 - r$ is called the **decay factor**.

REAL LIFE
Automobiles

EXAMPLE 4 *Modeling Exponential Decay*

You buy a new car for \$24,000. The value y of the car decreases by 16% each year.

a. Write an exponential decay model for the value of the car. Use the model to estimate the value after 2 years.

b. Graph the model.

c. Use the graph to estimate when the car will have a value of \$12,000.

SOLUTION

STUDENT HELP
INTERNET
HOMEWORK HELP
Visit our Web site www.mcdougallittell.com for extra examples.

a. Let t be the number of years since you bought the car. The exponential decay model is:

$y = a(1 - r)^t$ — Write exponential decay model.

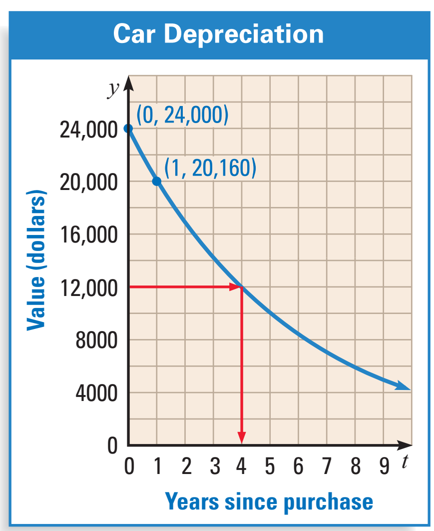
$= 24{,}000(1 - 0.16)^t$ — Substitute for a and r.

$= 24{,}000(0.84)^t$ — Simplify.

When $t = 2$, the value is $y = 24{,}000(0.84)^2 \approx \$16{,}934$.

b. The graph of the model is shown at the right. Notice that it passes through the points (0, 24,000) and (1, 20,160). The asymptote of the graph is the line $y = 0$.

c. Using the graph, you can see that the value of the car will drop to \$12,000 after about 4 years.



In Example 4 notice that the percent decrease, 16%, tells you how much value the car *loses* from one year to the next. The decay factor, 0.84, tells you what fraction of the car's value *remains* from one year to the next. The closer the percent decrease for some quantity is to 0%, the more the quantity is conserved or retained over time. The closer the percent decrease is to 100%, the more the quantity is used or lost over time.

Guided Practice

Vocabulary Check ✓

1. In the exponential decay model $y = 1500(0.65)^t$, identify the initial amount, the decay factor, and the percent decrease.

Concept Check ✓

2. What is the asymptote of the graph of the function $y = 2\left(\frac{1}{5}\right)^{x-2} + 3$?

3. For what values of b does $y = b^x$ represent exponential decay?

Skill Check ✓

Graph the function. State the domain and range.

4. $y = -(0.5)^x$
5. $y = 2\left(\frac{1}{3}\right)^x$
6. $y = 4\left(\frac{2}{3}\right)^x$
7. $y = -5\left(\frac{2}{3}\right)^{x-2}$
8. $y = -4(0.25)^{x+1}$
9. $y = 5\left(\frac{1}{2}\right)^x + 2$

10. **RADIOACTIVE DECAY** The amount y (in grams) of a sample of iodine-131 after t days is given by $y = 50(0.92)^t$.
    a. Identify the initial amount of the substance.
    b. What percent of the substance decays each day?

Practice and Applications

STUDENT HELP

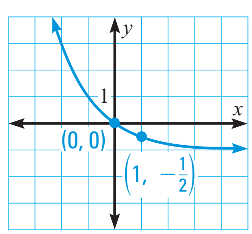
Extra Practice to help you master skills is on p. 950.

IDENTIFYING FUNCTIONS **Tell whether the function represents *exponential growth* or *exponential decay*.**

11. $f(x) = 4\left(\frac{3}{8}\right)^x$
12. $f(x) = 10 \cdot 3^x$
13. $f(x) = 8 \cdot 7^{-x}$
14. $f(x) = 8 \cdot 7^x$
15. $f(x) = 5\left(\frac{1}{8}\right)^{-x}$
16. $f(x) = 3\left(\frac{4}{3}\right)^x$
17. $f(x) = 8\left(\frac{2}{3}\right)^x$
18. $f(x) = 5(0.25)^{-x}$

MATCHING GRAPHS **Match the function with its graph.**

19. $y = (0.25)^x$
20. $y = -3^{x-1} + 3$
21. $y = -\left(\frac{1}{3}\right)^{x-1} + 3$
22. $y = \left(\frac{1}{2}\right)^{x-1}$
23. $y = -(0.25)^x$
24. $y = (0.5)^x - 1$

A.


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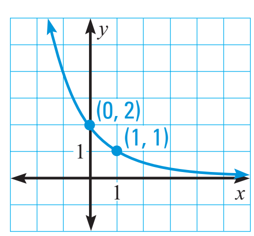

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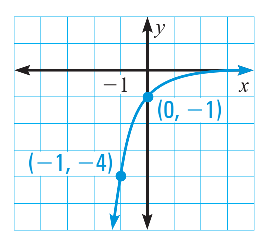

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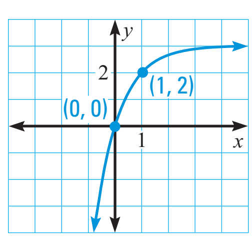

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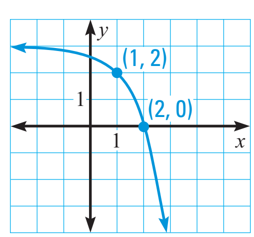

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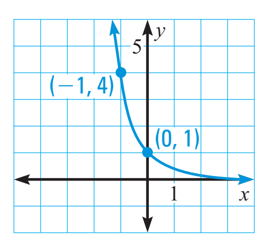

STUDENT HELP

HOMEWORK HELP
Example 1: Exs. 11–18
Example 2: Exs. 19, 23, 25–33
Example 3: Exs. 20–22, 24, 34–42
Example 4: Exs. 43–56

GRAPHING FUNCTIONS **Graph the function.**

25. $y = 3\left(\frac{1}{2}\right)^x$

26. $y = 2\left(\frac{1}{5}\right)^x$

27. $y = -2\left(\frac{1}{4}\right)^x$

28. $y = -5\left(\frac{1}{2}\right)^x$

29. $y = 4\left(\frac{1}{3}\right)^x$

30. $y = 5\left(\frac{1}{4}\right)^x$

31. $y = -3\left(\frac{2}{3}\right)^x$

32. $y = -5(0.75)^x$

33. $y = 3\left(\frac{3}{8}\right)^x$

GRAPHING FUNCTIONS **Graph the function. State the domain and range.**

34. $y = -\left(\frac{1}{2}\right)^x + 1$

35. $y = \left(\frac{2}{3}\right)^{x-1}$

36. $y = 4\left(\frac{1}{2}\right)^{x+1}$

37. $y = \left(\frac{1}{3}\right)^{x-2}$

38. $y = 2\left(\frac{1}{3}\right)^{x-1}$

39. $y = (0.25)^x + 3$

40. $y = -3\left(\frac{1}{3}\right)^{x-1}$

41. $y = \left(\frac{1}{3}\right)^x - 2$

42. $y = \left(\frac{2}{3}\right)^x - 1$

WRITING MODELS **In Exercises 43–45, write an exponential decay model that describes the situation.**

43. **STEREO SYSTEM** You buy a stereo system for \$780. Each year t, the value V of the stereo system decreases by 5%.

44. **BEVERAGES** You drink a beverage with 120 milligrams of caffeine. Each hour h, the amount c of caffeine in your system decreases by about 12%.

45. **MEDICINE** An adult takes 400 milligrams of ibuprofen. Each hour h, the amount i of ibuprofen in the person's system decreases by about 29%.

46. **RADIOACTIVE DECAY** One hundred grams of plutonium is stored in a container. The amount P (in grams) of plutonium present after t years can be modeled by this equation:

$$P = 100(0.99997)^t$$

How much plutonium is present after 20,000 years?

RECORD ALBUMS **In Exercises 47–49, use the following information.**
The number A (in millions) of record albums sold each year in the United States from 1982 to 1993 can be modeled by

$$A = 265(0.39)^t$$

where t represents the number of years since 1982.

INTERNET **DATA UPDATE** of Recording Industry Association of America data at www.mcdougallittell.com

47. Identify the initial amount, the decay factor, and the annual percent decrease.

48. Graph the model.

49. Estimate when the number of records sold was 1 million.

DEPRECIATION **In Exercises 50–52, use the following information.**
You buy a new car for \$22,000. The value of the car decreases by 12.5% each year.

50. Write an exponential decay model for the value of the car. Use the model to estimate the value after 3 years.

51. Graph the model.

52. Estimate when the car will have a value of \$8000.

COMPUTERS In Exercises 53–55, use the following information. You buy a new computer for $2100. The value of the computer decreases by about 50% annually.

53. Write an exponential decay model for the value of the computer. Use the model to estimate the value after 2 years.

54. Graph the model.

55. Estimate when the computer will have a value of $600.

56. **SCIENCE CONNECTION** During normal breathing, about 12% of the air in the lungs is replaced after one breath. Write an exponential decay model for the amount of the original air left in the lungs if the initial amount of air in the lungs is 500 milliliters. How much of the original air is present after 240 breaths?

Test Preparation

57. **MULTI-STEP PROBLEM** A new automobile worth $18,354 depreciates by about 17% each year. The payoff amount on a loan after making n monthly payments is given by the model

$$A(n) = \left(A_0 - \frac{P}{r}\right)(1 + r)^n + \frac{P}{r}$$

where A_0 is the original amount of the loan, P is the monthly payment, and r is the monthly interest rate expressed as a decimal.

a. Write an exponential decay model for the value V of the automobile t years after it is purchased.

b. Write a model for the payoff amount on a loan of $18,354 with a monthly payment of $280 and an *annual* interest rate of 8.5%. (*Hint:* The model for the payoff amount uses the monthly interest rate, not the annual interest rate.)

c. *Writing* Make a table showing the value of the car and the payoff amount on the loan for 5 years. When would it make sense to sell the car? Explain.

★ Challenge

58. **CRITICAL THINKING** Is the product of two exponential decay functions always another exponential decay function? Is the quotient of two exponential decay functions always another exponential decay function? Justify your answers.

MIXED REVIEW

GRAPHING FUNCTIONS Graph the function. (Review 7.5)

59. $y = (x + 1)^{1/3}$ **60.** $y = \sqrt[3]{x} + 1$ **61.** $y = -3x^{1/3}$

62. $y = \sqrt{x} + 4$ **63.** $y = -\sqrt{x + 5}$ **64.** $y = \sqrt[3]{x} + \frac{1}{4}$

USING A DATA SET Find the mean, the median, the mode, and the range for the set of data. (Review 7.7)

65. 11, 18, 13, 15, 17, 15, 23, 20, 12 **66.** 25, 30, 32, 42, 31, 33, 36, 22

67. **FINANCE** You deposit $2000 in a bank account. Find the balance after 4 years for each of the following situations. **(Review 8.1 for 8.3)**

a. The account pays 7% annual interest compounded quarterly.

b. The account pays 5% annual interest compounded monthly.