8.3 The Number *e*

What you should learn

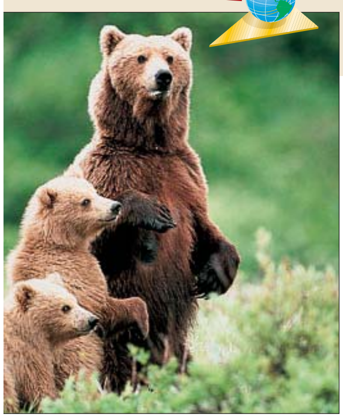
GOAL 1 Use the number e as the base of exponential functions.

GOAL 2 Use the natural base e in **real-life** situations, such as finding the air pressure on Mount Everest in **Ex. 79**.

Why you should learn it

▼ To solve **real-life** problems, such as finding the number of listed endangered species in **Example 5**.

REAL LIFE



The grizzly bear was first listed as threatened in 1975 and remains an endangered species today.

GOAL 1 USING THE NATURAL BASE *e*

The history of mathematics is marked by the discovery of special numbers such as counting numbers, zero, negative numbers, π, and imaginary numbers. In this lesson you will study one of the most famous numbers of modern times. Like π and i, the number e is denoted by a letter. The number is called the **natural base *e***, or the **Euler number**, after its discoverer, Leonhard Euler (1707–1783).

ACTIVITY Developing Concepts

Investigating the Natural Base *e*

1. Copy the table and use a calculator to complete the table.

n	10^1	10^2	10^3	10^4	10^5	10^6
$\left(1+\frac{1}{n}\right)^n$	2.594	?	?	?	?	?

2. Do the values in the table appear to be approaching a fixed decimal number? If so, what is the number rounded to three decimal places?

In the activity you may have discovered that as n gets larger and larger, the expression $\left(1+\frac{1}{n}\right)^n$ gets closer and closer to 2.71828 . . . , which is the value of e.

THE NATURAL BASE *e*

The natural base e is irrational. It is defined as follows:

As n approaches $+\infty$, $\left(1+\frac{1}{n}\right)^n$ approaches $e \approx 2.718281828459$.

EXAMPLE 1 *Simplifying Natural Base Expressions*

Simplify the expression.

a. $e^3 \cdot e^4$ **b.** $\frac{10e^3}{5e^2}$ **c.** $(3e^{-4x})^2$

SOLUTION

a. $e^3 \cdot e^4 = e^{3+4}$
$= e^7$

b. $\frac{10e^3}{5e^2} = 2e^{3-2}$
$= 2e$

c. $(3e^{-4x})^2 = 3^2 e^{(-4x)(2)}$
$= 9e^{-8x} = \frac{9}{e^{8x}}$

FOCUS ON PEOPLE

REAL LIFE **LEONHARD EULER** continued his mathematical research despite losing sight in one eye in 1735. He published more than 500 books and papers during his lifetime. Euler's use of *e* appeared in his book *Mechanica*, published in 1736.

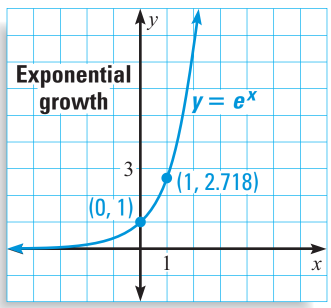
EXAMPLE 2 *Evaluating Natural Base Expressions*

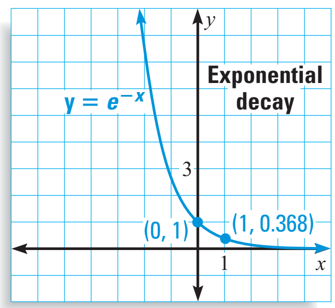
Use a calculator to evaluate the expression: **a.** e^2 **b.** $e^{-0.06}$

SOLUTION

EXPRESSION	KEYSTROKES	DISPLAY
a. e^2	2nd [e^x] 2 ENTER	7.389056
b. $e^{-0.06}$	2nd [e^x] (–) .06 ENTER	0.941765

A function of the form $f(x) = ae^{rx}$ is called a *natural base exponential function*. If $a > 0$ and $r > 0$, the function is an exponential growth function, and if $a > 0$ and $r < 0$, the function is an exponential decay function. The graphs of the basic functions $y = e^x$ and $y = e^{-x}$ are shown below.





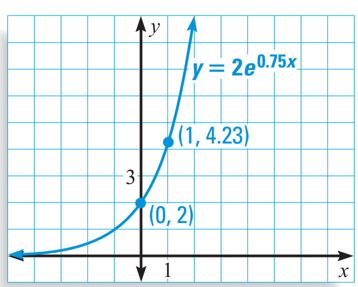
EXAMPLE 3 *Graphing Natural Base Functions*

Graph the function. State the domain and range.

a. $y = 2e^{0.75x}$ **b.** $y = e^{-0.5(x-2)} + 1$

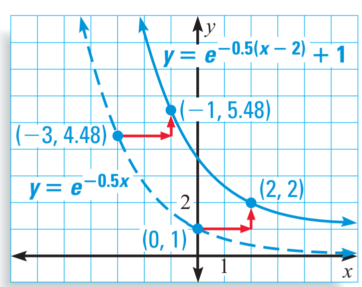
SOLUTION

a. Because $a = 2$ is positive and $r = 0.75$ is positive, the function is an exponential growth function. Plot the points (0, 2) and (1, 4.23) and draw the curve.



The domain is all real numbers, and the range is all positive real numbers.

b. Because $a = 1$ is positive and $r = -0.5$ is negative, the function is an exponential decay function. Translate the graph of $y = e^{-0.5x}$ to the right 2 units and up 1 unit.



The domain is all real numbers, and the range is $y > 1$.

GOAL 2 USING *e* IN REAL LIFE

In Lesson 8.1 you learned that the amount A in an account earning interest compounded n times per year for t years is given by

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

where P is the principal and r is the annual interest rate expressed as a decimal. As n approaches positive infinity, the compound interest formula approximates the following formula for *continuously compounded interest:*

$$A = Pe^{rt}$$

REAL LIFE Finance

EXAMPLE 4 *Finding the Balance in an Account*

You deposit \$1000 in an account that pays 8% annual interest compounded continuously. What is the balance after 1 year?

SOLUTION

Note that $P = 1000$, $r = 0.08$, and $t = 1$. So, the balance at the end of 1 year is:

$$A = Pe^{rt} = 1000e^{0.08(1)} \approx \$1083.29$$

In Example 4 of Lesson 8.1, you found that the balance from daily compounding is \$1083.28. So, continuous compounding earned only an additional \$.01.

EXAMPLE 5 *Using an Exponential Model*

ENDANGERED SPECIES Since 1972 the U.S. Fish and Wildlife Service has kept a list of endangered species in the United States. For the years 1972–1998, the number s of species on the list can be modeled by

$$s = 119.6e^{0.0917t}$$

where t is the number of years since 1972.

a. What was the number of endangered species in 1972?

b. Graph the model.

c. Use the graph to estimate when the number of endangered species reached 1000.

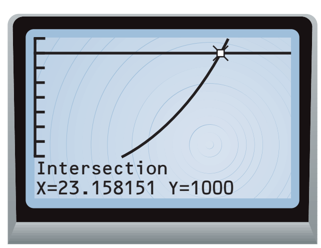
SOLUTION

a. In 1972, when $t = 0$, the model gives:

$$s = 119.6e^{0} = 119.6$$

So, there were about 120 endangered species on the list in 1972.

b. The graph of the model is shown.



c. Use the *Intersect* feature to determine that s reaches 1000 when $t \approx 23$, which is about 1995.

FOCUS ON CAREERS

MARINE BIOLOGIST
A marine biologist studies salt-water plants and animals. Those who work for the U.S. Fish and Wildlife Service help maintain populations of manatees, walruses, and other endangered species.

INTERNET CAREER LINK
www.mcdougallittell.com

GUIDED PRACTICE

Vocabulary Check ✓

1. What is the Euler number? Give an approximation of the Euler number rounded to three decimal places.

Concept Check ✓

2. Tell whether the function $f(x) = \frac{1}{4}e^{2x}$ is an example of *exponential growth* or *exponential decay*. Explain.
3. Is it possible to express e as a ratio of two integers? Explain.

Skill Check ✓

Simplify the expression.

4. $e^2 \cdot e^6$
5. $e^{-2} \cdot 3e^7$
6. $(2e^{5x})^2$
7. $(4e^{-2})^3$
8. $\left(\frac{1}{2}e^{-2}\right)^4$
9. $\sqrt{36e^{4x}}$
10. $\frac{e^x}{e^{2x}}$
11. $\frac{12e^4}{36e^{-2}}$
12. What is the horizontal asymptote of the graph of $f(x) = 2e^x - 2$?

Graph the function.

13. $y = e^{-2x}$
14. $y = \frac{1}{2}e^x$
15. $y = \frac{1}{8}e^{2x}$
16. **ENDANGERED SPECIES** Use the model in Example 5 to estimate the number of endangered species in 1998.

PRACTICE AND APPLICATIONS

STUDENT HELP

Extra Practice to help you master skills is on p. 950.

SIMPLIFYING EXPRESSIONS **Simplify the expression.**

17. $e^2 \cdot e^4$
18. $e^{-3} \cdot e^5$
19. $(3e^{-3x})^{-1}$
20. $(3e^{4x})^2$
21. $3e^{-2} \cdot e^6$
22. $\left(\frac{1}{4}e^{-2}\right)^3$
23. $e^x \cdot e^{-3x} \cdot e^5$
24. $\sqrt{4e^{2x}}$
25. $(100e^{0.5x})^{-2}$
26. $e^x \cdot 4e^{2x+1}$
27. $\frac{e^x}{2e}$
28. $\frac{5e^x}{e^{5x}}$
29. $\sqrt[3]{27e^{6x}}$
30. $(32e^{-4x})^3$
31. $\frac{6e^{3x}}{4e}$
32. $\sqrt[3]{64e^{9x}}$

EVALUATING EXPRESSIONS **Use a calculator to evaluate the expression. Round the result to three decimal places.**

33. e^3
34. $e^{-2/3}$
35. $e^{1.7}$
36. $e^{1/2}$
37. $e^{-1/4}$
38. $e^{3.2}$
39. e^8
40. e^{-3}
41. e^{-4}
42. $2e^{1/2}$
43. $-4e^{-3}$
44. $0.5e^{3.2}$
45. $-1.2e^5$
46. $0.02e^{-0.3}$
47. $225e^{-50}$
48. $-8.95e^{1/5}$

STUDENT HELP

HOMEWORK HELP
Example 1: Exs. 17–32
Example 2: Exs. 33–48
Example 3: Exs. 49–75
Example 4: Exs. 76–78
Example 5: Exs. 79, 80

GROWTH OR DECAY? **Tell whether the function is an example of *exponential growth* or *exponential decay*.**

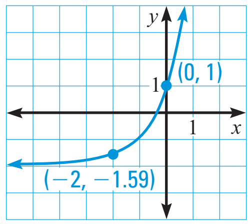
49. $f(x) = 5e^{-3x}$
50. $f(x) = \frac{1}{8}e^{5x}$
51. $f(x) = e^{-4x}$
52. $f(x) = \frac{1}{6}e^{2x}$
53. $f(x) = \frac{1}{4}e^{2x}$
54. $f(x) = e^{-8x}$
55. $f(x) = e^{3x}$
56. $f(x) = \frac{1}{4}e^{-x}$
57. $f(x) = e^{-6x}$
58. $f(x) = \frac{3}{8}e^{7x}$
59. $f(x) = e^{-9x}$
60. $f(x) = e^{8x}$

MATCHING GRAPHS **Match the function with its graph.**

61. $y = 3e^{0.5x}$ **62.** $y = \frac{1}{3}e^{0.5x}$ **63.** $y = \frac{1}{2}e^{-(x-1)}$

64. $y = e^{-x} + 1$ **65.** $y = 3e^{-x} - 2$ **66.** $y = 3e^{x} - 2$

A.



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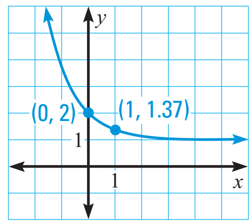

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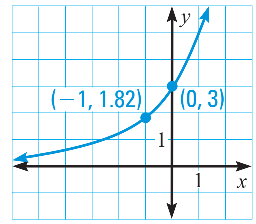

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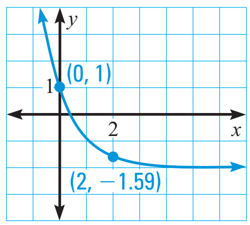
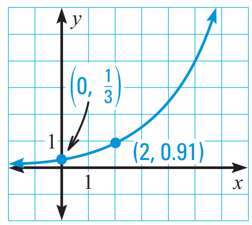
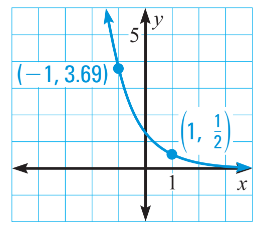


E.



F.



STUDENT HELP

INTERNET HOMEWORK HELP Visit our Web site www.mcdougallittell.com for help with Exs. 67–75.

GRAPHING FUNCTIONS **Graph the function. State the domain and range.**

67. $y = e^{-x}$ **68.** $y = 4e^{x}$ **69.** $y = \frac{1}{3}e^{x}$

70. $y = 3e^{2x} + 2$ **71.** $y = 1.5e^{-0.5x}$ **72.** $y = 0.1e^{2x} - 4$

73. $y = \frac{1}{3}e^{x-2} - 1$ **74.** $y = \frac{4}{3}e^{x-3} + 1$ **75.** $y = 0.5e^{-2(x-1)} - 2$

76. **CONTINUOUS COMPOUNDING** You deposit \$975 in an account that pays 5.5% annual interest compounded continuously. What is the balance after 6 years?

77. **COMPARING FORMULAS** You deposit \$2500 in an account that pays 6% annual interest. Use the formulas at the top of page 482 to calculate the account balance after one year when the interest is compounded annually, semiannually, quarterly, monthly, and continuously. What do you notice? Explain.

78. *Writing* Compare the effects of compounding interest continuously and compounding interest daily using the formulas $A = Pe^{rt}$ and

$$A = P\left(1 + \frac{r}{365}\right)^{365t}.$$

FOCUS ON PEOPLE

REAL LIFE MOUNT EVEREST In 1953 Sir Edmund Hillary of New Zealand and Tenzing Norgay, a Nepalese Sherpa tribesman, became the first people to reach the top of Mount Everest.

79. **MOUNT EVEREST** The air pressure P at sea level is about 14.7 pounds per square inch. As the altitude h (in feet above sea level) increases, the air pressure decreases. The relationship between air pressure and altitude can be modeled by:

$$P = 14.7e^{-0.00004h}$$

Mount Everest in Tibet and Nepal rises to a height of 29,028 feet above sea level. What is the air pressure at the peak of Mount Everest?

80. **RATE OF HEALING** The area of a wound decreases exponentially with time. The area A of a wound after t days can be modeled by

$$A = A_0e^{-0.05t}$$

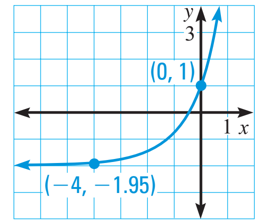
where A_0 is the initial wound area. If the initial wound area is 4 square centimeters, how much of the wound area is present after 14 days?

Test Preparation

81. MULTIPLE CHOICE What is the simplified form of $\sqrt[3]{\frac{8(81e^{11}x)}{3e^5x^{-2}}}$?

Ⓐ $6e^2\sqrt[3]{x}$ Ⓑ $6x\sqrt[3]{e^6}$ Ⓒ $6\sqrt[3]{e^{16}x}$ Ⓓ $\frac{6e^2}{x}$ Ⓔ $6e^2x$

82. MULTIPLE CHOICE Which function is graphed at the right?



Ⓐ $f(x) = 3e^{x-2}$ Ⓑ $f(x) = 3e^x - 2$

Ⓒ $f(x) = 3e^{-x} - 2$ Ⓓ $f(x) = 3e^{-(x+2)}$

Ⓔ $f(x) = 3e^{x+2}$

★ Challenge

83. CRITICAL THINKING Find a value of n for which $\left(1 + \frac{1}{n}\right)^n$ gives the value of e correct to 9 decimal places. Explain the process you used to find your answer.

MIXED REVIEW

FINDING INVERSE FUNCTIONS Find an equation for the inverse of the function. (Review 7.4 for 8.4)

84. $f(x) = -3x$ **85.** $f(x) = 6x + 7$ **86.** $f(x) = -5x - 24$

87. $f(x) = \frac{1}{2}x - 10$ **88.** $f(x) = -14x + 7$ **89.** $f(x) = -\frac{1}{5}x - 13$

SOLVING EQUATIONS Solve the equation. (Review 7.6)

90. $\sqrt{x} = 20$ **91.** $\sqrt[3]{5x - 4} + 7 = 10$ **92.** $2(x + 4)^{2/3} = 8$

93. $\sqrt{x^2 - 4} = x - 2$ **94.** $\sqrt{x + 3} = \sqrt{2x - 1}$ **95.** $\sqrt{3x - 5} - 3\sqrt{x} = 0$

QUIZ 1

Self-Test for Lessons 8.1–8.3

Graph the function. State the domain and range. (Lessons 8.1, 8.2)

1. $y = 4^x - 1$ **2.** $y = 3^{x+1} + 2$ **3.** $y = \frac{1}{2} \cdot 5^{x-1}$

4. $y = -2\left(\frac{1}{6}\right)^x$ **5.** $y = \left(\frac{5}{8}\right)^x + 2$ **6.** $y = -2 \cdot 6^{x-3} + 3$

Simplify the expression. (Lesson 8.3)

7. $2e^3 \cdot e^4$ **8.** $4e^{-5} \cdot e^7$ **9.** $(-3e^{2x})^2$ **10.** $(5e^{-3})^{-4x}$

11. $\frac{3e^x}{4e}$ **12.** $\frac{6e^x}{e^{5x}}$ **13.** $\sqrt{16e^2x}$ **14.** $\sqrt[3]{125e^{6x}}$

15. Graph the function $f(x) = -4e^{2x}$. **(Lesson 8.3)**

16. RADIOACTIVE DECAY One hundred grams of radium is stored in a container. The amount R (in grams) of radium present after t years can be modeled by $R = 100e^{-0.00043t}$. Graph the model. How much of the radium is present after 10,000 years? **(Lesson 8.3)**