8.5 Properties of Logarithms

What you should learn

GOAL 1 Use properties of logarithms.

GOAL 2 Use properties of logarithms to solve **real-life** problems, such as finding the energy needed for molecular transport in **Exs. 77–79**.

Why you should learn it

▼ To model **real-life** quantities, such as the loudness of different sounds in **Example 5**.

REAL LIFE

Airport workers wear hearing protection because of the loudness of jet engines.

GOAL 1 USING PROPERTIES OF LOGARITHMS

Because of the relationship between logarithms and exponents, you might expect logarithms to have properties similar to the properties of exponents you studied in Lesson 6.1.

ACTIVITY Developing Concepts: Investigating a Property of Logarithms

1. Copy and complete the table one row at a time.

$\log_b u$	$\log_b v$	$\log_b uv$
$\log 10 = ?$	$\log 100 = ?$	$\log 1000 = ?$
$\log 0.1 = ?$	$\log 0.01 = ?$	$\log 0.001 = ?$
$\log_2 4 = ?$	$\log_2 8 = ?$	$\log_2 32 = ?$

2. Use the completed table to write a conjecture about the relationship among $\log_b u$, $\log_b v$, and $\log_b uv$.

In the activity you may have discovered one of the properties of logarithms listed below.

PROPERTIES OF LOGARITHMS

Let b, u, and v be positive numbers such that $b \neq 1$.

PRODUCT PROPERTY $\log_b uv = \log_b u + \log_b v$

QUOTIENT PROPERTY $\log_b \frac{u}{v} = \log_b u - \log_b v$

POWER PROPERTY $\log_b u^n = n \log_b u$

EXAMPLE 1 *Using Properties of Logarithms*

Use $\log_5 3 \approx 0.683$ and $\log_5 7 \approx 1.209$ to approximate the following.

a. $\log_5 \frac{3}{7}$ **b.** $\log_5 21$ **c.** $\log_5 49$

SOLUTION

a. $\log_5 \frac{3}{7} = \log_5 3 - \log_5 7 \approx 0.683 - 1.209 = -0.526$

b. $\log_5 21 = \log_5 (3 \cdot 7) = \log_5 3 + \log_5 7 \approx 0.683 + 1.209 = 1.892$

c. $\log_5 49 = \log_5 7^2 = 2 \log_5 7 \approx 2(1.209) = 2.418$

You can use the properties of logarithms to expand and condense logarithmic expressions.

EXAMPLE 2 *Expanding a Logarithmic Expression*

Expand $\log_2 \frac{7x^3}{y}$. Assume x and y are positive.

SOLUTION

$$\log_2 \frac{7x^3}{y} = \log_2 7x^3 - \log_2 y \quad \textbf{Quotient property}$$

$$= \log_2 7 + \log_2 x^3 - \log_2 y \quad \textbf{Product property}$$

$$= \log_2 7 + 3\log_2 x - \log_2 y \quad \textbf{Power property}$$

STUDENT HELP

Study Tip
When you are expanding or condensing an expression involving logarithms, you may assume the variables are positive.

EXAMPLE 3 *Condensing a Logarithmic Expression*

Condense $\log 6 + 2\log 2 - \log 3$.

SOLUTION

$$\log 6 + 2\log 2 - \log 3 = \log 6 + \log 2^2 - \log 3 \quad \textbf{Power property}$$

$$= \log (6 \cdot 2^2) - \log 3 \quad \textbf{Product property}$$

$$= \log \frac{6 \cdot 2^2}{3} \quad \textbf{Quotient property}$$

$$= \log 8 \quad \textbf{Simplify.}$$

....

Logarithms with any base other than 10 or e can be written in terms of common or natural logarithms using the *change-of-base formula*.

CHANGE-OF-BASE FORMULA

Let u, b, and c be positive numbers with $b \neq 1$ and $c \neq 1$. Then:

$$\log_c u = \frac{\log_b u}{\log_b c}$$

In particular, $\log_c u = \frac{\log u}{\log c}$ and $\log_c u = \frac{\ln u}{\ln c}$.

EXAMPLE 4 *Using the Change-of-Base Formula*

Evaluate the expression $\log_3 7$ using common and natural logarithms.

SOLUTION

Using common logarithms: $\log_3 7 = \frac{\log 7}{\log 3} \approx \frac{0.8451}{0.4771} \approx 1.771$

Using natural logarithms: $\log_3 7 = \frac{\ln 7}{\ln 3} \approx \frac{1.946}{1.099} \approx 1.771$

GOAL 2 USING LOGARITHMIC PROPERTIES IN REAL LIFE

EXAMPLE 5 *Using Properties of Logarithms*

REAL LIFE Acoustics

The loudness L of a sound (in decibels) is related to the intensity I of the sound (in watts per square meter) by the equation

$$L = 10 \log \frac{I}{I_0}$$

where I_0 is an intensity of 10^{-12} watt per square meter, corresponding roughly to the faintest sound that can be heard by humans.

a. Two roommates each play their stereos at an intensity of 10^{-5} watt per square meter. How much louder is the music when both stereos are playing, compared with when just one stereo is playing?

b. Generalize the result from part (a) by using I for the intensity of each stereo.

Decibel level	Example
130	Jet airplane takeoff
120	Riveting machine
110	Rock concert
100	Boiler shop
90	Subway train
80	Average factory
70	City traffic
60	Conversational speech
50	Average home
40	Quiet library
30	Soft whisper
20	Quiet room
10	Rustling leaf
0	Threshold of hearing

SOLUTION

Let L_1 be the loudness when one stereo is playing and let L_2 be the loudness when both stereos are playing.

a. Increase in loudness $= L_2 - L_1$

$= 10 \log \frac{2 \cdot 10^{-5}}{10^{-12}} - 10 \log \frac{10^{-5}}{10^{-12}}$ — **Substitute for L_2 and L_1.**

$= 10 \log (2 \cdot 10^7) - 10 \log 10^7$ — **Simplify.**

$= 10(\log 2 + \log 10^7 - \log 10^7)$ — **Product property**

$= 10 \log 2$ — **Simplify.**

≈ 3 — **Use a calculator.**

▶ The sound is about 3 decibels louder.

b. Increase in loudness $= L_2 - L_1$

$$= 10 \log \frac{2I}{10^{-12}} - 10 \log \frac{I}{10^{-12}}$$

$$= 10\left(\log \frac{2I}{10^{-12}} - \log \frac{I}{10^{-12}}\right)$$

$$= 10\left(\log 2 + \log \frac{I}{10^{-12}} - \log \frac{I}{10^{-12}}\right)$$

$$= 10 \log 2$$

$$\approx 3$$

▶ Again, the sound is about 3 decibels louder. This result tells you that the loudness increases by 3 decibels when both stereos are played regardless of the intensity of each stereo individually.

FOCUS ON CAREERS

REAL LIFE **SOUND TECHNICIAN**

Sound technicians operate technical equipment to amplify, enhance, record, mix, or reproduce sound. They may work in radio or television recording studios or at live performances.

INTERNET CAREER LINK
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GUIDED PRACTICE

Vocabulary Check ✓

1. Give an example of the property of logarithms.

a. product property **b.** quotient property **c.** power property

Concept Check ✓

2. Which is equivalent to $\log\left(\frac{7}{9}\right)^2$? Explain.

A. $2(\log 7 - \log 9)$ **B.** $\frac{2\log 7}{\log 9}$ **C.** Neither A nor B

3. Which is equivalent to $\log_8(5x^2 + 3)$? Explain.

A. $\log_8 5x^2 + \log_8 3$ **B.** $\log_8 5x^2 \cdot \log_8 3$ **C.** Neither A nor B

4. Describe two ways to find the value of $\log_6 11$ using a calculator.

Skill Check ✓

Use a property of logarithms to evaluate the expression.

5. $\log_3(3 \cdot 9)$ **6.** $\log_2 4^5$ **7.** $\log_3 \frac{1}{3}$ **8.** $\log_5\left(\frac{1}{5}\right)^3$

Use $\log_2 7 \approx 2.81$ and $\log_2 21 \approx 4.39$ to approximate the value of the expression.

9. $\log_2 3$ **10.** $\log_2 49$ **11.** $\log_2 147$ **12.** $\log_2 441$

13. **SOUND INTENSITY** Use the loudness of sound equation in Example 5 to find the difference in the loudness of an average office with an intensity of 1.26×10^{-7} watt per square meter and a broadcast studio with an intensity of 3.16×10^{-10} watt per square meter.

PRACTICE AND APPLICATIONS

STUDENT HELP

Extra Practice to help you master skills is on p. 951.

EVALUATING EXPRESSIONS **Use a property of logarithms to evaluate the expression.**

14. $\log_2(4 \cdot 16)$ **15.** $\ln e^{-2}$ **16.** $\log_2 4^3$ **17.** $\log_5 125$

18. $\log_3 9^4$ **19.** $\log \frac{1}{10}$ **20.** $\ln \frac{1}{e^3}$ **21.** $\log(0.01)^3$

APPROXIMATING EXPRESSIONS **Use $\log 5 \approx 0.699$ and $\log 15 \approx 1.176$ to approximate the value of the expression.**

22. $\log 3$ **23.** $\log 25$ **24.** $\log 75$ **25.** $\log 125$

26. $\log \frac{1}{5}$ **27.** $\log 225$ **28.** $\log \frac{1}{15}$ **29.** $\log \frac{1}{3}$

STUDENT HELP

HOMEWORK HELP
Example 1: Exs. 14–29
Example 2: Exs. 30–45
Example 3: Exs. 46–57
Example 4: Exs. 58–73
Example 5: Exs. 74–85

EXPANDING EXPRESSIONS **Expand the expression.**

30. $\log_2 9x$ **31.** $\ln 22x$ **32.** $\log 4x^5$ **33.** $\log_6 x^6$

34. $\log_4 \frac{4}{3}$ **35.** $\log_3 25$ **36.** $\log_6 \frac{10}{3}$ **37.** $\ln 3xy^3$

38. $\log 6x^3yz$ **39.** $\log_8 64x^2$ **40.** $\ln x^{1/2}y^3$ **41.** $\log_3 12^{5/6}x^9$

42. $\log \sqrt{x}$ **43.** $\ln \frac{3y^4}{x^3}$ **44.** $\log \sqrt[4]{x^3}$ **45.** $\log_2 \sqrt{4x}$

Condensing Expressions Condense the expression.

46. $\log_5 8 - \log_5 12$

47. $\ln 16 - \ln 4$

48. $2 \log x + \log 5$

49. $4 \log_{16} 12 - 4 \log_{16} 2$

50. $3 \ln x + 5 \ln y$

51. $7 \log_4 2 + 5 \log_4 x + 3 \log_4 y$

52. $\ln 20 + 2 \ln \frac{1}{2} + \ln x$

53. $\log_3 2 + \frac{1}{2} \log_3 y$

54. $10 \log x + 2 \log 10$

55. $3(\ln 3 - \ln x) + (\ln x - \ln 9)$

56. $2(\log_6 15 - \log_6 5) + \frac{1}{2} \log_6 \frac{1}{25}$

57. $\frac{1}{4} \log_5 81 - \left(2 \log_5 6 - \frac{1}{2} \log_5 4\right)$

Change-of-Base Formula Use the change-of-base formula to evaluate the expression.

58. $\log_5 7$

59. $\log_7 12$

60. $\log_3 16$

61. $\log_9 25$

62. $\log_2 5$

63. $\log_6 9$

64. $\log_3 17$

65. $\log_5 32$

66. $\log_2 125$

67. $\log_6 24$

68. $\log_4 19$

69. $\log_{16} 81$

70. $\log_8 \frac{22}{7}$

71. $\log_9 \frac{5}{16}$

72. $\log_2 \frac{4}{15}$

73. $\log_5 \frac{32}{3}$

Photography In Exercises 74–76, use the following information.

The f-stops on a 35 millimeter camera control the amount of light that enters the camera. Let s be a measure of the amount of light that strikes the film and let f be the f-stop. Then s and f are related by this equation:

$$s = \log_2 f^2$$

74. Expand the expression for s.

75. The table shows the first eight f-stops on a 35 millimeter camera. Copy and complete the table. Then describe the pattern.

f	1.414	2.000	2.828	4.000	5.657	8.000	11.314	16.000
s	?	?	?	?	?	?	?	?

76. Many 35 millimeter cameras have nine f-stops. What do you think the ninth f-stop is? Explain your reasoning.

Science Connection In Exercises 77–79, use the following information.

The energy E (in kilocalories per gram-molecule) required to transport a substance from the outside to the inside of a living cell is given by

$$E = 1.4(\log C_2 - \log C_1)$$

where C_2 is the concentration of the substance inside the cell and C_1 is the concentration outside the cell.

77. Condense the expression for E.

78. The concentration of a particular substance inside a cell is twice the concentration outside the cell. How much energy is required to transport the substance from outside to inside the cell?

79. The concentration of a particular substance inside a cell is six times the concentration outside the cell. How much energy is required to transport the substance from outside to inside the cell?

Focus on Applications

PHOTOGRAPHY Photographers use f-stops to achieve the desired amount of light in a photo. The smaller the f-stop number, the more light the lens transmits.

APPLICATION LINK
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Student Help

HOMEWORK HELP
Visit our Web site www.mcdougallittell.com for help with Exs. 77–79.

FOCUS ON APPLICATIONS

RALPH E. ALLISON developed the first single zero-point audiometer in 1937, making the equipment usable for doctors who had previously used tuning forks to test hearing.

ACOUSTICS In Exercises 80–85, use the table and the loudness of sound equation from Example 5.

80. The intensity of the sound made by a propeller aircraft is 0.316 watts per square meter. Find the decibel level of a propeller aircraft. To what sound in the table from Example 5 is a propeller aircraft's sound most similar?

81. The intensity of the sound made by Niagara Falls is 0.003 watts per square meter. Find the decibel level of Niagara Falls. To what sound in the table from Example 5 is the sound of Niagara Falls most similar?

82. Three groups of people are in a room, and each group is having a conversation at an intensity of 1.4×10^{-7} watt per square meter. What is the decibel level of the combined conversations in the room?

83. Five cars are in a parking garage, and the sound made by each running car is at an intensity of 3.16×10^{-4} watt per square meter. What is the decibel level of the sound produced by all five cars in the parking garage?

84. A certain sound has an intensity of I watts per square meter. By how many decibels does the sound increase when the intensity is tripled?

85. A certain sound has an intensity of I watts per square meter. By how many decibels does the sound decrease when the intensity is halved?

86. CRITICAL THINKING Tell whether this statement is *true* or *false*: $\log(u + v) = \log u + \log v$. If true, prove it. If false, give a counterexample.

87. *Writing* Let n be an integer from 1 to 20. Use only the fact that $\log 2 \approx 0.3010$ and $\log 3 \approx 0.4771$ to find as many values of $\log n$ as you possibly can. Show how you obtained each value. What can you conclude about the values of n for which you *cannot* find $\log n$?

Test Preparation

88. MULTIPLE CHOICE Which of the following is *not* correct?

(A) $\log_2 24 = \log_2 6 + \log_2 4$ (B) $\log_2 24 = \log_2 72 - \log_2 3$

(C) $\log_2 24 = \log_2 8 + \log_2 16$ (D) $\log_2 24 = 2\log_2 2 + \log_2 6$

89. MULTIPLE CHOICE Which of the following is equivalent to $\log_5 8$?

(A) $\dfrac{\log 5}{\log 8}$ (B) $\dfrac{\log 8}{\log 5}$ (C) $\dfrac{\ln 8}{\ln 5}$ (D) $\dfrac{\ln 13}{\ln 5}$ (E) Both B and C

90. MULTIPLE CHOICE Which of the following is equivalent to $4\log_3 5$?

(A) $\log_3 20$ (B) $\log_3 625$ (C) $\log_3 60$ (D) $\log_3 243$ (E) Both B and C

★ Challenge

91. LOGICAL REASONING Use the given hint and properties of exponents to prove each property of logarithms.

a. Product property (*Hint:* Let $x = \log_b u$ and let $y = \log_b v$. Then $u = b^x$ and $v = b^y$ so that $\log_b uv = \log_b (b^x \cdot b^y)$.)

b. Quotient property (*Hint:* Let $x = \log_b u$ and let $y = \log_b v$. Then $u = b^x$ and $v = b^y$ so that $\log_b \frac{u}{v} = \log_b \frac{b^x}{b^y}$.)

c. Power property (*Hint:* Let $x = \log_b u$. Then $u = b^x$ and $u^n = b^{nx}$ so that $\log_b u^n = \log_b (b^{nx})$.)

d. Change-of-base formula (*Hint:* Let $x = \log_b u$, $y = \log_b c$, and $z = \log_c u$. Then $u = b^x$, $c = b^y$, and $u = c^z$ so that $b^x = c^z$.)

EXTRA CHALLENGE
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MIXED REVIEW

SIMPLIFYING EXPRESSIONS Simplify the expression. (Review 6.1)

92. $3y^2 \cdot y^2$ **93.** $(y^4)^3$ **94.** $(x^3y)^4$ **95.** $(-3x^2)^2$

96. $4x^{-1}y$ **97.** $xy^{-2}x$ **98.** $\frac{x^3}{x^{-1}}$ **99.** $\frac{4x^2y^7}{8xy^{-1}}$

SOLVING RADICAL EQUATIONS Solve the equation. Check for extraneous solutions. (Review 7.6 for 8.6)

100. $\sqrt[4]{x+2} + 9 = 14$ **101.** $\sqrt[3]{3x-4} = \sqrt[3]{x+10}$

102. $\sqrt{3x+7} = x + 3$ **103.** $(5x)^{1/2} - 18 = 32$

EVALUATING EXPRESSIONS Use a calculator to evaluate the expression. Round the result to three decimal places. (Review 8.3, 8.4 for 8.6)

104. e^9 **105.** e^{-12} **106.** $e^{1.7}$ **107.** $e^{-5.632}$

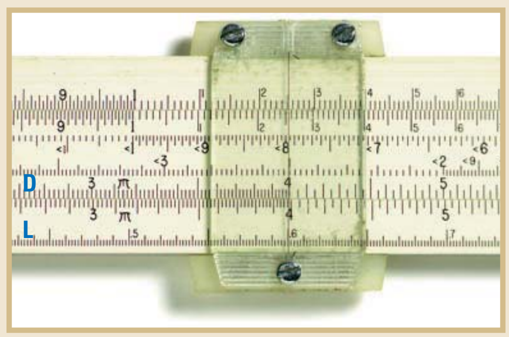
108. log 15 **109.** log 1.729 **110.** ln 16 **111.** ln 5.89

MATH & History — Logarithms

APPLICATION LINK www.mcdougallittell.com

THEN

IN 1614, John Napier published his discovery of logarithms. This discovery allowed calculations with exponents to be performed more easily. In 1632 William Oughtred set two logarithmic scales side by side to form the first slide rule. Because the slide rule could be used to multiply, divide, raise to powers, and take roots, it eliminated the need for many tedious paper-and-pencil calculations.



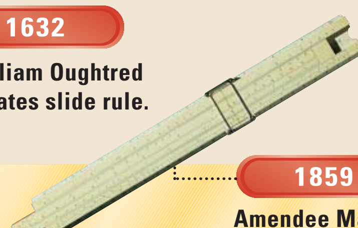
1. To approximate the logarithm of a number, look at the number on the D row and the corresponding value on the L row of the slide rule shown above. For example, log 4 ≈ 0.6. Approximate log 3 and log 5.
2. Use the product property of logarithms to find log 15.

NOW

TODAY, calculators have replaced the use of slide rules but not the use of logarithms. Logarithms are still used for scaling purposes, such as the decibel scale and the Richter scale, because the numbers involved span many orders of magnitude.

1617 John Napier invents Napier's Bones.

1632 William Oughtred creates slide rule.



1859 Amendee Manheim creates modern slide rule.

1999 Modern day calculator