8.6 Solving Exponential and Logarithmic Equations

What you should learn

GOAL 1 Solve exponential equations.

GOAL 2 Solve logarithmic equations, as applied in **Example 8**.

Why you should learn it

▼ To solve **real-life** problems, such as finding the diameter of a telescope's objective lens or mirror in **Ex. 69**.

REAL LIFE

GOAL 1 SOLVING EXPONENTIAL EQUATIONS

One way to solve exponential equations is to use the property that if two powers with the *same base* are equal, then their exponents must be equal.

$$\text{For } b > 0 \text{ and } b \neq 1, \text{ if } b^x = b^y, \text{ then } x = y.$$

EXAMPLE 1 Solving by Equating Exponents

Solve $4^{3x} = 8^{x+1}$.

SOLUTION

$4^{3x} = 8^{x+1}$ **Write original equation.**

$(2^2)^{3x} = (2^3)^{x+1}$ **Rewrite each power with base 2.**

$2^{6x} = 2^{3x+3}$ **Power of a power property**

$6x = 3x + 3$ **Equate exponents.**

$x = 1$ **Solve for *x*.**

▶ The solution is 1.

✓ **CHECK** Check the solution by substituting it into the original equation.

$4^{3 \cdot 1} \stackrel{?}{=} 8^{1+1}$ **Substitute 1 for *x*.**

$64 = 64$ ✓ **Solution checks.**

When it is not convenient to write each side of an exponential equation using the same base, you can solve the equation by taking a logarithm of each side.

EXAMPLE 2 Taking a Logarithm of Each Side

Solve $2^x = 7$.

SOLUTION

$2^x = 7$ **Write original equation.**

$\log_2 2^x = \log_2 7$ **Take $\log_2$ of each side.**

$x = \log_2 7$ **$\log_b b^x = x$**

$x = \dfrac{\log 7}{\log 2} \approx 2.807$ **Use change-of-base formula and a calculator.**

▶ The solution is about 2.807. Check this in the original equation.

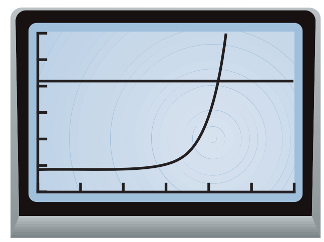
EXAMPLE 3 *Taking a Logarithm of Each Side*

Solve $10^{2x-3} + 4 = 21$.

SOLUTION

$10^{2x-3} + 4 = 21$	Write original equation.
$10^{2x-3} = 17$	Subtract 4 from each side.
$\log 10^{2x-3} = \log 17$	Take common log of each side.
$2x - 3 = \log 17$	$\log 10^x = x$
$2x = 3 + \log 17$	Add 3 to each side.
$x = \frac{1}{2}(3 + \log 17)$	Multiply each side by $\frac{1}{2}$.
$x \approx 2.115$	Use a calculator.

▶ The solution is about 2.115.



✓**CHECK** Check the solution algebraically by substituting into the original equation. Or, check it graphically by graphing both sides of the equation and observing that the two graphs intersect at $x \approx 2.115$.

Newton's law of cooling states that the temperature T of a cooling substance at time t (in minutes) can be modeled by the equation

$$T = (T_0 - T_R)e^{-rt} + T_R$$

where T_0 is the initial temperature of the substance, T_R is the room temperature, and r is a constant that represents the cooling rate of the substance.

REAL LIFE
Cooking

EXAMPLE 4 *Using an Exponential Model*

You are cooking *aleecha*, an Ethiopian stew. When you take it off the stove, its temperature is 212°F. The room temperature is 70°F and the cooling rate of the stew is $r = 0.046$. How long will it take to cool the stew to a serving temperature of 100°F?

STUDENT HELP

INTERNET
HOMEWORK HELP
Visit our Web site www.mcdougallittell.com for extra examples.

SOLUTION

You can use Newton's law of cooling with $T = 100$, $T_0 = 212$, $T_R = 70$, and $r = 0.046$.

$T = (T_0 - T_R)e^{-rt} + T_R$	Newton's law of cooling
$100 = (212 - 70)e^{-0.046t} + 70$	Substitute for T, T_0, T_R, and r.
$30 = 142e^{-0.046t}$	Subtract 70 from each side.
$0.211 \approx e^{-0.046t}$	Divide each side by 142.
$\ln 0.211 \approx \ln e^{-0.046t}$	Take natural log of each side.
$-1.556 \approx -0.046t$	$\ln e^x = \log_e e^x = x$
$33.8 \approx t$	Divide each side by -0.046.

▶ You should wait about 34 minutes before serving the stew.

GOAL 2 SOLVING LOGARITHMIC EQUATIONS

To solve a logarithmic equation, use this property for logarithms with the *same base:*

For positive numbers b, x, and y where $b \neq 1$, $\log_b x = \log_b y$ if and only if $x = y$.

EXAMPLE 5 *Solving a Logarithmic Equation*

Solve $\log_3 (5x - 1) = \log_3 (x + 7)$.

SOLUTION

$\log_3 (5x - 1) = \log_3 (x + 7)$ **Write original equation.**

$5x - 1 = x + 7$ **Use property stated above.**

$5x = x + 8$ **Add 1 to each side.**

$x = 2$ **Solve for *x*.**

▶ The solution is 2.

✓**CHECK** Check the solution by substituting it into the original equation.

$\log_3 (5x - 1) = \log_3 (x + 7)$ **Write original equation.**

$\log_3 (5 \cdot 2 - 1) \stackrel{?}{=} \log_3 (2 + 7)$ **Substitute 2 for *x*.**

$\log_3 9 = \log_3 9$ ✓ **Solution checks.**

When it is not convenient to write both sides of an equation as logarithmic expressions with the same base, you can *exponentiate* each side of the equation.

For $b > 0$ and $b \neq 1$, if $x = y$, then $b^x = b^y$.

EXAMPLE 6 *Exponentiating Each Side*

Solve $\log_5 (3x + 1) = 2$.

SOLUTION

$\log_5 (3x + 1) = 2$ **Write original equation.**

$5^{\log_5 (3x + 1)} = 5^2$ **Exponentiate each side using base 5.**

$3x + 1 = 25$ $b^{\log_b x} = x$

$x = 8$ **Solve for *x*.**

▶ The solution is 8.

✓**CHECK** Check the solution by substituting it into the original equation.

$\log_5 (3x + 1) = 2$ **Write original equation.**

$\log_5 (3 \cdot 8 + 1) \stackrel{?}{=} 2$ **Substitute 8 for *x*.**

$\log_5 25 \stackrel{?}{=} 2$ **Simplify.**

$2 = 2$ ✓ **Solution checks.**

Because the domain of a logarithmic function generally does not include all real numbers, you should be sure to check for extraneous solutions of logarithmic equations. You can do this algebraically or graphically.

EXAMPLE 7 *Checking for Extraneous Solutions*

STUDENT HELP

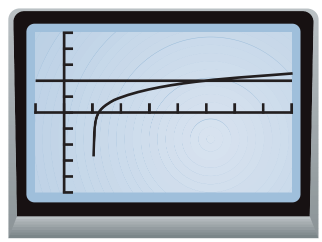
Look Back
For help with the zero product property, see p. 257.

Solve $\log 5x + \log (x - 1) = 2$. Check for extraneous solutions.

SOLUTION

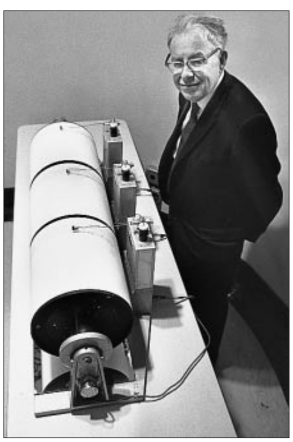
$\log 5x + \log (x - 1) = 2$	Write original equation.
$\log [5x(x - 1)] = 2$	Product property of logarithms
$10^{\log (5x^2 - 5x)} = 10^2$	Exponentiate each side using base 10.
$5x^2 - 5x = 100$	$10^{\log x} = x$
$x^2 - x - 20 = 0$	Write in standard form.
$(x - 5)(x + 4) = 0$	Factor.
$x = 5$ or $x = -4$	Zero product property

The solutions appear to be 5 and -4. However, when you check these in the original equation or use a graphic check as shown at the right, you can see that $x = 5$ is the only solution.



▶ The solution is 5.

FOCUS ON PEOPLE



CHARLES RICHTER developed the Richter scale in 1935 as a mathematical means of comparing the sizes of earthquakes. For large earthquakes, seismologists use a different measure called moment magnitude.

EXAMPLE 8 *Using a Logarithmic Model*

SEISMOLOGY The moment magnitude M of an earthquake that releases energy E (in ergs) can be modeled by this equation:

$$M = 0.291 \ln E + 1.17$$

On May 22, 1960, a powerful earthquake took place in Chile. It had a moment magnitude of 9.5. How much energy did this earthquake release?

▶ Source: U.S. Geological Survey National Earthquake Information Center

SOLUTION

$M = 0.291 \ln E + 1.17$	Write model for moment magnitude.
$9.5 = 0.291 \ln E + 1.17$	Substitute 9.5 for M.
$8.33 = 0.291 \ln E$	Subtract 1.17 from each side.
$28.625 \approx \ln E$	Divide each side by 0.291.
$e^{28.625} \approx e^{\ln E}$	Exponentiate each side using base e.
$2.702 \times 10^{12} \approx E$	$e^{\ln x} = e^{\log_e x} = x$

▶ The earthquake released about 2.7 trillion ergs of energy.

GUIDED PRACTICE

Vocabulary Check ✓

1. Give an example of an exponential equation and a logarithmic equation.

Concept Check ✓

2. How is solving a logarithmic equation similar to solving an exponential equation? How is it different?
3. Why do logarithmic equations sometimes have extraneous solutions?

Skill Check ✓

Solve the equation.

4. $3^x = 14$ **5.** $5^x = 8$ **6.** $9^{2x} = 3^{x-6}$

7. $10^{3x-4} = 0.1$ **8.** $2^{3x} = 4^{x-1}$ **9.** $10^{3x-1} + 4 = 32$

Solve the equation.

10. $\log x = 2.4$ **11.** $\log x = 3$ **12.** $\log_3 (2x - 1) = 3$

13. $12 \ln x = 44$ **14.** $\log_2 (x + 2) = \log_2 x^2$ **15.** $\log 3x + \log (x + 2) = 1$

ERROR ANALYSIS **In Exercises 16 and 17, describe the error.**

16.

$4^{x+1} = 8^x$

$\log_4 4^{x+1} = \log_4 8^x$

$x + 1 = x \log_4 8$

$x + 1 = 2x$

$1 = x$

17.

$\log_2 5x = 8$

$e^{\log_2 5x} = e^8$

$5x = e^8$

$x = \frac{1}{5}e^8$

18. **EARTHQUAKES** An earthquake that took place in Alaska on March 28, 1964, had a moment magnitude of 9.2. Use the equation given in Example 8 to determine how much energy this earthquake released.

PRACTICE AND APPLICATIONS

STUDENT HELP

Extra Practice to help you master skills is on p. 951.

CHECKING SOLUTIONS **Tell whether the *x*-value is a solution of the equation.**

19. $\ln x = 27, x = e^{27}$ **20.** $5 - \log_4 2x = 3, x = 8$

21. $\ln 5x = 4, x = \frac{1}{4}e^5$ **22.** $\log_5 \frac{1}{2}x = 17, x = 2e^{17}$

23. $5e^x = 15, x = \ln 3$ **24.** $e^x + 2 = 18, x = \log_2 16$

SOLVING EXPONENTIAL EQUATIONS **Solve the equation.**

STUDENT HELP

HOMEWORK HELP
Example 1–3: Exs. 23–42
Example 4: Exs. 62–68
Examples 5–7: Exs. 19–22, 43–60
Example 8: Exs. 69, 70

25. $10^{x-3} = 100^{4x-5}$ **26.** $25^{x-1} = 125^{4x}$ **27.** $3^{x-7} = 27^{2x}$

28. $36^{x-9} = 6^{2x}$ **29.** $8^{5x} = 16^{3x+4}$ **30.** $e^{-x} = 6$

31. $2^x = 15$ **32.** $1.2e^{-5x} + 2.6 = 3$ **33.** $4^x - 5 = 3$

34. $-5e^{-x} + 9 = 6$ **35.** $10^{2x} + 3 = 8$ **36.** $0.25^x - 0.5 = 2$

37. $\frac{1}{4}(4)^{2x} + 1 = 5$ **38.** $\frac{2}{3}e^{4x} + \frac{1}{3} = 4$ **39.** $10^{-12x} + 6 = 100$

40. $4 - 2e^x = -23$ **41.** $3^{0.1x} - 4 = 5$ **42.** $-16 + 0.2(10)^x = 35$

SOLVING LOGARITHMIC EQUATIONS **Solve the equation. Check for extraneous solutions.**

43. $\ln(4x + 1) = \ln(2x + 5)$

44. $\log_2 x = -1$

45. $4 \log_3 x = 28$

46. $16 \ln x = 30$

47. $\frac{1}{2} \log_6 16x = 3$

48. $1 - 2 \ln x = -4$

49. $2 \ln(-x) + 7 = 14$

50. $\log_5(2x + 15) = \log_5 3x$

51. $\ln x + \ln(x - 2) = 1$

52. $\ln x + \ln(x + 3) = 1$

53. $\log_8(11 - 6x) = \log_8(1 - x)$

54. $15 + 2 \log_2 x = 31$

55. $-5 + 2 \ln 3x = 5$

56. $\log(5 - 3x) = \log(4x - 9)$

57. $6.5 \log_5 3x = 20$

58. $\ln(x + 5) = \ln(x - 1) - \ln(x + 1)$

59. $\ln(5.6 - x) = \ln(18.4 - 2.6x)$

60. $10 \ln 100x - 3 = 117$

61. *Writing* Solve the equation $4^{3x} = 8^{x+1}$ in Example 1 by taking the common logarithm of each side of the equation. Do you prefer this method to the method shown in Example 1? Why or why not?

62. **COOKING** You are cooking chili. When you take it off the stove, it has a temperature of 205°F. The room temperature is 68°F and the cooling rate of the chili is $r = 0.03$. How long will it take to cool to a serving temperature of 95°F?

63. **FINANCE** You deposit \$2000 in an account that pays 2% annual interest compounded quarterly. How long will it take for the balance to reach \$2400?

64. **RADIOACTIVE DECAY** You have 20 grams of phosphorus-32 that decays 5% per day. How long will it take for half of the original amount to decay?

65. **DOUBLING TIME** You deposit \$500 in an account that pays 2.5% annual interest compounded continuously. How long will it take for the balance to double?

66. **HISTORY CONNECTION** The first permanent English colony in America was established in Jamestown, Virginia, in 1607. From 1620 through 1780, the population P of colonial America can be modeled by the equation

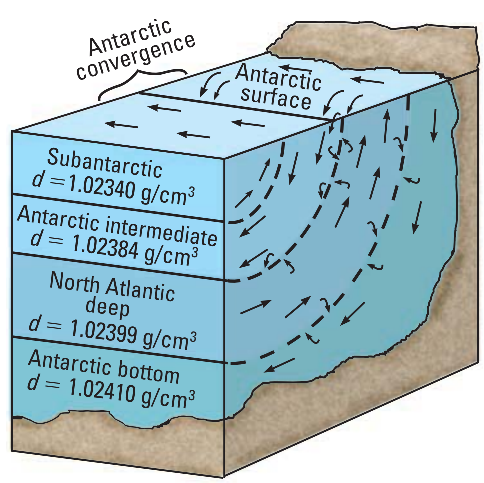
$$P = 8863(1.04)^t$$

where t is the number of years since 1620. When was the population of colonial America about 345,000?

67. **OCEANOGRAPHY** Oceanographers use the density d (in grams per cubic centimeter) of seawater to obtain information about the circulation of water masses and the rates at which waters of different densities mix. For water with a salinity of 30%, the density is related to the water temperature T (in degrees Celsius) by this equation:

$$d = 1.0245 - e^{0.1266T - 7.828}$$

Use the equation to find the temperature of each layer of water whose density is given in the diagram.



FOCUS ON APPLICATIONS

REAL LIFE **APPARENT MAGNITUDE** of a star is a number indicating the brightness of the star as seen from Earth. The greater the apparent magnitude, the fainter the star.

INTERNET APPLICATION LINK www.mcdougallittell.com

68. **MUON DECAY** A muon is an elementary particle that is similar to an electron, but much heavier. Muons are unstable—they very quickly decay to form electrons and other particles. In an experiment conducted in 1943, the number m of muon decays (of an original 5000 muons) was related to the time t (in microseconds) by this model:

$$m = e^{6.331 - 0.403t}$$

After how many microseconds were 204 decays recorded?

69. **ASTRONOMY** The relationship between a telescope's limiting magnitude (the apparent magnitude of the dimmest star that can be seen with the telescope) and the diameter of the telescope's objective lens or mirror can be modeled by

$$M = 5 \log D + 2$$

where M is the limiting magnitude and D is the diameter (in millimeters) of the lens or mirror. If a telescope can reveal stars with a magnitude of 12, what is the diameter of its objective lens or mirror? ▶ Source: *Practical Astronomy*

70. **ALTIMETER** An altimeter is an instrument that finds the height above sea level by measuring the air pressure. The height and the air pressure are related by the model

$$h = -8005 \ln \frac{P}{101{,}300}$$

where h is the height (in meters) above sea level and P is the air pressure (in pascals). What is the air pressure when the height is 4000 meters above sea level?

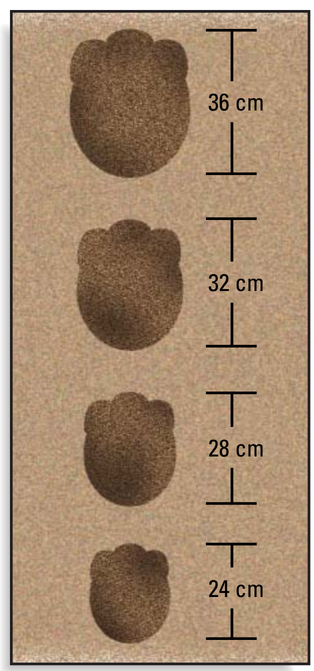
Test Preparation

71. **MULTI-STEP PROBLEM** A simple technique that biologists use to estimate the age of an African elephant is to measure the length of the elephant's footprint and then calculate its age using the equation

$$l = 45 - 25.7e^{-0.09a}$$

where l is the length of the footprint (in centimeters) and a is the age (in years).

▶ Source: *Journal of Wildlife Management*



a. Use the equation to find the ages of the elephants whose footprints are shown.

b. Solve the equation for a, and use this equation to find the ages of the elephants whose footprints are shown.

c. *Writing* Compare the methods you used in parts (a) and (b). Which method do you prefer? Explain.

★ Challenge

SOLVING EQUATIONS **Solve the equation.**

72. $2^{x+3} = 5^{3x-1}$

73. $10^{5x+2} = 5^{4-x}$

74. $\log_3 (x - 6) = \log_9 2x$

75. $\log_4 x = \log_8 4x$

76. *Writing* In Exercises 72–75 you solved exponential and logarithmic equations with different bases. Describe general methods for solving such equations.

EXTRA CHALLENGE www.mcdougallittell.com

MIXED REVIEW

MAKING SCATTER PLOTS **Draw a scatter plot of the data. Then approximate an equation of the best-fitting line.** (Review 2.5 for 8.7)

77.

x	−2	−1	−0.5	0	0.5	1	2	3	3.5	4
y	1.25	1.5	1.5	2	1.75	2	2.5	2.5	2.75	3.25

78.

x	−4	−3	−2.5	−2	−1.5	−1	0	1	1.5	2
y	1.5	1.75	1.75	2.25	2	2.25	2.75	2.75	3	3.5

THE SUBSTITUTION METHOD **Solve the linear system using the substitution method.** (Review 3.2 for 8.7)

79. $2x - y = 3$
$3x - 2y = 2$

80. $2x + y = 4$
$x + y = 3$

81. $x + 4y = -24$
$x - 4y = 24$

82. $x - 3y = -3$
$2x + y = 8$

83. $2x + y = -1$
$-4x - 2y = -5$

84. $-x + 6y = -32$
$7x - 2y = 24$

FACTORING **Factor the polynomial by grouping.** (Review 6.4)

85. $3x^3 - 6x^2 + 4x - 8$

86. $2x^3 - 5x^2 + 16x - 40$

87. $7x^3 + 4x^2 + 35x + 20$

88. $4x^3 - 3x^2 + 8x - 6$

QUIZ 2

Self-Test for Lessons 8.4–8.6

Evaluate the expression without using a calculator. (Lesson 8.4)

1. $\log_2 8$

2. $\log_5 625$

3. $\log_8 512$

4. Find the inverse of the function $y = \ln (x + 3)$. (Lesson 8.4)

Graph the function. State the domain and range. (Lesson 8.4)

5. $y = 1 + \log_4 x$

6. $y = \log_4 (x + 3)$

7. $y = 2 + \log_6 (x - 2)$

Use a property of logarithms to evaluate the expression. (Lesson 8.5)

8. $\log_3 (3 \cdot 27)$

9. $\log_2 \frac{1}{2}$

10. $\ln e^2$

11. Expand the expression $\log_4 x^{1/2}y^4$. (Lesson 8.5)

12. Condense the expression $2 \log_6 14 + 3 \log_6 x - \log_6 7$. (Lesson 8.5)

13. Use the change-of-base formula to evaluate the expression $\log_4 22$. (Lesson 8.5)

Solve the equation. (Lesson 8.6)

14. $3e^x - 1 = 14$

15. $3 \log_2 x = 28$

16. $\ln (2x + 7) = \ln (x - 4)$

17. **EARTHQUAKES** An earthquake that took place in Indonesia on February 1, 1938, had a moment magnitude of 8.5. Use the model $M = 0.291 \ln E + 1.17$, where M is the moment magnitude and E is the energy (in ergs) of an earthquake, to determine how much energy the Indonesian earthquake released. (Lesson 8.6)