9.4 Multiplying and Dividing Rational Expressions

What *you should learn*

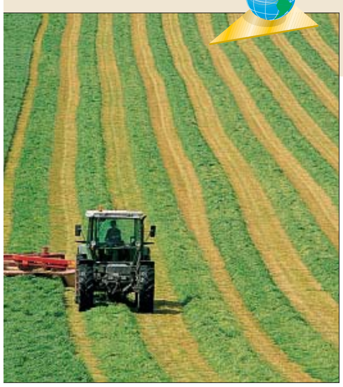
GOAL 1 Multiply and divide rational expressions.

GOAL 2 Use rational expressions to model **real-life** quantities, such as the heat generated by a runner in **Exs. 50 and 51**.

Why *you should learn it*

▼ To solve **real-life** problems, such as finding the average number of acres per farm in **Exs. 52 and 53**.

REAL LIFE



GOAL 1 Working with Rational Expressions

A rational expression is in **simplified form** provided its numerator and denominator have no common factors (other than ± 1). To simplify a rational expression, apply the following property.

SIMPLIFYING RATIONAL EXPRESSIONS

Let a, b, and c be nonzero real numbers or variable expressions. Then the following property applies:

$$\frac{a\cancel{c}}{b\cancel{c}} = \frac{a}{b}$$ **Divide out common factor *c*.**

Simplifying a rational expression usually requires two steps. First, factor the numerator and denominator. Then, divide out any factors that are common to both the numerator and denominator. Here is an example:

$$\frac{x^2 + 5x}{x^2} = \frac{\cancel{x}(x + 5)}{\cancel{x} \cdot x} = \frac{x + 5}{x}$$

Notice that you can divide out common factors in the second expression above, but you cannot divide out like terms in the third expression.

EXAMPLE 1 *Simplifying a Rational Expression*

Simplify: $\dfrac{x^2 - 4x - 12}{x^2 - 4}$

Solution

$$\frac{x^2 - 4x - 12}{x^2 - 4} = \frac{(x + 2)(x - 6)}{(x + 2)(x - 2)}$$ **Factor numerator and denominator.**

$$= \frac{\cancel{(x + 2)}(x - 6)}{\cancel{(x + 2)}(x - 2)}$$ **Divide out common factor.**

$$= \frac{x - 6}{x - 2}$$ **Simplified form**

....

The rule for multiplying rational expressions is the same as the rule for multiplying numerical fractions: multiply numerators, multiply denominators, and write the new fraction in simplified form.

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$ ← **Simplify $\frac{ac}{bd}$ if possible.**

EXAMPLE 2 Multiplying Rational Expressions Involving Monomials

Multiply: $\frac{5x^2y}{2xy^3} \cdot \frac{6x^3y^2}{10y}$

SOLUTION

$$\frac{5x^2y}{2xy^3} \cdot \frac{6x^3y^2}{10y} = \frac{30x^5y^3}{20xy^4}$$ Multiply numerators and denominators.

$$= \frac{3 \cdot \cancel{10} \cdot \cancel{x} \cdot x^4 \cdot \cancel{y^3}}{2 \cdot \cancel{10} \cdot \cancel{x} \cdot y \cdot \cancel{y^3}}$$ Factor and divide out common factors.

$$= \frac{3x^4}{2y}$$ Simplified form

EXAMPLE 3 Multiplying Rational Expressions Involving Polynomials

Multiply: $\frac{4x - 4x^2}{x^2 + 2x - 3} \cdot \frac{x^2 + x - 6}{4x}$

SOLUTION

$$\frac{4x - 4x^2}{x^2 + 2x - 3} \cdot \frac{x^2 + x - 6}{4x}$$

$$= \frac{4x(1 - x)}{(x - 1)(x + 3)} \cdot \frac{(x + 3)(x - 2)}{4x}$$ Factor numerators and denominators.

$$= \frac{4x(1 - x)(x + 3)(x - 2)}{(x - 1)(x + 3)(4x)}$$ Multiply numerators and denominators.

$$= \frac{4x(-1)(x - 1)(x + 3)(x - 2)}{(x - 1)(x + 3)(4x)}$$ Rewrite $(1 - x)$ as $(-1)(x - 1)$.

$$= \frac{\cancel{4x}(-1)\cancel{(x - 1)}\cancel{(x + 3)}(x - 2)}{\cancel{(x - 1)}\cancel{(x + 3)}\cancel{(4x)}}$$ Divide out common factors.

$$= -x + 2$$ Simplified form

EXAMPLE 4 Multiplying by a Polynomial

Multiply: $\frac{x + 3}{8x^3 - 1} \cdot (4x^2 + 2x + 1)$

STUDENT HELP

Look Back
For help with factoring a difference of two cubes, see p. 345.

SOLUTION

$$\frac{x + 3}{8x^3 - 1} \cdot (4x^2 + 2x + 1)$$

$$= \frac{x + 3}{8x^3 - 1} \cdot \frac{4x^2 + 2x + 1}{1}$$ Write polynomial as rational expression.

$$= \frac{(x + 3)(4x^2 + 2x + 1)}{(2x - 1)(4x^2 + 2x + 1)}$$ Factor and multiply numerators and denominators.

$$= \frac{(x + 3)\cancel{(4x^2 + 2x + 1)}}{(2x - 1)\cancel{(4x^2 + 2x + 1)}}$$ Divide out common factors.

$$= \frac{x + 3}{2x - 1}$$ Simplified form

To divide one rational expression by another, multiply the first expression by the reciprocal of the second expression.

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}$$ ⟵ **Simplify $\frac{ad}{bc}$ if possible.**

EXAMPLE 5 *Dividing Rational Expressions*

Divide: $\frac{5x}{3x - 12} \div \frac{x^2 - 2x}{x^2 - 6x + 8}$

SOLUTION

$$\frac{5x}{3x - 12} \div \frac{x^2 - 2x}{x^2 - 6x + 8} = \frac{5x}{3x - 12} \cdot \frac{x^2 - 6x + 8}{x^2 - 2x}$$ **Multiply by reciprocal.**

$$= \frac{5x}{3(x - 4)} \cdot \frac{(x - 2)(x - 4)}{x(x - 2)}$$ **Factor.**

$$= \frac{5\cancel{x}\cancel{(x - 2)}\cancel{(x - 4)}}{3\cancel{(x - 4)}\cancel{(x)}\cancel{(x - 2)}}$$ **Divide out common factors.**

$$= \frac{5}{3}$$ **Simplified form**

EXAMPLE 6 *Dividing by a Polynomial*

Divide: $\frac{6x^2 + 7x - 3}{6x^2} \div (2x^2 + 3x)$

SOLUTION

$$\frac{6x^2 + 7x - 3}{6x^2} \div (2x^2 + 3x) = \frac{6x^2 + 7x - 3}{6x^2} \cdot \frac{1}{2x^2 + 3x}$$

$$= \frac{(3x - 1)(2x + 3)}{6x^2} \cdot \frac{1}{x(2x + 3)}$$

$$= \frac{(3x - 1)\cancel{(2x + 3)}}{(6x^2)(x)\cancel{(2x + 3)}}$$

$$= \frac{3x - 1}{6x^3}$$

EXAMPLE 7 *Multiplying and Dividing*

Simplify: $\frac{x}{x + 5} \cdot (3x - 5) \div \frac{9x^2 - 25}{x + 5}$

SOLUTION

$$\frac{x}{x + 5} \cdot (3x - 5) \div \frac{9x^2 - 25}{x + 5} = \frac{x}{x + 5} \cdot \frac{3x - 5}{1} \cdot \frac{x + 5}{9x^2 - 25}$$

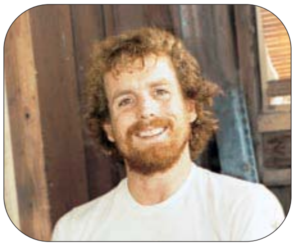
$$= \frac{x\cancel{(3x - 5)}\cancel{(x + 5)}}{\cancel{(x + 5)}\cancel{(3x - 5)}(3x + 5)}$$

$$= \frac{x}{3x + 5}$$

STUDENT HELP

INTERNET **HOMEWORK HELP**
Visit our Web site www.mcdougallittell.com for extra examples.

FOCUS ON PEOPLE



REAL LIFE **GREGORY ROBERTSON** made a daring rescue while skydiving in 1987. To reach a novice skydiver in trouble, he increased his velocity by diving headfirst towards the other diver. Just 10 seconds before he would have hit the ground, he was able to deploy both of their chutes.

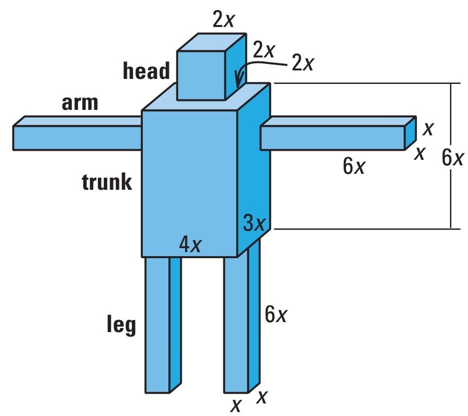
GOAL 2 USING RATIONAL EXPRESSIONS IN REAL LIFE

EXAMPLE 8 *Writing and Simplifying a Rational Model*

SKYDIVING A falling skydiver accelerates until reaching a constant falling speed, called the *terminal velocity*. Because of air resistance, the ratio of a skydiver's volume to his or her cross-sectional surface area affects the terminal velocity: the larger the ratio, the greater the terminal velocity.

a. The diagram shows a simplified geometric model of a skydiver with maximum cross-sectional surface area. Use the diagram to write a model for the ratio of volume to cross-sectional surface area for a skydiver.

b. Use the result of part (a) to compare the terminal velocities of two skydivers: one who is 60 inches tall and one who is 72 inches tall.



SOLUTION

a. The volume and cross-sectional surface area of each part of the skydiver are given in the table below. (Assume that the front side of the skydiver's body is parallel with the ground when falling.)

Body part	Volume	Cross-sectional surface area
Arm or leg	$V = 6x^3$	$S = 6x(x) = 6x^2$
Head	$V = 8x^3$	$S = 2x(2x) = 4x^2$
Trunk	$V = 72x^3$	$S = 6x(4x) = 24x^2$

Using these volumes and cross-sectional surface areas, you can write the ratio as:

$$\frac{\text{Volume}}{\text{Surface area}} = \frac{4(6x^3) + 8x^3 + 72x^3}{4(6x^2) + 4x^2 + 24x^2}$$

$$= \frac{104x^3}{52x^2}$$

$$= 2x$$

b. The overall height of the geometric model is $14x$. For the skydiver whose height is 60 inches, $14x = 60$, so $x \approx 4.3$. For the skydiver whose height is 72 inches, $14x = 72$, so $x \approx 5.1$. The ratio of volume to cross-sectional surface area for each skydiver is:

60 inch skydiver: $\frac{\text{Volume}}{\text{Surface area}} = 2x \approx 2(4.3) = 8.6$

72 inch skydiver: $\frac{\text{Volume}}{\text{Surface area}} = 2x \approx 2(5.1) = 10.2$

▶ The taller skydiver has the greater terminal velocity.

GUIDED PRACTICE

Vocabulary Check ✓

1. Explain how you know when a rational expression is in simplified form.

Concept Check ✓

2. **ERROR ANALYSIS** Explain what is wrong with the simplification of the rational expression shown.

$$\frac{5x^2 + 8x + 3}{5x^2 + 5x} = \frac{(5x + 3)(x + 1)}{5x(x + 1)}$$
$$= \frac{5x + 3}{5x}$$
$$= \frac{3}{1} = 3$$

Skill Check ✓

If possible, simplify the rational expression.

3. $\frac{4x^2}{4x^3 + 12x}$
4. $\frac{x^2 + 4x - 5}{x^2 - 1}$
5. $\frac{x^2 + 10x - 4}{x^2 + 10x}$
6. $\frac{6x^2 - 4x - 3}{3x^2 + x}$
7. $\frac{x^2 - 9}{2x + 1}$
8. $\frac{2x^3 - 32x}{x^2 + 8x + 16}$

Perform the indicated operation. Simplify the result.

9. $\frac{16x^3}{5y^9} \cdot \frac{x^5y^8}{80x^3y}$
10. $\frac{7x^4y^3}{5xy} \cdot \frac{2x^7}{21y^5}$
11. $\frac{x^2 + x - 6}{2x^2} \cdot \frac{2x + 8}{x^2 + 7x + 12}$
12. $\frac{144}{4xy} \div \frac{54y^3}{3x^3y}$
13. $\frac{16xy}{3x^5y^5} \div \frac{8x^2}{9xy^7}$
14. $\frac{5x^2 + 10x}{x^2 - x - 6} \div \frac{15x^3 + 45x^2}{x^2 - 9}$

15. **SKYDIVING** Look back at Example 8 on page 557. Some skydivers wear "wings" to increase their surface area. Suppose a skydiver who is 65 inches tall is wearing wings that add $18x^2$ of surface area and an insignificant amount of volume. Calculate the skydiver's volume to surface area ratio with and without the wings.

PRACTICE AND APPLICATIONS

STUDENT HELP

Extra Practice to help you master skills is on p. 953.

SIMPLIFYING **If possible, simplify the rational expression.**

16. $\frac{3x^3}{12x^2 + 9x}$
17. $\frac{x^2 - x - 6}{x^2 + 8x + 16}$
18. $\frac{x^2 - 3x + 2}{x^2 + 5x - 6}$
19. $\frac{x^2 + 2x - 4}{x^2 + x - 6}$
20. $\frac{x^2 - 2x - 3}{x^2 - 7x + 12}$
21. $\frac{3x^2 - 3x - 6}{x^2 - 4}$
22. $\frac{x - 2}{x^3 - 8}$
23. $\frac{x^3 - 27}{x^3 + 3x^2 + 9x}$
24. $\frac{x^2 + 6x + 9}{x^2 - 9}$
25. $\frac{15x^2 - 8x - 18}{-20x^2 + 14x + 12}$
26. $\frac{x^3 - 2x^2 + x - 2}{3x^2 - 3x - 8}$
27. $\frac{x^3 + 3x^2 - 2x - 6}{x^3 + 27}$

MULTIPLYING **Multiply the rational expressions. Simplify the result.**

STUDENT HELP

HOMEWORK HELP
Example 1: Exs. 16–27
Examples 2–4: Exs. 28–35, 44–49
Examples 5–7: Exs. 36–49
Example 8: Exs. 50–55

28. $\frac{4xy^3}{x^2y} \cdot \frac{y}{8x}$
29. $\frac{80x^4}{y^3} \cdot \frac{xy}{5x^2}$
30. $\frac{2x^2 - 10}{x + 1} \cdot \frac{x + 2}{3x^2 - 15}$
31. $\frac{x - 3}{2x - 8} \cdot \frac{6x^2 - 96}{x^2 - 9}$
32. $\frac{x^2 - x - 6}{4x^3} \cdot \frac{x + 1}{x^2 + 5x + 6}$
33. $\frac{2x^2 - 2}{x^2 - 6x - 7} \cdot (x^2 - 10x + 21)$
34. $\frac{x^3 + 5x^2 - x - 5}{x^2 - 25} \cdot (x + 1)$
35. $\frac{x - 3}{-x^3 + 3x^2} \cdot (x^2 + 2x + 1)$

DIVIDING **Divide the rational expressions. Simplify the result.**

36. $\frac{32x^3y}{y^9} \div \frac{8x^4}{y^6}$

37. $\frac{2xyz}{x^2z^2} \div \frac{6y^3}{3xz}$

38. $\frac{3x^2 + x - 2}{x^2 + 3x + 2} \div \frac{2x}{x + 2}$

39. $\frac{x^2 - 14x + 48}{x^2 - 6x} \div (3x - 24)$

40. $\frac{2x^2 - 12x}{x^2 - 7x + 6} \div \frac{2x}{3x - 3}$

41. $\frac{x^2 + 8x + 16}{x + 2} \div \frac{x^2 + 6x + 8}{x^2 - 4}$

42. $\frac{x^2 + 6x - 7}{3x^2} \div \frac{x + 7}{6x}$

43. $(x^2 + 6x - 27) \div \frac{3x^2 + 27x}{x + 5}$

COMBINED OPERATIONS **Perform the indicated operations. Simplify the result.**

44. $(x - 5) \div \frac{x^2 - 11x + 30}{x^2 + 7x + 12} \cdot (x - 6)$

45. $\frac{x^2 - x - 12}{8x^2} \div \frac{x^3 + 3x^2}{8x^3 - 2x^2} \div \frac{4x - 1}{x + 2}$

46. $\frac{x^2 + 11x}{x - 2} \div (3x^2 + 6x) \cdot \frac{x^2 - 4}{x + 11}$

47. $\frac{2x^2 + x - 15}{2x^2 - 11x - 21} \cdot (6x + 9) \div \frac{2x - 5}{3x - 21}$

48. $(x^3 + 8) \cdot \frac{x - 2}{x^2 - 2x + 4} \div \frac{x^2 - 4}{x - 6}$

49. $\frac{x^2 + 12x + 20}{4x^2 - 9} \cdot \frac{6x^3 - 9x^2}{x^3 + 10x^2} \cdot (2x + 3)$

HEAT GENERATION **In Exercises 50 and 51, use the following information.**
Almost all of the energy generated by a long-distance runner is released in the form of heat. The rate of heat generation h_g and the rate of heat released h_r for a runner of height H can be modeled by

$$h_g = k_1H^3V^2 \qquad \text{and} \qquad h_r = k_2H^2$$

where k_1 and k_2 are constants and V is the runner's speed.

50. Write the ratio of heat generated to heat released.

51. When the ratio of heat generated to heat released equals 1, how is height related to velocity? Does this mean that a taller or a shorter runner has an advantage?

FARMLAND **In Exercises 52 and 53, use the following information.**
From 1987 to 1996, the total acres of farmland L (in millions) and the total number of farms F (in hundreds of thousands) in the United States can be modeled by

$$L = \frac{43.3t + 999}{0.0482t + 1} \qquad \text{and} \qquad F = \frac{0.101t^2 + 2.20}{0.0500t^2 + 1}$$

where t represents the number of years since 1987. ▶ Source: U.S. Bureau of the Census

52. Write a model for the average number of acres A per farm as a function of the year.

53. What was the average number of acres per farm in 1993?

WEIGHT IN GOLD **In Exercises 54 and 55, use the following information.**
From 1990 to 1996, the price P of gold (in dollars per ounce) and the weight W of gold mined (in millions of ounces) in the United States can be modeled by

$$P = \frac{53.4t^2 - 243t + 385}{0.00146t^3 + 0.122t^2 - 0.586t + 1}$$

$$W = -0.0112t^5 + 0.193t^4 - 1.17t^3 + 2.82t^2 - 1.76t + 10.4$$

where t represents the number of years since 1990. ▶ Source: U.S. Bureau of the Census

54. Write a model for the total value V of gold mined as a function of the year.

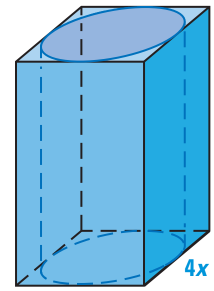
55. What was the total value of gold mined in the United States in 1994?

FOCUS ON CAREERS

REAL LIFE **FARMER**
In 1996 there were 1.3 million farmers and farm managers in the United States. In addition to knowing about crops and animals, farmers must keep up with changing technology and possess strong business skills.

INTERNET **CAREER LINK**
www.mcdougallittell.com

56. **GEOMETRY CONNECTION** Use the diagram at the right. Find the ratio of the volume of the rectangular prism to the volume of the inscribed cylinder. Write your answer in simplified form.



Test Preparation

57. **MULTI-STEP PROBLEM** The surface area S and the volume V of a tin can are given by $S = 2\pi r^2 + 2\pi rh$ and $V = \pi r^2 h$ where r is the radius of the can and h is the height of the can. One measure of the *efficiency* of a tin can is the ratio of its surface area to its volume.

a. Find a general formula (in simplified form) for the ratio $\frac{S}{V}$.

b. Find the efficiency of a can when $h = 2r$.

c. Calculate the efficiency of each can.

- A soup can with $r = 2\frac{5}{8}$ inches and $h = 3\frac{7}{8}$ inches.
- A 2 pound coffee can with $r = 5\frac{1}{8}$ inches and $h = 6\frac{1}{2}$ inches.
- A 3 pound coffee can with $r = 6\frac{3}{16}$ inches and $h = 7$ inches.

d. *Writing* Rank the three cans in part (c) by efficiency (most efficient to least efficient). Explain your rankings.

★ **Challenge**

58. Find two rational functions $f(x)$ and $g(x)$ such that $f(x) \cdot g(x) = x^2$ and $\frac{f(x)}{g(x)} = \frac{(x-1)^2}{(x+2)^2}$.

59. Find two rational functions $f(x)$ and $g(x)$ such that $f(x) \cdot g(x) = x - 1$ and $\frac{f(x)}{g(x)} = \frac{(x+1)^2(x-1)}{x^4}$.

EXTRA CHALLENGE
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MIXED REVIEW

GCFs AND LCMs **Find the greatest common factor and least common multiple of each pair of numbers.** (Skills Review, p. 908)

60. 96, 160	**61.** 120, 165	**62.** 48, 108
63. 72, 84	**64.** 238, 51	**65.** 480, 600

MULTIPLYING POLYNOMIALS **Find the product.** (Review 6.3 for 9.5)

66. $x(x^2 + 7x - 1)$	**67.** $(x + 7)(x - 1)$	**68.** $(x + 10)(x - 3)$
69. $(x + 3)(x^2 + 3x + 2)$	**70.** $(2x - 2)(x^3 - 4x^2)$	**71.** $x(x^2 - 4)(5 - 6x^3)$

BICYCLE DEPRECIATION **In Exercises 72 and 73, use the following information.** You bought a new mountain bike for \$800. The value of the bike decreases by about 14% each year. (Review 8.2)

72. Write an exponential decay model for the value of the bike. Use the model to estimate the value after 4 years.

73. Graph the model. Use the graph to estimate when the bike will be worth \$300.