VOCABULARY

arithmetic mean (p. 760)
arithmetic sequence (p. 759)
arithmetic series (p. 761)
Binomial Theorem (p. 803)
common difference (p. 759)
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comparison test (p. 789)
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exponential series (p. 808)
Fibonacci sequence (p. 806)
fractal geometry (p. 817)
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trigonometric series (p. 808)

UNDERSTANDING AND USING THE VOCABULARY

Choose the letter of the term that best matches each statement or phrase.

1. each succeeding term is formulated from one or more previous terms
2. ratio of successive terms in a geometric sequence
3. used to determine convergence
4. can have infinitely many terms
5. $e^{ix} = \cos \alpha + \boldsymbol{i} \sin \alpha$
6. the terms between any two nonconsecutive terms of a geometric sequence
7. indicated sum of the terms of an arithmetic sequence
8. $n! = n(n - 1)(n - 2) \cdot \cdots \cdot 1$
9. used to demonstrate the validity of a conjecture based on the truth of a first case, the assumption of truth of a *k*th case, and the demonstration of truth for the $(k + 1)$th case
10. an infinite series with a sum or limit

a. term
b. mathematical induction
c. arithmetic series
d. recursive formula
e. *n* factorial
f. geometric mean
g. sigma notation
h. convergent
i. common ratio
j. infinite sequence
k. Euler's Formula
l. prisoner point
m. ratio test
n. limit

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SKILLS AND CONCEPTS

OBJECTIVES AND EXAMPLES	REVIEW EXERCISES

Lesson 12-1 Find the nth term and arithmetic means of an arithmetic sequence.

- Find the 35th term in the arithmetic sequence $-5, -1, 3, \ldots$.
  Begin by finding the common difference d.
  $d = -1 - (-5)$ or 4
  Use the formula for the nth term.
  $a_n = a_1 + (n - 1)d$
  $a_{35} = -5 + (35 - 1)(4)$ or 131

11. Find the next four terms of the arithmetic sequence 3, 4.3, 5.6, ….

12. Find the 20th term of the arithmetic sequence for which $a_1 = 5$ and $d = -3$.

13. Form an arithemetic sequence that has three arithmetic means between 6 and -4.

Lesson 12-1 Find the sum of n terms of an arithmetic series.

- The sum S_n of the first n terms of an arithmetic series is given by $S_n = \frac{n}{2}(a_1 + a_n)$.

14. What is the sum of the first 14 terms in the arithmetic series $-30 - 23 - 16 - \cdots$?

15. Find n for the arithmetic series for which $a_1 = 2$, $d = 1.4$, and $S_n = 250.2$.

Lesson 12-2 Find the nth term and geometric means of a geometric sequence.

- Find an approximation for the 12th term of the sequence $-8, 4, -2, 1, \ldots$.
  First, find the common ratio.
  $a_2 \div a_1 = 4 \div (-8)$ or -0.5
  Use the formula for the nth term.
  $a_{12} = -8(-0.5)^{12-1}$ *$a_n = a_1r^{n-1}$*
  $= -8(-0.5)^{11}$ or about 0.004

16. Find the next three terms of the geometric sequence 49, 7, 1, ….

17. Find the 15th term of the geometric sequence for which $a_1 = 2.2$ and $r = 2$.

18. If $r = 0.2$ and $a_7 = 8$, what is the first term of the geometric sequence?

19. Write a geometric sequence that has three geometric means between 0.2 and 125.

Lesson 12-2 Find the sum of n terms of a geometric series.

- Find the sum of the first 12 terms of the geometric series $4 + 10 + 25 + 62.5 + \cdots$.
  First find the common ratio.
  $a_2 \div a_1 = 10 \div 4$ or 2.5
  Now use the formula for the sum of a finite geometric series.
  $S_n = \dfrac{a_1 - a_1r^n}{1 - r}$
  $S_{12} = \dfrac{4 - 4(2.5)^{12}}{1 - 2.5}$ *$n = 12, a_1 = 4, r = 2.5$*
  $S_{12} \approx 158{,}943.05$ *Use a calculator.*

20. What is the sum of the first nine terms of the geometric series $1.2 - 2.4 + 4.8 - \cdots$?

21. Find the sum of the first eight terms of the geometric series $4 + 4\sqrt{2} + 8 + \cdots$.

OBJECTIVES AND EXAMPLES

Lesson 12-3 Find the limit of the terms and the sum of an infinite geometric series.

- Find $\lim_{n\to\infty} \frac{2n^2+5}{3n^2}$.

$$\lim_{n\to\infty} \frac{2n^2+5}{3n^2} = \lim_{n\to\infty}\left(\frac{2}{3}+\frac{5}{3n^2}\right)$$
$$= \lim_{n\to\infty}\frac{2}{3} + \lim_{n\to\infty}\frac{5}{3}\cdot\lim_{n\to\infty}\frac{1}{n^2}$$
$$= \frac{2}{3}+\frac{5}{3}\cdot 0$$

Thus, the limit is $\frac{2}{3}$.

REVIEW EXERCISES

Find each limit, or state that the limit does not exist and explain your reasoning.

22. $\lim_{n\to\infty} \frac{3n}{4n+1}$ **23.** $\lim_{n\to\infty} \frac{6n-3}{n}$

24. $\lim_{n\to\infty} \frac{2^n n^3}{3n^3}$ **25.** $\lim_{n\to\infty} \frac{4n^3-3n}{n^4-4n^3}$

26. Write $5.\overline{123}$ as a fraction.

27. Find the sum of the infinite series $1260 + 504 + 201.6 + 80.64 + \cdots$, or state that the sum does not exist and explain your reasoning.

Lesson 12-4 Determine whether a series is convergent or divergent.

- Use the ratio test to determine whether the series $3 + \frac{3^2}{2!} + \frac{3^3}{3!} + \frac{3^4}{4!}$ is convergent or divergent.

The nth term a_n of this series has a general form of $\frac{3^n}{n!}$ and $a_{n+1} = \frac{3^{n+1}}{(n+1)!}$. Find $\lim_{n\to\infty}\frac{a_{n+1}}{a_n}$.

$$r = \lim_{n\to\infty} \frac{\frac{3^{n+1}}{(n+1)!}}{\frac{3^n}{n!}}$$
$$r = \lim_{n\to\infty}\left[\frac{3^{n+1}}{(n+1)!}\cdot\frac{n!}{3^n}\right]$$
$$r = \lim_{n\to\infty}\frac{3}{n+1} \text{ or } 0$$

Since $r < 0$, the series is convergent.

28. Use the ratio test to determine whether the series $\frac{1}{5} + \frac{2^2}{5^2} + \frac{3^2}{5^3} + \frac{4^2}{5^4} + \cdots$ is *convergent* or *divergent*.

29. Use the comparison test determine whether the series $\frac{6}{1} + \frac{7}{2} + \frac{8}{3} + \frac{9}{4} + \cdots$ is *convergent* or *divergent*.

30. Determine whether the series $2 + 1 + \frac{2}{3} + \frac{1}{2} + \frac{2}{5} + \frac{1}{3} + \frac{2}{7} + \cdots$ is *convergent* or *divergent*.

Lesson 12-5 Use sigma notation.

- Write $\sum_{n=1}^{3}(n^2-1)$ in expanded form and then find the sum.

$$\sum_{n=1}^{3}(n^2-1) = (1^2-1)+(2^2-1)+(3^2-1)$$
$$= 0+3+8 \text{ or } 11$$

Write each expression in expanded form and then find the sum.

31. $\sum_{a=5}^{9}(3a-3)$ **32.** $\sum_{k=1}^{\infty}(0.4)^k$

Express each series using sigma notation

33. $-1 + 1 + 3 + 5 + \cdots$

34. $2 + 5 + 10 + 17 + \cdots + 82$

OBJECTIVES AND EXAMPLES

Lesson 12-6 Use the Binomial Theorem to expand binomials.

Find the fourth term of $(2x - y)^6$.

$$(2x - y)^6 = \sum_{r=0}^{6} \frac{6!}{r!(6-r)!}(2x)^{6-r}(-y)^r$$

To find the fourth term, evaluate the general term for $r = 3$.

$$\frac{6!}{3!(6-3)!}(2x)^{6-3}(-y)^3$$

$$= \frac{6 \cdot 5 \cdot 4 \cdot 3!}{3! \cdot 3!}(2x)^3(-y^3) \text{ or } -160x^3y^3$$

REVIEW EXERCISES

Use the Binomial Theorem to expand each binomial.

35. $(a - 4)^6$ **36.** $(2r + 3s)^4$

Find the designated term of each binomial expansion.

37. 5th term of $(x - 2)^{10}$

38. 3rd term of $(4m + 1)^8$

39. 8th term of $(x + 3y)^{10}$

40. 6th term of $(2c - d)^{12}$

Lesson 12-7 Use Euler's Formula to write the exponential form of a complex number.

Write $\sqrt{3} - \boldsymbol{i}$ in exponential form.

Write the polar form of $\sqrt{3} - \boldsymbol{i}$.

$r = \sqrt{(\sqrt{3})^2 + (-1)^2}$ or 2, and

$\theta = \text{Arctan } \frac{-1}{\sqrt{3}}$ or $\frac{5\pi}{6}$

$$\sqrt{3} - \boldsymbol{i} = 2\left(\cos \frac{5\pi}{6} + \boldsymbol{i} \sin \frac{5\pi}{6}\right) = 2e^{\boldsymbol{i}\frac{5\pi}{6}}$$

Write each expression or complex number in exponential form.

41. $2\left(\cos \frac{3\pi}{4} + \boldsymbol{i} \sin \frac{3\pi}{4}\right)$

42. $4\boldsymbol{i}$

43. $2 - 2\boldsymbol{i}$

44. $3\sqrt{3} + 3\boldsymbol{i}$

Lesson 12-8 Iterate functions using real and complex numbers.

Find the first three iterates of the function $f(z) = 2z$ if the initial value is $3 - \boldsymbol{i}$.

$z_0 = 3 - \boldsymbol{i}$

$z_1 = 2(3 - \boldsymbol{i})$ or $6 - 2\boldsymbol{i}$

$z_2 = 2(6 - 2\boldsymbol{i})$ or $12 - 4\boldsymbol{i}$

$z_3 = 2(12 - 4\boldsymbol{i})$ or $24 - 8\boldsymbol{i}$

Find the first four iterates of each function using the given initial value. If necessary, round your answers to the nearest hundredth.

45. $f(x) = 6 - 3x$, $x_0 = 2$

46. $f(x) = x^2 + 4$, $x_0 = -3$

Find the first three iterates of the function $f(z) = 0.5z + (4 - 2i)$ for each initial value.

47. $z_0 = 4\boldsymbol{i}$ **48.** $z_0 = -8$

49. $z_0 = -4 + 6\boldsymbol{i}$ **50.** $z_0 = 12 - 8\boldsymbol{i}$

Lesson 12-9 Use mathematical induction to prove the validity of mathematical statements.

Proof by mathematical induction:

1. First, verify that the conjecture S_n is valid for the first possible case, usually $n = 1$.
2. Then, assume that S_n is valid for $n = k$ and use this assumption to prove that it is also valid for $n = k + 1$.

Use mathematical induction to prove that each proposition is valid for all positive integral values of n.

51. $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$

52. $3 + 8 + 15 + \cdots + n(n - 2) = \frac{n(n+1)(2n+7)}{6}$

53. $9^n - 4^n$ is divisible by 5.

CONTENTS

APPLICATIONS AND PROBLEM SOLVING

54. Physics If an object starting at rest falls in a vacuum near the surface of Earth, it will fall 16 feet during the first second, 48 feet during the next second, 80 feet during the third second, and so on. *(Lesson 12-1)*

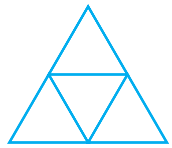
a. How far will the object fall during the twelfth second?

b. How far will the object have fallen after twelve seconds?

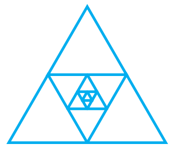
55. Budgets A major corporation plans to cut the budget on one of its projects by 3 percent each year. If the current budget for the project to be cut is $160 million, what will the budget for that project be in 10 years? *(Lesson 12-2)*

56. Geometry If the midpoints of the sides of an equilateral triangle are joined by straight lines, the new figure will also be an equilateral triangle. *(Lesson 12-3)*

a. If the original triangle has a perimeter of 6 units, find the perimeter of the new triangle.



b. If this process is continued to form a sequence of "nested" triangles, what will be the sum of the perimeters of all the triangles?



ALTERNATIVE ASSESSMENT

OPEN-ENDED ASSESSMENT

1. A sequence has a common difference of 3.

   a. Is this sequence arithmetic or geometric? Explain.

   b. Form a sequence that has this common difference. Write a recursive formula for your sequence.

2. Write a general expression for an infinite sequence that has no limit. Explain your reasoning.

PORTFOLIO

Explain the difference between a convergent and a divergent series. Give an example of each type of series and show why it is that type of series.

Additional Assessment See p. A67 for Chapter 12 practice test.

Unit 4 interNET Project

THE UNITED STATES CENSUS BUREAU

That's a lot of people!

- Use the Internet to find the population of the United States from at least 1900 through 2000. Write a sequence using the population for each ten-year interval, for example, 1900, 1910, and so on.
- Write a formula for an arithmetic sequence that provides a reasonable model for the population sequence.
- Write a formula for a geometric sequence that provides a reasonable model for the population sequence.
- Use your models to predict the U.S. population for the year 2050. Write a one-page paper comparing the arithmetic and geometric sequences you used to model the population data. Discuss which formula you think best models the data.

Percent Problems

A few common words and phrases used in percent problems, along with their translations into mathematical expressions, are listed below.

what ➡ x (variable)
of ➡ $\times$ (multiply)
is ➡ $=$ (equals)

What percent of A is B? ➡ $\frac{x}{100} \cdot A = B$

What is A percent more than B? ➡ $x = B + \frac{A}{100} \cdot B$

What is A percent less than B? ➡ $x = B - \frac{A}{100} \cdot B$

percent of change (increase or decrease) ➡ $\frac{\text{amount of change}}{\text{original amount}} \times 100$

THE PRINCETON REVIEW

TEST-TAKING TIP

With common percents, like 10%, 25%, or 50%, it is faster to use the fraction equivalents and mental math than a calculator.

ACT EXAMPLE

1. If c is positive, what percent of $3c$ is 9?

A $\frac{c}{100}\%$ **B** $\frac{300}{c}\%$ **C** $\frac{9}{c}\%$ **D** 3% **E** $\frac{c}{3}\%$

HINT Use variables just as you would use numbers.

Solution Start by translating the question into an equation.

What percent of $3c$ is 9?

$$\frac{x}{100} \cdot 3c = 9$$

Now solve the equation for x, not c.

$$\frac{x}{100} = \frac{9}{3c} \quad \textit{Divide each side by 3c.}$$

$$\frac{x}{100} = \frac{3}{c} \quad \textit{Simplify.}$$

$$x = \frac{300}{c} \quad \textit{Solve for x.}$$

The answer is choice **B**.

Alternate Solution "Plug in" 3 for c. The question becomes "what percent of 9 is 9?" The answer is 100%, so check each expression choice to see if it is equal to 100% when $c = 3$.

Choice A: $\frac{c}{100}\% = \frac{3}{100}\%$

Choice B: $\frac{300}{c}\% = 100\%$

Choice **B** is correct.

SAT EXAMPLE

2. An electronics store offers a 25% discount on all televisions during a sale week. How much must a customer pay for a television marked at $240?

A $60 **B** $300 **C** $230.40

D $180 **E** $215

HINT A discount is a decrease in the price of an item. So, the question asked is "What is 25% less than 240?"

Solution Start by translating this question into an equation.

What is 25% less than 240? ➡ $x = 240 - \frac{25}{100} \cdot 240$

Now simplify the right-hand side of the equation.

$$x = 240 - \frac{1}{4} \cdot 240 \text{ or } 180$$

Choice **D** is correct.

Alternate Solution If there is a 25% discount, a customer will pay (100 − 25)% or 75% of the marked price.

What is 75% of the marked price? ➡ $x = 0.75 \cdot 240$ or 180

The answer is choice **D**.

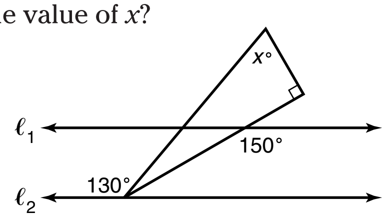
After you work each problem, record your answer on the answer sheet provided or on a piece of paper.

Multiple Choice

1. Shanika has a collection of 80 tapes. If 40% of her records are jazz tapes and the rest are blues tapes, how many blues tapes does she have?

A 32 **B** 40 **C** 42 **D** 48 **E** 50

2. If ℓ_1 is parallel to ℓ_2 in the figure below, what is the value of x?



A 20 **B** 50 **C** 70 **D** 80 **E** 90

3. There are k gallons of gasoline available to fill a tank. After d gallons have been pumped, then, in terms of k and d, what percent of the gasoline has been pumped?

A $\frac{100d}{k}\%$ **B** $\frac{k}{100d}\%$ **C** $\frac{100k}{d}\%$

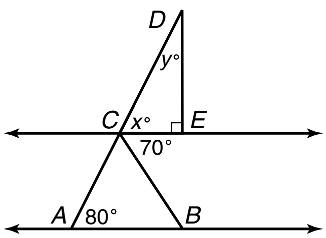
D $\frac{k}{100(k-d)}\%$ **E** $\frac{100(k-d)}{k}\%$

4. In 1985, Andrei had a collection of 48 baseball caps. Since then he has given away 13 caps, purchased 17 new caps, and traded 6 of his caps to Pierre for 8 of Pierre's caps. Since 1985, what has been the net percent increase in Andrei's collection?

A 6% **B** $12\frac{1}{2}\%$ **C** $16\frac{2}{3}\%$

D 25% **E** $28\frac{1}{2}\%$

5. In the figure below, $AB = AC$ and AD is a line segment. What is the value of $x - y$?



Note: Figure is NOT drawn to scale.

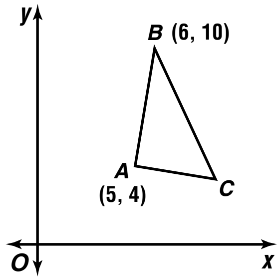
A 10 **B** 20 **C** 30 **D** 70 **E** 90

6. At the beginning of 2000, the population of Rockville was 204,000, and the population of Springfield was 216,000. If the population of each city increased by exactly 20% in 2000, how many more people lived in Springfield than in Rockville at the end of 2000?

A 9,600 **B** 10,000 **C** 12,000

D 14,400 **E** 20,000

7. In the figure, the slope of $\overline{AC}$ is $-\frac{1}{6}$, and $m\angle C = 30°$. What is the length of $\overline{BC}$?



A $\sqrt{37}$

B $\sqrt{111}$

C 2

D $2\sqrt{37}$

E It cannot be determined from the information given.

8. If $x + 6 > 0$ and $1 - 2x > -1$, then x could equal each of the following EXCEPT ?

A -6 **B** -4 **C** -2 **D** 0 **E** $\frac{1}{2}$

9. The percent increase from 99 to 100 is which of the following?

A greater than 1

B 1

C less than 1, but more than $\frac{1}{2}$

D less than $\frac{1}{2}$, but more than 0

E 0

10. Grid-In One fifth of the cars in a parking lot are blue, and $\frac{1}{2}$ of the blue cars are convertibles. If $\frac{1}{4}$ of the convertibles in the parking lot are blue, then what percent of the cars in the lot are neither blue nor convertibles?

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