Combinatorics And Probability

CHAPTER OBJECTIVES

- **Solve problems involving combinations and permutations.** *(Lessons 13-1, 13-2)*
- **Distinguish between independent and dependent events and between mutually exclusive and mutually inclusive events.** *(Lessons 13-1, 13-4, 13-5)*
- **Find probabilities.** *(Lessons 13-3, 13-4, 13-5, 13-6)*
- **Find odds for the success and failure of an event.** *(Lesson 13-3)*

13-1 Permutations and Combinations

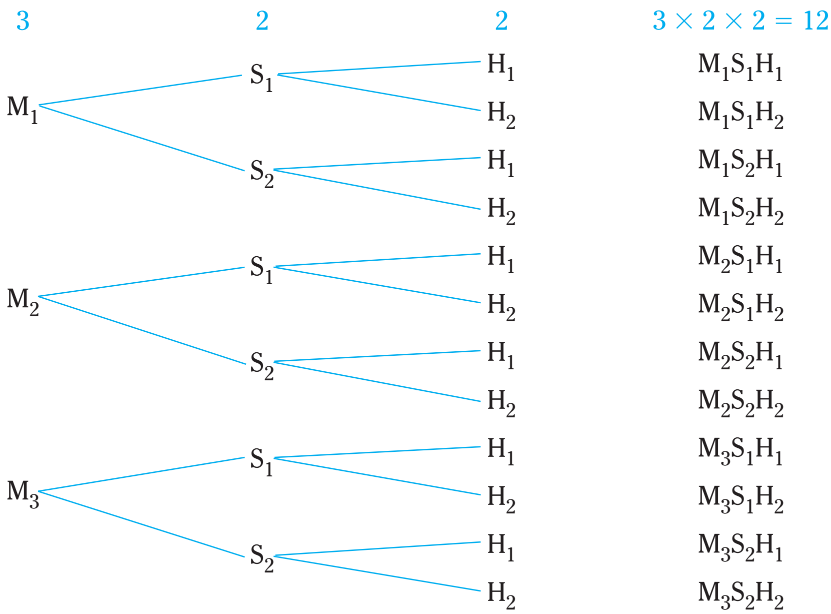
OBJECTIVES

- Solve problems related to the Basic Counting Principle.
- Distinguish between dependent and independent events.
- Solve problems involving permutations or combinations.

Real World Application

EDUCATION Ivette is a freshman at the University of Miami. She is planning her fall schedule for next year. She has a choice of three mathematics courses, two science courses, and two humanities courses. She can only select one course from each area. How many course schedules are possible?

Let M_1, M_2, and M_3 represent the three math courses, S_1 and S_2 the science courses, and H_1 and H_2 the humanities courses. Once Ivette makes a selection from the three mathematics courses she has two choices for her science course. Then, she has two choices for humanities. A **tree diagram** is often used to show all the choices.



Ivette has 12 possible schedules from which to choose.

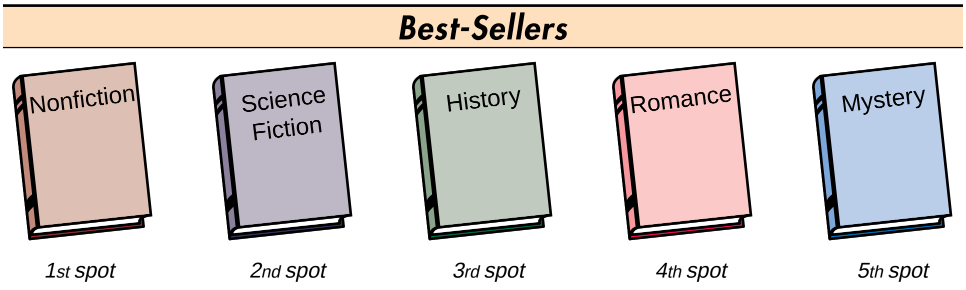
The choice of selecting a mathematics course does *not* affect the choice of ways to select a science or humanities course. Thus, these three choices are called **independent events.** Events that do affect each other are called **dependent events.** An example of dependent events would be the order in which runners finish a race. The first place runner affects the possibilities for second place, the second place runner affects the possibilities for third place, and so on.

The branch of mathematics that studies different possibilities for the arrangement of objects is called **combinatorics.** The example of choosing possible course schedules illustrates a rule of combinatorics known as the **Basic Counting Principle.**

Basic Counting Principle	Suppose one event can be chosen in p different ways, and another independent event can be chosen in q different ways. Then the two events can be chosen successively in $p \cdot q$ ways.

This principle can be extended to any number of independent events. For example, in the previous application, the events are chosen in $p \times q \times r$ or $3 \times 2 \times 2$ different ways.

Example 1 **Vickie works for a bookstore. Her manager asked her to arrange a set of five best-sellers for a display. The display is to be set up as shown below. The display set is made up of one book from each of 5 categories. There are 4 nonfiction, 4 science fiction, 3 history, 3 romance, and 3 mystery books from which to choose.**



a. Are the choices for each book independent or dependent events?

Since the choice of one type of book does not affect the choice of another type of book, the events are independent.

b. How many different ways can Vickie choose the books for the display?

Vickie has four choices for the first spot in the display, four choices for the second spot, and three choices for each of the next three spots.

1st spot		2nd spot		3rd spot		4th spot		5th spot
4	·	4	·	3	·	3	·	3

This can be represented as $4 \cdot 4 \cdot 3 \cdot 3 \cdot 3$ or 432 different arrangements.

There are 432 possible ways for Vickie to choose books for the display.

The arrangement of objects in a certain order is called a **permutation.** In a permutation, the order of the objects is very important. The symbol $P(n, n)$ denotes the number of permutations of n objects taken all at once. The symbol $P(n, r)$ denotes the number of permutations of n objects taken r at a time.

Permutations $P(n, n)$ and $P(n, r)$	The number of permutations of n objects, taken n at a time is defined as $P(n, n) = n!$. The number of permutations of n objects, taken r at a time is defined as $P(n, r) = \frac{n!}{(n - r)!}$.

Recall that n! is read "n factorial" and, $n! = n(n - 1)(n - 2)\ldots(1)$.

Example 2 **During a judging of a horse show at the Fairfield County Fair, there are three favorite horses: Rye Man, Oiler, and Sea of Gus.**

a. Are the selection of first, second and third place from the three horses independent or dependent events?

b. Assuming there are no ties and the three favorites finish in the top three places, how many ways can the horses win first, second and third places?

a. The choice of a horse for first place does affect the choice for second and third places. For example, if Rye Man is first, then is impossible for it to finish second or third. Therefore, the events are dependent.

b. Since order is important, this situation is a permutation.

Method 1: Tree diagram

There are three possibilities for first place, two for second, and one for third as shown in the tree diagram below. If Rye Man finishes first, then either Oiler or Sea of Gus will finish second. If Oiler finishes second, then Sea of Gus must finish third. Likewise, if Sea of Gus finishes second, then Oiler finishes third.

first	Rye Man		Oiler		Sea of Gus	
second	Oiler	Sea of Gus	Rye Man	Sea of Gus	Rye Man	Oiler
third	Sea of Gus	Oiler	Sea of Gus	Rye Man	Oiler	Rye Man

There are 6 possible ways the horses can win.

Method 2: Permutation formula

This situation depicts three objects taken three at a time and can be represented as $P(3, 3)$.

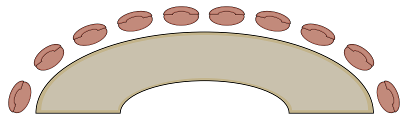
$$P(3, 3) = 3!$$
$$= 3 \cdot 2 \cdot 1 \text{ or } 6$$

Thus, there are 6 ways the horses can win first, second, and third place.

Example 3 **The board of directors of B.E.L.A. Technology Consultants is composed of 10 members.**

a. How many different ways can all the members sit at the conference table as shown?

b. In how many ways can they elect a chairperson, vice-chairperson, treasurer, and secretary, assuming that one person cannot hold more than one office?



a. Since order is important, this situation is a permutation. Also, the 10 members are being taken all at once so the situation can be represented as $P(10, 10)$.

$$P(10, 10) = 10!$$
$$= 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \text{ or } 3{,}628{,}800$$

There are 3,628,800 ways that the 10 board members can sit at the table.

b. This is a permutation of 10 people being chosen 4 at a time.

$$P(10, 4) = \frac{10!}{(10-4)!}$$
$$= \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6!}{6!}$$
$$= 5040$$

There are 5040 ways in which the offices can be filled.

Suppose that in the situation presented in Example 1, Vickie needs to select three types of books from the five types available. There are $P(5, 3)$ or 60 possible arrangements. She can arrange them as shown in the table below.

Arrangement	Type		
1	nonfiction	science fiction	history
2	nonfiction	history	science fiction
3	nonfiction	romance	mystery
4	nonfiction	mystery	romance
5	science fiction	nonfiction	history
6	science fiction	history	nonfiction
7	science fiction	romance	mystery
8	science fiction	mystery	romance
9	history	nonfiction	science fiction
10	history	science fiction	nonfiction
⋮	⋮	⋮	⋮
60	romance	mystery	nonfiction

Note that arrangements 1, 2, 5, 6, 9 and 10 contain the same three types of books. In each group of three books, there are 3! or 6 ways they can be arranged. Thus, if order is disregarded, there are $\frac{60}{3!}$ or 10 different groups of three types of books that can be selected from the five types. In this situation, called a **combination,** the order in which the books are selected is *not* a consideration.

A combination of *n* objects taken *r* at a time is calculated by dividing the number of permutations by the number of arrangements containing the same elements and is denoted by $C(n, r)$.

Combination $C(n, r)$

The number of combinations of *n* objects taken *r* at a time is defined as

$$C(n, r) = \frac{n!}{(n-r)!\,r!}.$$

The main difference between a permutation and a combination is whether order is considered (as in permutation) or not (as in combination). For example, for objects E, F, G, and H taken two at a time, the permutations and combinations are listed below.

Permutations				Combinations	
EF	FE	GE	HE	EF	FG
EG	FG	GF	HF	EG	FH
EH	FH	GH	HG	EH	GH

Note that in permutations, EF is different from FE. But in combinations, EF is the same as FE.

Example 4 Real World Application

ART In 1999, The National Art Gallery in Washington, D.C., opened an exhibition of the works of John Singer Sargent (1856–1925). The gallery's curator wanted to select four paintings out of twenty on display to showcase the work of the artist. How many groups of four paintings can be chosen?

"Oyster Gatherers of Cancale," 1878

Since order is not important, the selection is a combination of 20 objects taken 4 at a time. It can be represented as $C(20, 4)$.

$$C(20, 4) = \frac{20!}{(20-4)!\,4!}$$
$$= \frac{20!}{16!\,4!}$$
$$= \frac{20 \cdot 19 \cdot 18 \cdot 17 \cdot 16!}{16!\,4!}$$
$$= \frac{20 \cdot 19 \cdot 18 \cdot 17}{4 \cdot 3 \cdot 2 \cdot 1}$$
$$= 4845$$

Express 20! as $20 \cdot 19 \cdot 18 \cdot 17 \cdot 16!$ since $\frac{16!}{16!} = 1$.

There are 4845 possible groups of paintings.

Example 5 **At Grant Senior High School, there are 15 names on the ballot for junior class officers. Five will be selected to form a class committee.**

a. How many different committees of 5 can be formed?

b. In how many ways can a committee of 5 be formed if each student has a different responsibility?

c. If there are 8 girls and 7 boys on the ballot, how many committees of 2 boys and 3 girls can be formed?

a. Order is not important in this situation, so the selection is a combination of 15 people chosen 5 at a time.

$$C(15, 5) = \frac{15!}{(15-5)!\,5!}$$
$$= \frac{15!}{10!\,5!}$$
$$= \frac{15 \cdot 14 \cdot 13 \cdot 12 \cdot 11 \cdot 10!}{10!\,5!} \text{ or } 3003$$

There are 3003 different ways to form the committees of 5.

b. Order has to be considered in this situation because each committee member has a different responsibility.

$$P(15, 5) = \frac{15!}{(15-5)!}$$
$$= \frac{15!}{10!} \text{ or } 360{,}360$$

There are 360,360 possible committees.

c. Order is not important. There are three questions to consider.
How many ways can 2 boys be chosen from 7?
How many ways can 3 girls be chosen from 8?
Then, how many ways can 2 boys and 3 girls be chosen together?

Since the events are independent, the answer is the product of the combinations $C(7, 2)$ and $C(8, 3)$.

$$C(7, 2) \cdot C(8, 3) = \frac{7!}{(7-2)!\,2!} \cdot \frac{8!}{(8-3)!\,3!}$$
$$= \frac{7!}{5!\,2!} \cdot \frac{8!}{5!\,3!}$$
$$= 21 \cdot 56 \text{ or } 1176$$

There are 1176 possible committees.

Check for Understanding

Communicating Mathematics

Read and study the lesson to answer each question.

1. **Compare and contrast** permutations and combinations.
2. **Write** an expression for the number of ways, out of a standard 52-card deck, that 5-card hands can have 2 jacks and 3 queens.

3. **You Decide** Ms. Sloan asked her students how many ways 5 patients in a hospital could be assigned to 7 identical private rooms. Anita said that the problem dealt with computing $C(7, 5)$. Sam disagreed, saying that $P(7, 5)$ was the correct way to answer the question. Who is correct? Why?

4. **Draw** a tree diagram to illustrate all of the possible T-shirts available that come in sizes small, medium, large, and extra large and in the colors blue, green and gray.

Guided Practice

5. A restaurant offers the choice of an entrée, a vegetable, a dessert, and a drink for a lunch special. If there are 4 entrees, 3 vegetables, 5 desserts and 5 drinks available to choose from, how many different lunches are available?

6. Are choosing a movie to see and choosing a snack to buy dependent or independent events?

Find each value.

7. $P(6, 6)$
8. $P(5, 3)$
9. $\frac{P(12, 8)}{P(6, 4)}$
10. $C(7, 4)$
11. $C(20, 15)$
12. $C(4, 3) \cdot C(5, 2)$

13. If a group of 10 students sits in the same row in an auditorium, how many possible ways can they be arranged?

14. How many baseball lineups of 9 players can be formed from a team that has 15 members if all players can play any position?

15. **Postal Service** The U.S. Postal Service uses 5-digit ZIP codes to route letters and packages to their destinations.
    a. How many ZIP codes are possible if the numbers 0 through 9 are used for each of the 5 digits?
    b. Suppose that when the first digit is 0, the second, third, and fourth digits cannot be 0. How many 5-digit ZIP codes are possible if the first digit is 0?
    c. In 1983, the U.S. Postal Service introduced the ZIP +4, which added 4 more digits to the existing 5-digit ZIP codes. Using the numbers 0 through 9, how many additional ZIP codes were possible?

EXERCISES

Practice

16. If you toss a coin, then roll a die, and then spin a 4-colored spinner with equal sections, how many outcomes are possible?

17. How many ways can 7 classes be scheduled, if each class is offered in each of 7 periods?

18. Find the number of different 7-digit telephone numbers where:
    a. the first digit cannot be zero.
    b. only even digits are used.
    c. the complete telephone numbers are multiples of 10.
    d. the first three digits are 593 in that order.

State whether the events are *independent* or *dependent.*

19. selecting members for a team

20. tossing a penny, rolling a die, then tossing a dime

21. deciding the order in which to complete your homework assignments

Find each value.

22. $P(8, 8)$	**23.** $P(6, 4)$	**24.** $P(5, 3)$
25. $P(7, 4)$	**26.** $P(9, 5)$	**27.** $P(10, 7)$
28. $\frac{P(6, 3)}{P(4, 2)}$	**29.** $\frac{P(6, 4)}{P(5, 3)}$	**30.** $\frac{P(6, 3) \cdot P(7, 5)}{P(9, 6)}$
31. $C(5, 3)$	**32.** $C(10, 5)$	**33.** $C(4, 2)$
34. $C(12, 4)$	**35.** $C(9, 9)$	**36.** $C(14, 7)$
37. $C(3, 2) \cdot C(8, 3)$	**38.** $C(7, 3) \cdot C(8, 5)$	**39.** $C(5, 1) \cdot C(4, 2) \cdot C(8, 2)$

40. A pizza shop has 14 different toppings from which to choose. How many different 4-topping pizzas can be made?

41. If you make a fruit salad using 5 different fruits and you have 14 different varieties from which to choose, how many different fruit salads can you make?

42. How many different 12-member juries can be formed from a group of 18 people?

43. A bag contains 3 red, 5 yellow, and 8 blue marbles. How many ways can 2 red, 1 yellow, and 2 blue marbles be chosen?

44. How many different ways can 11 paintings be displayed on a wall?

45. From a standard 52-card deck, find how many 5-card hands are possible that have:

a. 3 hearts and 2 clubs.

b. 1 ace, 2 jacks, and 2 kings.

c. all face cards.

Applications and Problem Solving

Real World Application

46. Home Security A home security company offers a security system that uses the numbers 0 through 9, inclusive, for a 5-digit security code.

a. How many different security codes are possible?

b. If no digits can be repeated, how many security codes are available?

c. Suppose the homeowner does not want to use 0 as one of the digits and wants only two of the digits to be odd. How many codes can be formed if the digits can be repeated? If no repetitions are allowed, how many codes are available?

47. Baseball How many different 9-player teams can be fielded if there are 3 players who can only play catcher, 4 players who can only play first base, 6 players who can only pitch, and 14 players who can play in any of the remaining 6 positions?

48. Transportation In a train yard, there are 12 flatcars, 10 tanker cars, 15 boxcars, and 5 livestock cars.

a. If the cars must be connected according to their final destinations, how many different ways can they be arranged?

b. How many ways can the train be made up if it is to have 30 cars?

c. How many trains can be formed with 3 livestock cars, 6 flatcars, 6 tanker cars, and 5 boxcars?

49. Critical Thinking Prove $P(n, n - 1) = P(n, n)$.

50. Entertainment Three couples have reserved seats for a Broadway musical. Find how many different ways they can sit if:

a. there are no seating restrictions.

b. two members of each couple wish to sit together.

51. Botany A researcher with the U.S. Department of Agriculture is conducting an experiment to determine how well certain crops can survive adverse weather conditions. She has gathered 6 corn plants, 3 wheat plants, and 2 bean plants. She needs to select four plants at random for the experiment.

a. In how many ways can this be done?

b. If exactly 2 corn plants must be included, in how many ways can the plants be selected?

52. Geometry How many lines are determined by 10 points, no 3 of which are collinear?

53. Critical Thinking There are 6 permutations of the digits 1, 6, and 7.

167 176 617 671 716 761

The average of these six numbers is $\frac{3108}{6} = 518$ which is equal to $37(1 + 6 + 7)$. If the digits are 0, 4, and 7, then the average of the six permutations is $\frac{2442}{6} = 407$ or $37(0 + 4 + 7)$.

a. Use this pattern to find the average of the six permutations of 2, 5, and 9.

b. Will this pattern hold for all sets of three digits? If so, prove it.

Mixed Review

54. Banking Cynthia has a savings account that has an annual yield of 5.8%. Find the balance of the account after each of the first three years if her initial balance is \$2140. *(Lesson 12-8)*

55. Find the sum of the first ten terms of the series $1^3 + 2^3 + 3^3 + \cdots$. *(Lesson 12-5)*

56. Solve $7.1^x = 83.1$ using logarithms. Round to the nearest hundredth. *(Lesson 11-6)*

57. Find the value of x to the nearest tenth such that $x = e^{0.346}$. *(Lesson 11-3)*

58. Communications A satellite dish tracks a satellite directly overhead. Suppose the graph of the equation $y = 4x^2$ models the shape of the dish when it is oriented in this position. Later in the day, the dish is observed to have rotated approximately 45°. Find an equation that models the new orientation of the dish. *(Lesson 10-7)*

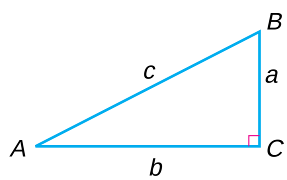
59. Graph the system of polar equations $r = 2$, and $r = 2 \cos 2\theta$. Then solve the system and compare the common solutions. *(Lesson 9-2)*

60. Find the initial vertical and horizontal velocities of a rock thrown with an initial velocity of 28 feet per second at an angle of 45° with the horizontal. *(Lesson 8-7)*

61. Solve $\sin 2x + 2 \sin x = 0$ for $0° \leq x \leq 360°$. *(Lesson 7-5)*

62. State the amplitude, period, and phase shift for the function $y = 8 \cos(\theta - 30°)$. *(Lesson 6-5)*

63. Given the triangle at the right, solve the triangle if $A = 27°$ and $b = 15.2$. Round angle measures to the nearest degree and side measures to the nearest tenth. *(Lesson 5-5)*



64. SAT/ACT Practice What is the number of degrees through which the hour hand of a clock moves in 2 hours 12 minutes?

A 66° **B** 72° **C** 126° **D** 732° **E** 792°

Extra Practice See p. A51.

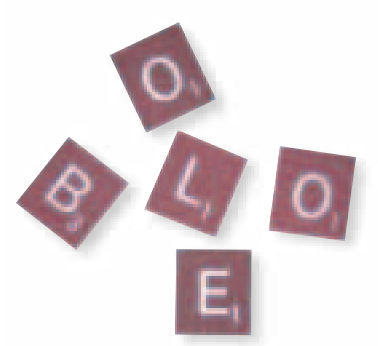
CONTENTS

13-2 Permutations with Repetitions and Circular Permutations

OBJECTIVES
- Solve problems involving permutations with repetitions.
- Solve problems involving circular permutations.

Real World Application

MARKETING Marketing professionals sometimes investigate the number of permutations and arrangements of letters to create company or product names. For example, the company JATACO was derived from the first initials of the owners Alan, Anthony, John, and Thomas. Suppose five high school students have developed a web site to help younger students better understand first year algebra. They decided to use the initials of their first names to create the title of their web site. The initials are: E, L, O, B, O. How many different five-letter words can be created with these letters?



Each "O" is marked a different color to differentiate it from the other. Some of the possible arrangements are listed below.

EOLBO	EOOLB	EOOBL	EOBOL
LEOOB	OBELO	OBLEO	BLOOE

The five letters can be arranged in $P(5, 5)$ or 5! ways. However, several of these 120 arrangements are the same unless the Os are colored. So without coloring the Os, there are repetitions in the 5! possible arrangements.

Permutations with Repetitions

The number of permutations of n objects of which p are alike and q are alike is

$$\frac{n!}{p!\ q!}.$$

Using this formula, we find there are only $\frac{5!}{2!}$ or 60 permutations of the five letters of which 2 are Os.

Example 1 **How many eight-letter patterns can be formed from the letters of the word *parabola*?**

The eight letters can be arranged in $P(8, 8)$ or 8! ways. However, several of these 40,320 arrangements have the same appearance since *a* appears 3 times.

The number of permutations of 8 letters of which 3 are the same is $\frac{8!}{3!}$ or 6720.

There are 6720 different eight-letter patterns that can be formed from the letters of the word *parabola.*

Example 2 **How many eleven-letter patterns can be formed from the letters of the word *Mississippi*?**

$$\frac{11!}{4!\,4!\,2!} = \frac{11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{4 \cdot 3 \cdot 2 \cdot 1 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot 2 \cdot 1}$$

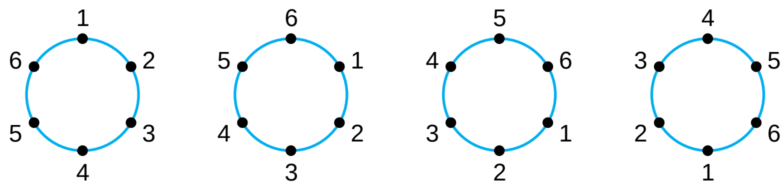
There are 11 letters in Mississippi.
4 i's   4 s's   2 p's

$$= 34{,}650$$

There are 34,650 eleven-letter patterns.

So far, you have been studying objects that are arranged in a line. Consider the problem of making distinct arrangements of six children riding on a merry-go-round at a playground. How many riding arrangements are possible?

Let the numbers 1 through 6 represent the children. Four possible arrangements are shown below.



When objects are arranged in a circle, some of the arrangements are alike. In the situation above, these similar arrangements fall into groups of six, each of which can be found by rotating the circle $\frac{1}{6}$ of a revolution. Thus, the number of distinct arrangements around the circular merry-go-round is $\frac{1}{6}$ of the total number of arrangements if the children stood in a line.

$$\frac{1}{6} \cdot 6! = \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{6}$$
$$= 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$$
$$= 5! \text{ or } (6-1)!$$

Thus, there are $(6 - 1)!$ or 120 distinct arrangements of the 6 children around the merry-go-round.

Circular Permutations	If n objects are arranged in a circle, then there are $\frac{n!}{n}$ or $(n - 1)!$ permutations of the n objects around the circle.

If the circular object looks the same when it is turned over, such as a plain key ring, then the number of permutations must be divided by 2.

Example 3 **At the Family Friendly Restaurant, nine bowls of food are placed on a circular, revolving tray in the center of the table. You can serve yourself from each of the bowls.**

a. Is the arrangement of the bowls on the tray a linear or circular permutation? Explain.

The arrangement is a circular permutation since the bowls form a circle on the tray and there is no reference point.

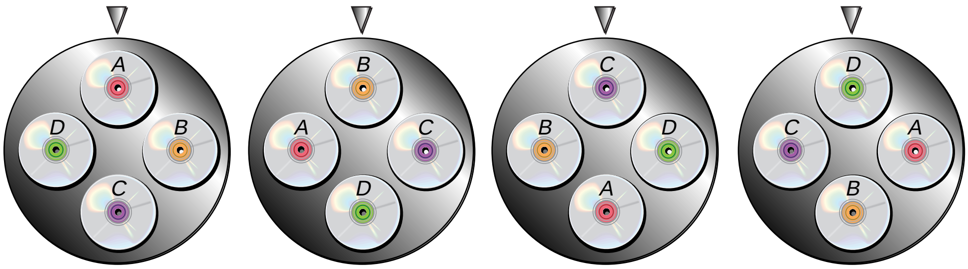
b. How many ways can the bowls be arranged?

There are nine bowls so the number of arrangements can be described by $(9 - 1)!$ or $8!$

$$8! = 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \text{ or } 40{,}320$$

There are 40,320 ways in which the bowls can be arranged on the tray.

Suppose a CD changer holds 4 CDs on a circular platter. Let each circle below represent the platter and the labeled points represent each CD. The arrow indicates which CD will be played.



These arrangements are different. In each one, a different CD is being played. Thus, there are $P(4, 4)$ or 24 arrangements relative to the playing position.

If n objects are arranged relative to a fixed point, then there are $n!$ permutations. Circular arrangements with fixed points of reference are treated as linear permutations.

Example 4 **Seven people are to be seated at a round table where one person is seated next to a window.**

a. Is the arrangement of the people around the table a linear or circular permutation? Explain.

b. How many possible arrangements of people relative to the window are there?

a. Since the people are seated around a table with a fixed reference point, the arrangement is a linear permutation.

b. There are seven people with a fixed reference point. So there are 7! or 5040 ways in which the people can be seated around the table.

Check for Understanding

Communicating Mathematics

Read and study the lesson to answer each question.

1. **Write** an explanation as to why a circular permutation is not computed the same as a linear permutation.
2. **Describe** two real-world situations involving permutations with repetitions.
3. Provide a **counterexample** for the following statement.
   The number of permutations for n objects in a circular arrangement is $(n - 1)!$.

Guided Practice **How many different ways can the letters of each word be arranged?**

4. *kangaroo*
5. *classical*

6. In how many ways can 2 red lights, 4 yellow lights, 5 blue lights, 1 green light, and 2 pink lights be arranged on a string of lights?

Determine whether each arrangement of objects is a linear or circular permutation. Then determine the number of arrangements for each situation.

7. 11 football players in a huddle
8. 8 jewels on a necklace
9. 12 decorative symbols around the face of a clock
10. 5 beads strung on a string arranged in a square pattern relative to a knot in the string

11. **Communication** Morse code is a system of dots, dashes, and spaces that telegraphers in the United States and Canada once used to send messages by wire. How many different arrangements are there of 5 dots, 2 dashes, and 2 spaces?

EXERCISES

Practice **How many different ways can the letters of each word be arranged?**

12. *pizzeria*
13. *California*
14. *calendar*
15. *centimeter*
16. *trigonometry*
17. *Tennessee*

18. How many different 7-digit phone numbers can have the digits 7, 3, 5, 2, 7, 3, and 2?
19. Five country CDs and four rap CDs are to be placed in a display window. How many ways can they be arranged if they are placed by category?
20. The table below shows the initial letter for each of the 50 states. How many different ways can you arrange the initial letters of the states?

Initial Letter	A	C	D	F	G	H	I	K	L	M	N	O	P	R	S	T	U	V	W
Number of States	4	3	1	1	1	1	4	2	1	8	8	3	1	1	2	2	1	1	3

Determine whether each arrangement of objects is a linear or circular permutation. Then determine the number of arrangements for each situation.

21. 12 gondolas on a Ferris wheel
22. a stack of 6 pennies, 3 nickels, 7 dimes, and 10 quarters
23. the placement of 9 specialty departments along the outside perimeter of a supermarket
24. a family of 5 seated around a rectangular table
25. 8 tools on a utility belt

Determine whether each arrangement of objects is a linear or circular permutation. Then determine the number of arrangements for each situation.

26. 6 houses on a cul-de-sac relative to the incoming street

27. 10 different beads on a string

28. a waiter placing 9 drinks along the edge of a circular tray

29. 14 keys on a key ring

30. 20 wooden dowels used as spokes for a wagon wheel

31. 32 horses on the outside edge of a carousel

32. 25 sections of a circular stadium relative to the main entrance

Applications and Problem Solving

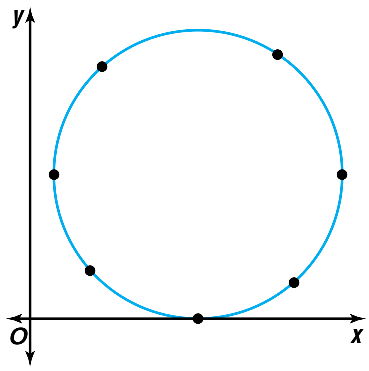
33. Biology A biologist needs to determine the number of possible arrangements of 4 kinds of molecules in a chain. If the chain contains 8 molecules with 2 of each kind, how many different arrangements are possible?

Real World Application

34. Geometry Suppose 7 points on the circle at the right are selected at random.

a. Using the letters A through G, how many ways can the points be named on the circle?

b. Relative to the point which lies on the *x*-axis, how many arrangements are possible?



35. Money Trish has a penny, 3 nickels, 4 dimes, and 3 quarters in her pocket. How many different arrangements are possible if she removes one coin at a time from her pocket?

36. Critical Thinking An anagram is a word or phrase made from another by rearranging its letters. For example, *now* can be changed to *won.* Consider the phrase "*calculating rules.*"

a. How many different ways can the letters in *calculating* be arranged?

b. Rearrange the letters and the space in the phrase to form the name of a branch of mathematics.

37. Auto Racing Most stock car races are held on oval-shaped tracks with cars from various manufacturers. Let *F, C,* and *P* represent three auto manufacturers.

a. Suppose for one race 20 *F* cars, 14 *C* cars, and 9 *P* cars qualified to be in a race. How many different starting line-ups based on manufacturer are possible?

b. If there are 43 cars racing, how many different ways could the cars be arranged on the track?

c. Relative to the leader of the race, how many different ways could the cars be arranged on the track?

38. Critical Thinking To break a code, Zach needs to find how many symbols there are in a particular sequence. He is told that there are 3 x's and some dashes. He is also told that there are 35 linear permutations of the symbols. What is the total number of symbols in the code sequence?

Mixed Review

39. Food Classic Pizza offers pepperoni, mushrooms, sausage, onions, peppers, and olives as toppings for their 7-inch pizza. How many different 3-topping pizzas can be made? *(Lesson 13-1)*

Extra Practice See p. A51.

40. Use the Binomial Theorem to expand $(5x - 1)^3$. *(Lesson 12-6)*

41. Solve $x < \log_2 413$ using logarithms. Round to the nearest hundredth. *(Lesson 11-5)*

42. Write the equation of the parabola with vertex at $(6, -1)$ and focus at $(3, -1)$. *(Lesson 10-5)*

43. Simplify $2(4 - 3\boldsymbol{i})(7 - 2\boldsymbol{i})$. *(Lesson 9-5)*

44. Find the cross product of $\vec{\mathbf{v}}\ \langle 2, 0, 3\rangle$ and $\vec{\mathbf{w}}\ \langle 2, 5, 0\rangle$. Verify that the resulting vector is perpendicular to $\vec{\mathbf{v}}$ and $\vec{\mathbf{w}}$. *(Lesson 8-4)*

45. **Manufacturing** A knife is held at a 45° angle to the vertical on a 16-inch diameter sharpening wheel. How far above the wheel must a lamp be placed so it will not be showered with sparks? *(Lesson 7-6)*

46. **SAT/ACT Practice** If $x^2 = 36$, then 2^{x-1} could equal

A 4 **B** 6 **C** 8 **D** 16 **E** 32

CAREER CHOICES

Actuary

Insurance companies, whether they cover property, liability, life, or health, need to determine how much to charge customers for coverage. If you are interested in statistics and probability, you may want to consider a career as an actuary.

Actuaries use statistical methods to determine the probability of such events as death, injury, unemployment, and property damage or loss. An actuary must estimate the number and amount of claims that may be made in order for the insurance company to set its insurance coverage rates for its customers. As an actuary, you can specialize in property and liability or life and health statistics. Most actuaries work for insurance companies, consulting firms, or the government.

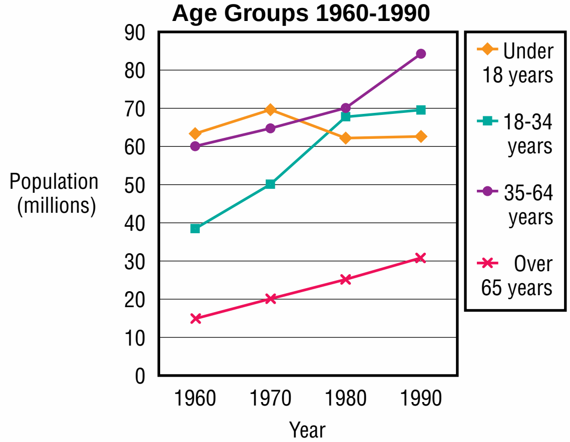
CAREER OVERVIEW

Degree Preferred:
bachelor's degree in actuarial science or mathematics

Related Courses:
mathematics, statistics, computer science, business courses

Outlook:
faster than average through the year 2006

Population of Various Age Groups 1960-1990



interNET CONNECTION For more information on careers in actuarial science, visit: **www.amc.glencoe.com**

13-3

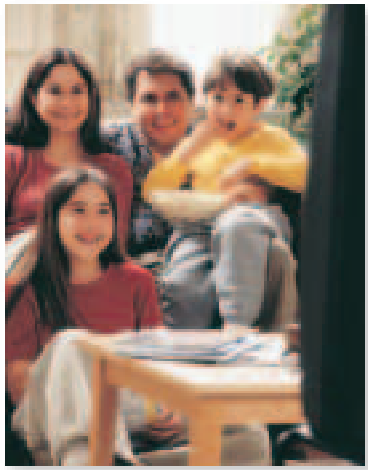
Probability and Odds

OBJECTIVES

- Find the probability of an event.
- Find the odds for the success and failure of an event.

Real World Application

MARKET RESEARCH To determine television ratings, Nielsen Media Research estimates how many people are watching any given television program. This is done by selecting a sample audience, having them record their viewing habits in a journal, and then counting the number of viewers for each program. There are about 100 million households in the U.S., and only 5000 are selected for the sample group. What is the probability of any one household being selected to participate? *This problem will be solved in Example 1.*



When we are uncertain about the occurrence of an event, we can measure the chances of its happening with **probability.** For example, there are 52 possible outcomes when selecting a card at random from a standard deck of playing cards. The set of all outcomes of an event is called the **sample space.** A desired outcome, drawing the king of hearts for example, is called a **success.** Any other outcome is called a **failure.** The probability of an event is the ratio of the number of ways an event can happen to the total number of outcomes in the sample space, which is the sum of successes and failures. There is one way to draw a king of hearts, and there are a total of 52 outcomes when selecting a card from a standard deck. So, the probability of selecting the king of hearts is $\frac{1}{52}$.

Probability of Success and of Failure

If an event can succeed in s ways and fail in f ways, then the probability of success $P(s)$ and the probability of failure $P(f)$ are as follows.

$$P(s) = \frac{s}{s+f} \qquad P(f) = \frac{f}{s+f}$$

Example 1 Real World Application

MARKET RESEARCH **What is the probability of any one household being chosen to participate for the Nielsen Media Research group?**

Use the probability formula. Since 5000 households are selected to participate $s = 5000$. The denominator, $s + f$, represents the total number of households, those selected, s, and those not selected, f. So, $s + f = 100{,}000{,}000$.

$$P(5000) = \frac{5000}{100{,}000{,}000} \text{ or } \frac{1}{20{,}000} \qquad P(s) = \frac{s}{s+f}$$

The probability of any one household being selected is $\frac{1}{20{,}000}$ or 0.005%.

An event that cannot fail has a probability of 1. An event that cannot succeed has a probability of 0. Thus, the probability of success $P(s)$ is always between 0 and 1 inclusive. That is, $0 \leq P(s) \leq 1$.

Example 2 **A bag contains 5 yellow, 6 blue, and 4 white marbles.**

a. What is the probability that a marble selected at random will be yellow?

b. What is the probability that a marble selected at random will *not* be white?

a. The probability of selecting a yellow marble is written $P(\text{yellow})$. There are 5 ways to select a yellow marble from the bag, and 6 + 4 or 10 ways not to select a yellow marble. So, $s = 5$ and $f = 10$.

$P(\text{yellow}) = \frac{5}{5 + 10}$ or $\frac{1}{3}$ *$P(s) = \frac{s}{s + f}$*

The probability of selecting a yellow marble is $\frac{1}{3}$

b. There are 4 ways to select a white marble. So there are 11 ways not to select a white marble.

$P(\text{not white}) = \frac{11}{4 + 11}$ or $\frac{11}{15}$

The probability of *not* selecting a white marble is $\frac{11}{15}$.

The counting methods you used for permutations and combinations are often used in determining probability.

Example 3 **A circuit board with 20 computer chips contains 4 chips that are defective. If 3 chips are selected at random, what is the probability that all 3 are defective?**

There are $C(4, 3)$ ways to select 3 out of 4 defective chips, and $C(20, 3)$ ways to select 3 out of 20 chips.

$P(\text{3 defective chips}) = \frac{C(4, 3)}{C(20, 3)}$ *← ways of selecting 3 defective chips* *← ways of selecting 3 chips*

$$= \frac{\frac{4!}{1!\,3!}}{\frac{20!}{17!\,3!}} \text{ or } \frac{1}{285}$$

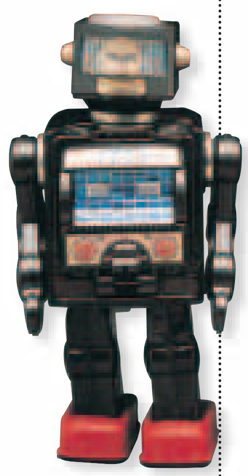
The probability of selecting three defective computer chips is $\frac{1}{285}$.

The sum of the probability of success and the probability of failure for any event is always equal to 1.

$$P(s) + P(f) = \frac{s}{s + f} + \frac{f}{s + f}$$
$$= \frac{s + f}{s + f} \text{ or } 1$$

This property is often used in finding the probability of events. For example, the probability of drawing a king of hearts is $P(s) = \frac{1}{52}$, so the probability of not drawing the king of hearts is $P(f) = 1 - \frac{1}{52}$ or $\frac{51}{52}$. Because their sum is 1, $P(s)$ and $P(f)$ are called **complements**.

Example 4 **The CyberToy Company has determined that out of a production run of 50 toys, 17 are defective. If 5 toys are chosen at random, what is the probability that at least 1 is defective?**



The complement of selecting at least 1 defective toy is selecting no defective toys. That is, P(at least 1 defective toy) $= 1 - P$(no defective toys).

$$\begin{aligned} P(\text{at least 1 defective toy}) &= 1 - P(\text{no defective toys}). \\ &= 1 - \frac{C(33, 5)}{C(50, 5)} \quad \begin{matrix} \leftarrow \textit{ways of selecting 5 defective toys} \\ \leftarrow \textit{ways of selecting 5 toys} \end{matrix} \\ &= 1 - \frac{237{,}336}{2{,}118{,}760} \\ &\approx 0.8879835375 \quad \textit{Use a calculator.} \end{aligned}$$

The probability of selecting at least 1 defective toy is about 89%.

Another way to measure the chance of an event occurring is with **odds.** The probability of success of an event and its complement are used when computing the odds of an event.

Odds

The odds of the successful outcome of an event is the ratio of the probability of its success to the probability of its failure.

$$\text{Odds} = \frac{P(s)}{P(f)}$$

Example 5 **Katrina must select at random a chip from a box to determine which question she will receive in a mathematics contest. There are 6 blue and 4 red chips in the box. If she selects a blue chip, she will have to solve a trigonometry problem. If the chip is red, she will have to write a geometry proof.**

a. What is the probability that Katrina will draw a red chip?

b. What are the odds that Katrina will have to write a geometry proof?

a. The probability that Katrina will select a red chip is $\frac{4}{10}$ or $\frac{2}{5}$.

b. To find the odds that Katrina will have to write a geometry proof, you need to know the probability of a successful outcome and of a failing outcome.

Let s represent selecting a red chip and f represent not selecting a red chip.

$P(s) = \frac{2}{5}$ $\qquad$ $P(f) = 1 - \frac{2}{5}$ or $\frac{3}{5}$

Now find the odds.

$$\frac{P(s)}{P(f)} = \frac{\frac{2}{5}}{\frac{3}{5}} \text{ or } \frac{2}{3}$$

The odds that Katrina will choose a red chip and thus have to write a geometry proof is $\frac{2}{3}$. *The ratio $\frac{2}{3}$ is read "2 to 3."*

Sometimes when computing odds, you must find the sample space first. This can involve finding permutations and combinations.

Example 6 **Twelve male and 16 female students have been selected as equal qualifiers for 6 college scholarships. If the awarded recipients are to be chosen at random, what are the odds that 3 will be male and 3 will be female?**

First, determine the total number of possible groups.

$C(12, 3)$ *number of groups of 3 males*
$C(16, 3)$ *number of groups of 3 females*

Using the Basic Counting Principle we can find the number of possible groups of 3 males and 3 females.

$C(12, 3) \cdot C(16, 3) = \frac{12!}{9!\,3!} \cdot \frac{16!}{13!\,3!}$ or 123,200 possible groups

The total number of groups of 6 recipients out of the 28 who qualified is $C(28, 6)$ or 376,740. So, the number of groups that do not have 3 males and 3 females is 376,740 − 123,200 or 253,540.

Finally, determine the odds.

$P(s) = \frac{123{,}200}{376{,}740}$ $\quad$ $P(f) = \frac{253{,}540}{376{,}740}$

$$\text{odds} = \frac{\frac{123{,}200}{376{,}740}}{\frac{253{,}540}{376{,}740}} \text{ or } \frac{880}{1811}$$

Thus, the odds of selecting a group of 3 males and 3 females are $\frac{880}{1811}$ or close to $\frac{1}{2}$.

CHECK FOR UNDERSTANDING

Communicating Mathematics

Read and study the lesson to answer each question.

1. **Explain** how you would interpret $P(E) = \frac{1}{2}$.
2. Find two examples of the use of probability in newspapers or magazines. **Describe** how probability concepts are applied.
3. **Write** about the difference between the probability of the successful outcome of an event and the odds of the successful outcome of an event.
4. **You Decide** Mika has figured that his odds of winning the student council election are 3 to 2. Geraldo tells him that, based on those odds, the probability of his winning is 60%. Mika disagreed. Who is correct? Explain your answer.

Guided Practice

A box contains 3 tennis balls, 7 softballs, and 11 baseballs. One ball is chosen at random. Find each probability.

5. P(softball)
6. P(not a baseball)
7. P(golf ball)
8. In an office, there are 7 women and 4 men. If one person is randomly called on the phone, find the probability the person is a woman.

Of 7 kittens in a litter, 4 have stripes. Three kittens are picked at random. Find the odds of each event.

9. All three have stripes. **10.** Only 1 has stripes. **11.** One is not striped.

12. **Meteorology** A local weather forecast states that the probability of rain on Saturday is 80%. What are the odds that it will not rain Saturday? (*Hint*: Rewrite the percent as a fraction.)

EXERCISES

Practice

Using a standard deck of 52 cards, find each probability. *The face cards include kings, queens, and jacks.*

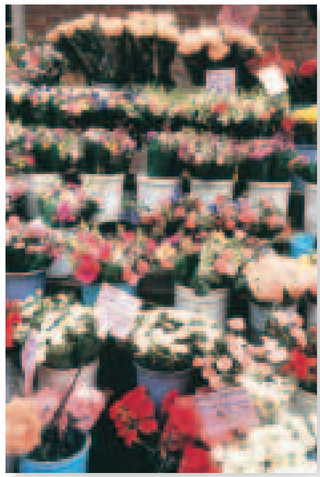
13. P(face card) **14.** P(a card of 6 or less)

15. P(a black, non-face card) **16.** P(not a face card)

One flower is randomly taken from a vase containing 5 red flowers, 2 white flowers, and 3 pink flowers. Find each probability.

17. P(red) **18.** P(white)

19. P(not pink) **20.** P(red or pink)



Jacob has 10 rap, 18 rock, 8 country, and 4 pop CDs in his music collection. Two are selected at random. Find each probability.

21. P(2 pop) **22.** P(2 country)

23. P(1 rap and 1 rock) **24.** P(not rock)

25. A number cube is thrown two times. What is the probability of rolling 2 fives?

A box contains 1 green, 2 yellow, and 3 red marbles. Two marbles are drawn at random without replacement. What are the odds of each event occurring?

26. drawing 2 red marbles **27.** not drawing yellow marbles

28. drawing 1 green and 1 red **29.** drawing two different colors

Of 27 students in a class, 11 have blue eyes, 13 have brown eyes, and 3 have green eyes. If 3 students are chosen at random what are the odds of each event occurring?

30. all three have blue eyes **31.** 2 have brown and 1 has blue eyes

32. no one has brown eyes **33.** only 1 has green eyes

34. The odds of winning a prize in a raffle with one raffle ticket are $\frac{1}{249}$. What is the probability of winning with one ticket?

35. The probability of being accepted to attend a state university is $\frac{4}{5}$. What are the odds of being accepted to this university?

36. From a deck of 52 cards, 5 cards are drawn. What are the odds of having three cards of one suit and the other two cards be another suit?

Applications and Problem Solving

37. **Weather** During a particular hurricane, hurricane trackers determine that the odds of it hitting the South Carolina coast are 1 to 4. What is the probability of this happening?

38. **Baseball** At one point in the 1999 season, Ken Griffey, Jr. had a batting average of 0.325. What are the odds that he would hit the ball the next time he came to bat?

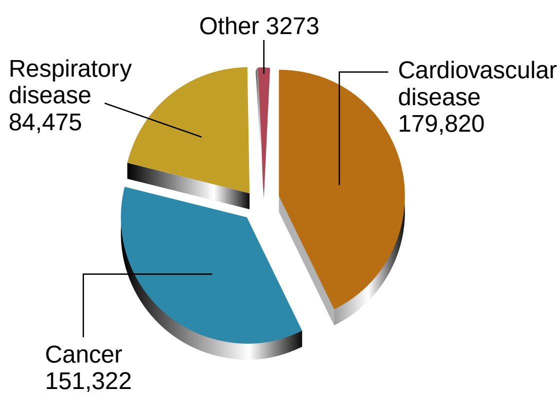
39. **Security** Kim uses a combination lock on her locker that has 3 wheels, each labeled with 10 digits from 0 to 9. The combination is a particular sequence with no digits repeating.
   a. What is the probability of someone guessing the correct combination?
   b. If the digits can be repeated, what are the odds against someone guessing the combination?

40. **Critical Thinking** Spencer is carrying out a survey of the bear population at Yellowstone National Park. He spots two bears—one has a light colored coat and the other has a dark coat.
   a. Assume that there are equal numbers of male and female bears in the park. What is the probability that both bears are male?
   b. If the lighter colored bear is male, what are the odds that both are male?

41. **Testing** Ms. Robinson gives her precalculus class 20 study problems. She will select 10 to answer on an upcoming test. Carl can solve 15 of the problems.
   a. Find the probability that Carl can solve all 10 problems on the test.
   b. Find the odds that Carl will know how to solve 8 of the problems.

42. **Mortality Rate** During 1990, smoking was linked to 418,890 deaths in the United States. The graph shows the diseases that caused these smoking-related deaths.
   a. Find the probability that a smoking-related death was the result of either cardiovascular disease or cancer.
   b. Determine the odds against a smoking-related death being caused by cancer.

Smoking-Related Deaths



43. **Critical Thinking** A plumber cuts a pipe in two pieces at a point selected at random. What is the probability that the length of the longer piece of pipe is at least 8 times the length of the shorter piece of pipe?

Mixed Review

44. A food vending machine has 6 different items on a revolving tray. How many different ways can the items be arranged on the tray? *(Lesson 13-2)*

45. The Foxtrail Condominium Association is electing board members. How many groups of 4 can be chosen from the 10 candidates who are running? *(Lesson 13-1)*

46. Find S_{14} for the arithmetic series for which $a_1 = 3.2$ and $d = 1.5$. *(Lesson 12-1)*

47. Simplify $7^{\log_7 2x}$. *(Lesson 11-4)*

48. **Landscaping** Carolina bought a new sprinkler to water her lawn. The sprinkler rotates 360° while spraying a stream of water. Carolina places the sprinkler in her yard so the ordered pair that represents its location is (7, 2), and the sprinkler sends out water that just barely reaches the point at (10, −8). Find an equation representing the farthestmost points the water can reach. *(Lesson 10-2)*

49. Find the product $3(\cos \pi + \boldsymbol{i} \sin \pi) \cdot 2\left(\cos \frac{\pi}{4} + \boldsymbol{i} \sin \frac{\pi}{4}\right)$. Then express it in rectangular form. *(Lesson 9-7)*

50. Find an ordered pair to represent $\vec{\mathbf{u}}$ if $\vec{\mathbf{u}} = \vec{\mathbf{v}} + \vec{\mathbf{w}}$, if $\vec{\mathbf{v}} = \langle 3, -5 \rangle$ and $\vec{\mathbf{w}} = \langle -4, 2 \rangle$. *(Lesson 8-2)*

51. **SAT Practice** What is the area of an equilateral triangle with sides $2s$ units long?
 - **A** s^2 units^2
 - **B** $\sqrt{3}s^2$ units^2
 - **C** $2s^2$ units^2
 - **D** $4s^2$ units^2
 - **E** $6s^2$ units^2

MID-CHAPTER QUIZ

Find each value. (Lesson 13-1)

1. $P(15, 5)$.
2. $C(20, 9)$.
3. Regular license plates in Ohio have three letters followed by four digits. How many different license plate arrangements are possible? (Lesson 13-1)
4. Suppose there are 12 runners competing in the finals of a track event. Awards are given to the top five finishers. How many top-five arrangements are possible? (Lesson 13-1)
5. An ice cream shop has 18 different flavors of ice cream, which can be ordered in a cup, sugar cone, or waffle cone. There is also a choice of six toppings. How many two-scoop servings with a topping are possible? (Lesson 13-1)
6. How many nine-letter patterns can be formed from the letters in the word *quadratic*? (Lesson 13-2)
7. How many different arrangements can be made with ten pieces of silverware laid in a row if three are identical spoons, four are identical forks, and three are identical knives? (Lesson 13-2)
8. Eight children are riding a merry-go-round. How many ways can they be seated? (Lesson 13-2)
9. Two cards are drawn at random from a standard deck of 52 cards. What is the probability that both are hearts? (Lesson 13-3)
10. A bowl contains four apples, three bananas, three oranges, and two pears. If two pieces of fruit are selected at random, what are the odds of selecting an orange and a banana? (Lesson 13-3)

Extra Practice See p. A51.

13-4

Probabilities of Compound Events

OBJECTIVES
- Find the probability of independent and dependent events.
- Identify mutually exclusive events.
- Find the probability of mutually exclusive and inclusive events.

Real World Application

TRANSPORTATION According to U.S. Department of Transportation statistics, the top ten airlines in the United States arrive on time 80% of the time. During their vacation, the Hiroshi family has direct flights to Washington, D.C., Chicago, Seattle, and San Francisco on different days. What is the probability that all their flights arrived on time?

Since the flights occur on different days, the four flights represent independent events. Let A represent an on-time arrival of an airplane.

$$P(\text{all flights on time}) = \underbrace{P(A)}_{\text{Flight 1}} \cdot \underbrace{P(A)}_{\text{Flight 2}} \cdot \underbrace{P(A)}_{\text{Flight 3}} \cdot \underbrace{P(A)}_{\text{Flight 4}}$$
$$= (0.80)^4 \quad A = 0.80$$
$$\approx 0.4096 \text{ or about } 41\%$$

Thus, the probability of all four flights arriving on time is about 41%.

This problem demonstrates that the probability of more than one independent event is the product of the probabilities of the events.

Probability of Two Independent Events

If two events, A and B, are independent, then the probability of both events occurring is the product of each individual probability.

$$P(A \text{ and } B) = P(A) \cdot P(B)$$

Example 1 **Using a standard deck of playing cards, find the probability of selecting a face card, replacing it in the deck, and then selecting an ace.**

Let A represent a face card for the first card drawn from the deck, and let B represent the ace in the second selection.

$$P(A) = \frac{12}{52} \text{ or } \frac{3}{13} \quad \frac{\text{12 face cards}}{\text{52 cards in a standard deck}}$$

$$P(B) = \frac{4}{52} \text{ or } \frac{1}{13} \quad \frac{\text{4 aces}}{\text{52 cards in a standard deck}}$$

The two draws are independent because when the card is returned to the deck, the outcome of the second draw is not affected by the first one.

$$P(A \text{ and } B) = P(A) \cdot P(B)$$
$$= \frac{3}{13} \cdot \frac{1}{13} \text{ or } \frac{3}{169}$$

The probability of selecting a face card first, replacing it, and then selecting an ace is $\frac{3}{169}$.

Example 2 **OCCUPATIONAL HEALTH** **Statistics collected in a particular coal-mining region show that the probability that a miner will develop black lung disease is $\frac{5}{11}$. Also, the probability that a miner will develop arthritis is $\frac{1}{5}$. If one health problem does not affect the other, what is the probability that a randomly-selected miner will not develop black lung disease but will develop arthritis?**

Real World Application

The events are independent since having black lung disease does not affect the existence of arthritis.

$$P(\text{not black lung disease and arthritis}) = [1 - P(\text{black lung disease})] \cdot P(\text{arthritis})$$
$$= \left(1 - \frac{5}{11}\right) \cdot \frac{1}{5} \text{ or } \frac{6}{55}$$

The probability that a randomly-selected miner will not develop black lung disease but will develop arthritis is $\frac{6}{55}$.

What do you think the probability of selecting two face cards would be if the first card drawn were not placed back in the deck? Unlike the situation in Example 1, these events are dependent because the outcome of the first event affects the second event. This probability is also calculated using the product of the probabilities.

first card — $P(\text{face card}) = \frac{12}{52}$

second card — $P(\text{face card}) = \frac{11}{51}$

$$P(\text{two face cards}) = \frac{12}{52} \cdot \frac{11}{51} \text{ or } \frac{11}{221}$$

Notice that when a face card is removed from the deck, not only is there one less face card, but also one less card in the deck.

Thus, the probability of selecting two face cards from a deck without replacing the cards is $\frac{11}{221}$ or about $\frac{1}{20}$.

Probability of Two Dependent Events

If two events, A and B, are dependent, then the probability of both events occurring is the product of each individual probability.

$$P(A \text{ and } B) = P(A) \cdot P(B \text{ following } A)$$

Example 3 **Tasha has 3 rock, 4 country, and 2 jazz CDs in her car. One day, before she starts driving, she pulls 2 CDs from her CD carrier without looking.**

a. Determine if the events are independent or dependent.

b. What is the probability that both CDs are rock?

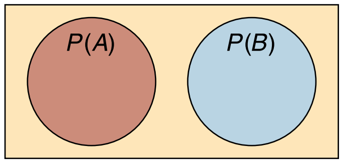
a. The events are dependent. This event is equivalent to selecting one CD, not replacing it, then selecting another CD.

b. Determine the probability.

$$P(\text{rock, rock}) = P(\text{rock}) \cdot P(\text{rock following first rock selection})$$
$$P(\text{rock, rock}) = \frac{3}{9} \cdot \frac{2}{8} \text{ or } \frac{1}{12}$$

The probability that Tasha will select two rock CDs is $\frac{1}{12}$.

There are times when two events cannot happen at the same time. For example, when tossing a number cube, what is the probability of tossing a 2 *or* a 5? In this situation, both events cannot happen at the same time. That is, the events are **mutually exclusive.** The probability of tossing a 2 or a 5 is $P(2) + P(5)$, which is $\frac{1}{6} + \frac{1}{6}$ or $\frac{2}{6}$.



Events A and B are mutually exclusive.

Note that the two events do not overlap, as shown in the Venn diagram. So, the probability of two mutually exclusive events occurring can be represented by the sum of the areas of the circles.

Probability of Mutually Exclusive Events

If two events, A and B, are mutually exclusive, then the probability that either A or B occurs is the sum of their probabilities.

$$P(A \text{ or } B) = P(A) + P(B)$$

Example 4

Lenard is a contestant in a game where if he selects a blue ball or a red ball he gets an all-expenses paid Caribbean cruise. Lenard must select the ball at random from a box containing 2 blue, 3 red, 9 yellow, and 10 green balls. What is the probability that he will win the cruise?

These are mutually exclusive events since Lenard cannot select a blue and a red ball at the same time. Find the sum of the individual probabilities.

$$P(\text{blue or red}) = P(\text{blue}) + P(\text{red})$$

$$= \frac{2}{24} + \frac{3}{24} \text{ or } \frac{5}{24} \quad P(\text{blue}) = \frac{2}{24},\ P(\text{red}) = \frac{3}{24}$$

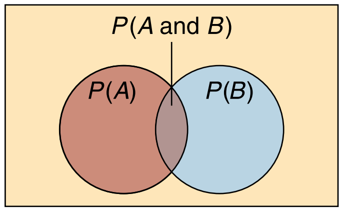
The probability that Lenard will win the cruise is $\frac{5}{24}$.

What is the probability of rolling two number cubes, in which the first number cube shows a 2 or the sum of the number cubes is 6 or 7? Since each number cube can land six different ways, and two number cubes are rolled, the sample space can be represented by making a chart. A **reduced sample space** is the subset of a sample space that contains only those outcomes that satisfy a given condition.

		Second Number Cube					
		1	2	3	4	5	6
First Number Cube	1	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
	2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
	3	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
	4	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
	5	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
	6	(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

It is possible to have the first number cube show a 2 *and* have the sum of the two number cubes be 6 or 7. Therefore, these events are not mutually exclusive. They are called **inclusive events.** In this case, you must adjust the formula for mutually exclusive events.

Note that the circles in the Venn diagram overlap. This area represents the probability of both events occurring at the same time. When the areas of the two circles are added, this overlapping area is counted twice. Therefore, it must be subtracted to find the correct probability of the two events.



Events A and B are inclusive events.

Let A represent the event "the first number cube shows a 2".

Let B represent the event "the sum of the two number cubes is 6 or 7".

$P(A) = \frac{6}{36}$ $\qquad$ $P(B) = \frac{11}{36}$

Note that (2, 4) and (2, 5) are counted twice, both as the first cube showing a 2 and as a sum of 6 or 7. To find the correct probability, you must subtract P(2 and sum of 6 or 7).

$$P(\text{2 or sum of 6 or 7}) = \underbrace{\frac{6}{36}}_{P(2)} + \underbrace{\frac{11}{36}}_{P(\text{sum of 6 or 7})} - \underbrace{\frac{2}{36}}_{P(\text{2 and sum of 6 or 7})} \text{ or } \frac{15}{36}$$

The probability of the first number cube showing a 2 or the sum of the number cubes being 6 or 7 is $\frac{15}{36}$ or $\frac{5}{12}$.

Probability of Inclusive Events

If two events, A and B, are inclusive, then the probability that either A or B occurs is the sum of their probabilities decreased by the probability of both occurring.

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

Examples

5 Kerry has read that the probability for a driver's license applicant to pass the road test the first time is $\frac{5}{6}$. He has also read that the probability of passing the written examination on the first attempt is $\frac{9}{10}$. The probability of passing both the road and written examinations on the first attempt is $\frac{4}{5}$.

a. Determine if the events are mutually exclusive or mutually inclusive.

Since it is possible to pass both the road examination and the written examination, these events are mutually inclusive.

b. What is the probability that Kerry can pass either examination on his first attempt?

$P(\text{passing road exam}) = \frac{5}{6}$ $\qquad$ $P(\text{passing written exam}) = \frac{9}{10}$

$P(\text{passing both exams}) = \frac{4}{5}$

$P(\text{passing either examination}) = \frac{5}{6} + \frac{9}{10} - \frac{4}{5} = \frac{56}{60} \text{ or } \frac{14}{15}$

The probability that Kerry will pass either test on his first attempt is $\frac{14}{15}$.

6 **There are 5 students and 4 teachers on the school publications committee. A group of 5 members is being selected at random to attend a workshop on school newspapers. What is the probability that the group attending the workshop will have at least 3 students?**

At least 3 students means the groups may have 3, 4, or 5 students. It is not possible to select a group of 3 students, a group of 4 students, and a group of 5 students in the same 5-member group. Thus, the events are mutually exclusive.

$$P(\text{at least 3 students}) = P(\text{3 students}) + P(\text{4 students}) + P(\text{5 students})$$

$$= \frac{C(5,3)\cdot C(4,2)}{C(9,5)} + \frac{C(5,4)\cdot C(4,1)}{C(9,5)} + \frac{C(5,5)\cdot C(4,0)}{C(9,5)}$$

$$= \frac{60}{126} + \frac{20}{126} + \frac{1}{126} \text{ or } \frac{9}{14}$$

The probability of at least 3 students going to the workshop is $\frac{9}{14}$.

CHECK FOR UNDERSTANDING

Communicating Mathematics

Read and study the lesson to answer each question.

1. **Describe** the difference between independent and dependent events.
2. a. **Draw** a Venn diagram to illustrate the event of selecting an ace or a diamond from a deck of cards.
   b. Are the events mutually exclusive? Explain why or why not.
   c. **Write** the formula you would use to determine the probability of these events.
3. *Math Journal* **Write** an example of two mutually exclusive events and two mutually inclusive events in your own life. **Explain** why the events are mutually exclusive or inclusive.

Guided Practice

Determine if each event is *independent* or *dependent.* Then determine the probability.

4. the probability of rolling a sum of 7 on the first toss of two number cubes and a sum of 4 on the second toss
5. the probability of randomly selecting two navy socks from a drawer that contains 6 black and 4 navy socks
6. There are 2 bottles of fruit juice and 4 bottles of sports drink in a cooler. Without looking, Desiree chose a bottle for herself and then one for a friend. What is the probability of choosing 2 bottles of the sports drink?

Determine if each event is *mutually exclusive* or *mutually inclusive.* Then determine each probability.

7. the probability of choosing a penny or a dime from 4 pennies, 3 nickels, and 6 dimes
8. the probability of selecting a boy or a blonde-haired person from 12 girls, 5 of whom have blonde hair, and 15 boys, 6 of whom have blonde hair
9. the probability of drawing a king or queen from a standard deck of cards

In a bingo game, balls numbered 1 to 75 are placed in a bin. Balls are randomly drawn and not replaced. Find each probability for the first 5 balls drawn.

10. P(selecting 5 even numbers)

11. P(selecting 5 two digit numbers)

12. P(5 odd numbers or 5 multiples of 4)

13. P(5 even numbers or 5 numbers less than 30)

14. **Business** A furniture importer has ordered 100 grandfather clocks from an overseas manufacturer. Four clocks are damaged in shipment, but the packaging shows no signs of damage. If a dealer buys 6 of the clocks without examining them first, what is the probability that none of the 6 clocks is damaged?

15. **Sports** A baseball team's pitching staff has 5 left-handed and 8 right-handed pitchers. If 2 pitchers are randomly chosen to warm up, what is the probability that at least one of them is right-handed? (*Hint:* Consider the order when selecting one right-handed and one left-handed pitcher.)

EXERCISES

Practice

Determine if each event is *independent* or *dependent.* Then determine the probability.

16. the probability of selecting a blue marble, not replacing it, then a yellow marble from a box of 5 blue marbles and 4 yellow marbles

17. the probability of randomly selecting two oranges from a bowl of 5 oranges and 4 tangerines, if the first selection is replaced

18. A green number cube and a red number cube are tossed. What is the probability that a 4 is shown on the green number cube and a 5 is shown on the red number cube?

19. the probability of randomly taking 2 blue notebooks from a shelf which has 4 blue and 3 black notebooks

20. A bank contains 4 nickels, 4 dimes, and 7 quarters. Three coins are removed in sequence, without replacement. What is the probability of selecting a nickel, a dime, and a quarter in that order?

21. the probability of removing 13 cards from a standard deck of cards and have all of them be red

22. the probability of randomly selecting a knife, a fork, and a spoon in that order from a kitchen drawer containing 8 spoons, 8 forks, and 12 table knives

23. the probability of selecting three different-colored crayons from a box containing 5 red, 4 black, and 7 blue crayons, if each crayon is replaced

24. the probability that a football team will win its next four games if the odds of winning each game are 4 to 3

For Exercises 25-33, determine if each event is *mutually exclusive* or *mutually inclusive.* Then determine each probability.

25. the probability of tossing two number cubes and either one shows a 4

26. the probability of selecting an ace or a red card from a standard deck of cards

27. the probability that if a card is drawn from a standard deck it is red or a face card

28. the probability of randomly picking 5 puppies of which at least 3 are male puppies, from a group of 5 male puppies and 4 female puppies.

29. the probability of two number cubes being tossed and showing a sum of 6 or a sum of 9.

30. the probability that a group of 6 people selected at random from 7 men and 7 women will have at least 3 women

31. the probability of at least 4 tails facing up when 6 coins are dropped on the floor

32. the probability that two cards drawn from a standard deck will both be aces or both will be black

33. from a collection of 6 rock and 5 rap CDs, the probability that at least 2 are rock from 3 randomly selected

Find the probability of each event using a standard deck of cards.

34. P(all red cards) if 5 cards are drawn without replacement

35. P(both kings or both aces) if 2 cards are drawn without replacement

36. P(all diamonds) if 10 cards are selected with replacement

37. P(both red or both queens) if 2 cards are drawn without replacement

There are 5 pennies, 7 nickels, and 9 dimes in an antique coin collection. If two coins are selected at random and the coins are not replaced, find each probability.

38. P(2 pennies)

39. P(2 nickels or 2 silver-colored coins)

40. P(at least 1 nickel)

41. P(2 dimes or 1 penny and 1 nickel)

There are 5 male and 5 female students in the executive council of the Douglas High School honor society. A committee of 4 members is to be selected at random to attend a conference. Find the probability of each group being selected.

42. P(all female)

43. P(all female or all male)

44. P(at least 3 females)

45. P(at least 2 females and at least 1 male)

Applications and Problem Solving

Real World Application

46. Computers A survey of the members of the Piper High School Computer Club shows that $\frac{2}{5}$ of the students who have home computers use them for word processing, $\frac{1}{3}$ use them for playing games, and $\frac{1}{4}$ use them for both word processing and playing games. What is the probability that a student with a home computer uses it for word processing or playing games?

47. Weather A weather forecaster states that the probability of rain is $\frac{3}{5}$, the probability of lightning is $\frac{2}{5}$, and the probability of both is $\frac{1}{5}$. What is the probability that a baseball game will be cancelled due to rain or lightning?

48. Critical Thinking Felicia and Martin are playing a game where the number cards from a single suit are selected. From this group, three cards are then chosen at random. What is the probability that the sum of the value of the cards will be an even number?

49. City Planning There are six women and seven men on a committee for city services improvement. A subcommittee of five members is being selected at random to study the feasibility of modernizing the water treatment facility. What is the probability that the committee will have at least three women?

50. Medicine A study of two doctors finds that the probability of one doctor correctly diagnosing a medical condition is $\frac{93}{100}$ and the probability the second doctor will correctly diagnose a medical condition is $\frac{97}{100}$. What is the probability that at least one of the doctors will make a correct diagnosis?

51. Disaster Relief During the 1999 hurricane season, Hurricanes Dennis, Floyd, and Irene caused extensive flooding and damage in North Carolina. After a relief effort, 2500 people in one supporting community were surveyed to determine if they donated supplies or money. Of the sample, 812 people said they donated supplies and 625 said they donated money. Of these people, 375 people said they donated both. If a member of this community were selected at random, what is the probability that this person donated supplies or money?

52. Critical Thinking If events A and B are inclusive, then $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$.

a. Draw a Venn diagram to represent $P(A \text{ or } B \text{ or } C)$.

b. Write a formula to find $P(A \text{ or } B \text{ or } C)$.

53. Product Distribution Ms. Kameko is the shipping manager of an Internet-based audio and video store. Over the past few months, she has determined the following probabilities for items customers might order.

Item	Probability
Action video	$\frac{4}{7}$
Pop/rock CD	$\frac{1}{2}$
Romance DVD	$\frac{5}{11}$
Action video and pop/rock CD	$\frac{2}{9}$
Pop/rock CD and romance DVD	$\frac{1}{7}$
Action video and romance DVD	$\frac{1}{4}$
Action video, pop/rock CD, and romance DVD	$\frac{1}{44}$

What is the probability, rounded to the nearest hundredth, that a customer will order an action video, pop/rock CD, or a romance DVD?

54. Critical Thinking There are 18 students in a classroom. The students are surveyed to determine their birthday (month and day only). Assume that 366 birthdays are possible.

a. What is the probability of any two students in the classroom having the same birthday?

b. Write an inequality that can be used to determine the probability of any two students having the same birthday to be greater than $\frac{1}{2}$.

c. Are there enough students in the classroom to have the probability in part **a** be greater than $\frac{1}{2}$? If not, at least how many more students would there need to be?

55. **Automotive Repairs** An auto club's emergency service has determined that when club members call to report that their cars will not start, the probability that the engine is flooded is $\frac{1}{2}$, the probability that the battery is dead is $\frac{2}{5}$, and the probability that both the engine is flooded and the battery is dead is $\frac{1}{10}$.

a. Are the events mutually exclusive or mutually inclusive?

b. Draw a Venn Diagram to represent the events.

c. What is the probability that the next member to report that a car will not start has a flooded engine or a dead battery?

Mixed Review

56. Two number cubes are tossed and their sum is 6. Find the probability that each cube shows a 3. *(Lesson 13-3)*

57. How many ways can 7 people be seated around a table? *(Lesson 13-2)*

58. **Sports** Ryan plays basketball every weekend. He averages 12 baskets per game out of 20 attempts. He has decided to try to make 15 baskets out of 20 attempts in today's game. How many ways can Ryan make 15 out of 20 baskets? *(Lesson 12-6)*

59. **Ecology** An underground storage container is leaking a toxic chemical. One year after the leak began, the chemical has spread 1200 meters from its source. After two years, the chemical has spread 480 meters more, and by the end of the third year it has reached an additional 192 meters. If this pattern continues, will the spill reach a well dug 2300 meters away? *(Lesson 12-4)*

60. Solve $12^{x+2} = 3^{x-4}$. *(Lesson 11-5)*

61. **Entertainment** A theater has been staging children's plays during the summer. The average attendance at each performance is 400 people and the cost of a ticket is $3. Next summer, they would like to increase the cost of the tickets, while maximizing their profits. The director estimates that for every $1 increase in ticket price, the attendance at each performance will decrease by 20. What price should the director propose to maximize their income, and what maximum income might be expected? *(Lesson 10-5)*

62. **Geology** A drumlin is an elliptical streamlined hill whose shape can be expressed by the equation $r = \ell \cos k\theta$ for $-\frac{\pi}{2k} \leq \theta \leq \frac{\pi}{2k}$, where ℓ is the length of the drumlin and $k > 1$ is a parameter that is the ratio of the length to the width. Suppose the area of a drumlin is 8270 square yards and the formula for area is $A = \frac{\ell^2\pi}{4k}$. Find the length of a drumlin modeled by $r = \ell \cos 7\theta$. *(Lesson 9-3)*

63. Write a vector equation describing a line passing through $P(1, -5)$ and parallel to $\vec{\mathbf{v}} = \langle -2, -4 \rangle$. *(Lesson 8-6)*

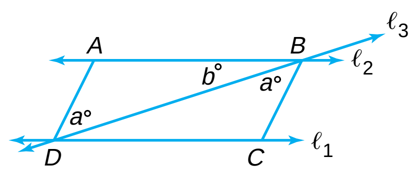
64. Solve $2 \tan x - 4 = 0$ for principal values of x. *(Lesson 7-5)*

65. **SAT/ACT Practice** If $a = 45$, which of the following statements must be true?

I. $\overline{AD} \parallel \overline{BC}$

II. ℓ_3 bisects $\angle ABC$.

III. $b = 45$



A None **B** I only

C I and II only **D** I and III only

E I, II, and III

Extra Practice See p. A52.

13-5

Conditional Probability

OBJECTIVE
- Find the probability of an event given the occurrence of another event.

Real World Application

MEDICINE Danielle Jones works in a medical research laboratory where a drug that promotes hair growth in balding men is being tested. The results of the preliminary tests are shown in the table.

	Number of Subjects	
	Using Drug	**Using Placebo**
Hair growth	1600	1200
No hair growth	800	400

Ms. Jones needs to find the probability that a subject's hair growth was a result of using the experimental drug. *This problem will be solved in Example 1.*

The probability of an event under the condition that some preceding event has occurred is called **conditional probability.** The conditional probability that event A occurs given that event B occurs can be represented by $P(A|B)$. *$P(A|B)$ is read "the probability of A given B."*

Conditional Probability

The conditional probability of event A, given event B, is defined as

$$P(A|B) = \frac{P(A \text{ and } B)}{P(B)} \text{ where } P(B) \neq 0.$$

Example 1 Real World Application

MEDICINE **Refer to the application above. What is the probability that a test subject's hair grew, given that he used the experimental drug?**

Let H represent hair growth and D represent experimental drug usage. We need to find $P(H|D)$.

$$P(H|D) = \frac{P(\text{used experimental drug and had hair growth})}{P(\text{used experimental drug})}$$

$$P(H|D) = \frac{\frac{1600}{4000}}{\frac{2400}{4000}}$$

← *P(used experimental drug and had hair growth)* $= \frac{1600}{4000}$

← *P(used experimental drug)* $= \frac{1600 + 800}{4000}$

$$P(H|D) = \frac{1600}{2400} \text{ or } \frac{2}{3}$$

The probability that a subject's hair grew, given that they used the experimental drug is $\frac{2}{3}$.

Example 2 **Denette tosses two coins. What is the probability that she has tossed 2 heads, given that she has tossed at least 1 head?**

Let event A be that the two coins come up heads.

Let event B be that there is at least one head.

$P(B) = \frac{3}{4}$ *Three of the four outcomes have at least one head.*

$P(A \text{ and } B) = \frac{1}{4}$ *One of the four outcomes has two heads.*

$$P(A|B) = \frac{P(A \text{ and } B)}{P(B)}$$

$$= \frac{\frac{1}{4}}{\frac{3}{4}}$$

$$= \frac{1}{4} \cdot \frac{4}{3} \text{ or } \frac{1}{3}$$

The probability of tossing two heads, given that at least one toss was a head is $\frac{1}{3}$.

Sample spaces and reduced sample spaces can be used to help determine the outcomes that satisfy a given condition.

Example 3 **Alfonso is conducting a survey of families with 3 children. If a family is selected at random, what is the probability that the family will have exactly 2 boys if the second child is a boy?**

The sample space is $S = \{BBB, BBG, BGB, BGG, GBB, GBG, GGB, GGG\}$ and includes all of the possible outcomes for a family with three children.

Determine the reduced sample spaces that satisfy the given conditions that there are exactly 2 boys and that the second child is a boy.

The condition that there are exactly 2 boys reduces the sample space to exclude the outcomes where there are 1, 3, or no boys.

Let X represent the event that there are two boys.

$X = \{BBG, BGB, GBB\}$

$P(X) = \frac{3}{8}$

The condition that the second child is a boy reduces the sample space to exclude the outcomes where the second child is a girl.

Let Y represent the event that the second child is a boy.

$Y = \{BBB, BBG, GBB, GBG\}$

$P(Y) = \frac{4}{8}$ or $\frac{1}{2}$

$(X \text{ and } Y)$ is the intersection of X and Y. $(X \text{ and } Y) = \{BBG, GBB\}$.

So, $P(X \text{ and } Y) = \frac{2}{8}$ or $\frac{1}{4}$.

(continued on the next page)

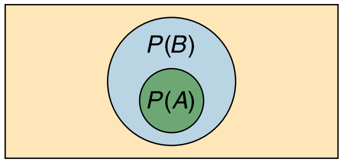
$$P(X \mid Y) = \frac{P(X \text{ and } Y)}{P(Y)}$$

$$= \frac{\frac{1}{4}}{\frac{1}{2}}$$

$$= \frac{1}{4} \cdot \frac{2}{1} \text{ or } \frac{1}{2}$$

The probability that a family with 3 children selected at random will have exactly 2 boys, given that the second child is a boy, is $\frac{1}{2}$.

In some situations, event A is a subset of event B. When this occurs, the probability that both event A and event B, $P(A \text{ and } B)$, occur is the same as the probability of event A occurring. Thus, in these situations $P(A \mid B) = \frac{P(A)}{P(B)}$.



Event A is a subset of event B.

Example 4 **A 12-sided dodecahedron has the numerals 1 through 12 on its faces. The die is rolled once, and the number on the top face is recorded. What is the probability that the number is a multiple of 4 if it is known that it is even?**

Let A represent the event that the number is a multiple of 4. Thus, $A = \{4, 8, 12\}$.

$$P(A) = \frac{3}{12} \text{ or } \frac{1}{4}$$

Let B represent the event that the number is even. So, $B = \{2, 4, 6, 8, 10, 12\}$.

$$P(B) = \frac{6}{12} \text{ or } \frac{1}{2}$$

In this situation, A is a subset of B.

$P(A \text{ and } B) = P(A) = \frac{1}{4}$ $\qquad$ $P(B) = \frac{1}{2}$

$$P(A \mid B) = \frac{P(A)}{P(B)}$$

$$= \frac{\frac{1}{4}}{\frac{1}{2}} \text{ or } \frac{1}{2}$$

The probability that a multiple of 4 is rolled, given that the number is even, is $\frac{1}{2}$.

Check for Understanding

Communicating Mathematics

Read and study the lesson to answer each question.

1. **Explain** the relationship between conditional probability and the probability of two independent events.

2. **Describe** the sample space for *P*(face card) if the card drawn is black.

3. *Math Journal* Find two real-world examples that use conditional probability. Explain how you know conditional probability is used.

Guided Practice

Find each probability.

4. Two number cubes are tossed. Find the probability that the numbers showing on the cubes match given that their sum is greater than five.

5. One card is drawn from a standard deck of cards. What is the probability that it is a queen if it is known to be a face card?

Three coins are tossed. Find the probability that they all land heads up for each known condition.

6. the first coin shows a head

7. at least one coin shows a head

8. at least two coins show heads

A pair of number cubes is thrown. Find each probability given that their sum is greater than or equal to 9.

9. *P*(numbers match)

10. *P*(sum is even)

11. *P*(numbers match or sum is even)

12. **Medicine** To test the effectiveness of a new vaccine, researchers gave 100 volunteers the conventional treatment and gave 100 other volunteers the new vaccine. The results are shown in the table below.

Treatment	Disease Prevented	Disease Not Prevented
New Vaccine	68	32
Conventional Treatment	62	38

a. What is the probability that the disease is prevented in a volunteer chosen at random?

b. What is the probability that the disease is prevented in a volunteer who was given the new vaccine?

c. What is the probability that the disease is prevented in a volunteer who was not given the new vaccine?

13. **Currency** A dollar-bill changer in a snack machine was tested with 100 \$1-bills. Twenty-five of the bills were counterfeit. The results of the test are shown in the chart at the right.

Bill	Accepted	Rejected
Legal	69	6
Counterfeit	1	24

a. What is the probability that a bill accepted by the changer is legal?

b. What is the probability that a bill is rejected given that it is legal?

c. What is the probability that a counterfeit bill is not rejected?

EXERCISES

Practice

Find each probability.

14. Two coins are tossed. What is the probability that one coin shows heads if it is known that at least one coin is tails?

15. A city council consists of six Democrats, two of whom are women, and six Republicans, four of whom are men. A member is chosen at random. If the member chosen is a man, what is the probability that he is a Democrat?

16. A bag contains 4 red chips and 4 blue chips. Another bag contains 2 red chips and 6 blue chips. A chip is randomly selected from one of the bags, and found to be blue. What is the probability that the chip is from the first bag?

17. Two boys and two girls are lined up at random. What is the probability that the girls are separated if a girl is at an end?

18. A five-digit number is formed from the digits 1, 2, 3, 4, and 5. What is the probability that the number ends in the digits 52, given that it is even?

19. Two game tiles, numbered 1 through 9, are selected at random from a box without replacement. If their sum is even, what is the probability that both numbers are odd?

A card is chosen at random from a standard deck of cards. Find each probability given that the card is black.

20. P(ace) **21.** P(4) **22.** P(face card)

23. P(queen of hearts) **24.** P(6 of clubs) **25.** P(jack or ten)

A container holds 3 green marbles and 5 yellow marbles. One marble is randomly drawn and discarded. Then a second marble is drawn. Find each probability.

26. the second marble is green, given that the first marble was green

27. the second marble is yellow, given that the first marble was green

28. the second marble is yellow, given that the first marble was yellow

Three fish are randomly removed from an aquarium that contains a trout, a bass, a perch, a catfish, a walleye, and a salmon. Find each probability.

29. P(salmon, given bass)

30. P(not walleye, given trout and perch)

31. P(bass and perch, given not catfish)

32. P(perch and trout, given neither bass nor walleye)

In Mr. Hewson's homeroom, 60% of the students have brown hair, 30% have brown eyes, and 10% have both brown hair and eyes. A student is excused early to go to a doctor's appointment.

33. If the student has brown hair, what is the probability that the student also has brown eyes?

34. If the student has brown eyes, what is the probability that the student does not have brown hair?

35. If the student does not have brown hair, what is the probability that the student does not have brown eyes?

In a game played with a standard deck of cards, each face card has a value of 10 points, each ace has a value of 1 point, and each number card has a value equal to its number. Two cards are drawn at random.

36. At least one card is an ace. What is the probability that the sum of the cards is 7 or less?

37. One card is the queen of diamonds. What is the probability that the sum of the cards is greater than 18?

Applications and Problem Solving

Real World Application

38. Health Care At Park Medical Center, in a sample group, there are 40 patients diagnosed with lung cancer, and 30 patients who are chronic smokers. Of these, there are 25 patients who have lung cancer and smoke.

a. Draw a Venn diagram to represent the situation.

b. If the medical center currently has 200 patients, and one of them is randomly selected for a medical study, what is the probability that the patient has lung cancer, given that the patient smokes?

39. Business The manager of a computer software store wants to know whether people who come in and ask questions are more likely to make a purchase than the average person. A survey of 500 people exiting the store found that 250 people bought something, 120 asked questions and bought something, and 30 people asked questions but did not buy anything. Based on the survey, determine whether a person who asks questions is more likely to buy something than the average person.

40. Critical Thinking In a game using two number cubes, a sum of 10 has not turned up in the past few rolls. A player believes that a roll of 10 is "due" to come up. Analyze the player's thinking.

41. Testing Winona's chances of passing a precalculus exam are $\frac{4}{5}$ if she studies, and only $\frac{1}{5}$ if she decides to take it easy. She knows that $\frac{2}{3}$ of her class studied for and passed the exam. What is the probability that Winona studied for it?

42. Manufacturing Three computer chip companies manufacture a product that enhances the 3-D graphic capacities of computer displays. The table below shows the number of functioning and defective chips produced by each company during one day's manufacturing cycle.

Company	Number of functioning chips	Number of defective chips
CyberChip Corp.	475	25
3-D Images, Inc.	279	21
MegaView Designs	180	20

a. What is the probability that a randomly selected chip is defective?

b. What is the probability that a defective chip came from 3-D Images, Inc.?

c. What is the probability that a randomly selected chip is functioning?

d. If you were a computer manufacturer, which company would you select to produce the most reliable graphic chip? Why?

43. **Critical Thinking** The probability of an event A is equal to the probability of the same event, given that event B has already occurred. Prove that A and B are independent events.

Mixed Review

44. **City Planning** There are 6 women and 7 men on the committee for city park enhancement. A subcommittee of five members is being selected at random to study the feasibility of redoing the landscaping in one of the parks. What is the probability that the committee will have at least three women? *(Lesson 13-4)*

45. Suppose there are 9 points on a circle. How many 4-sided closed figures can be formed by joining any 4 of these points? *(Lesson 13-1)*

46. Write $\sum_{b=1}^{\infty} 3(0.5)^b$ in expanded form. Then find the sum. *(Lesson 12-5)*

47. Compare and contrast the graphs of $y = 3^x$ and $y = -3^x$ *(Lesson 11-2)*

48. Graph the system of inequalities. *(Lesson 10-8)*

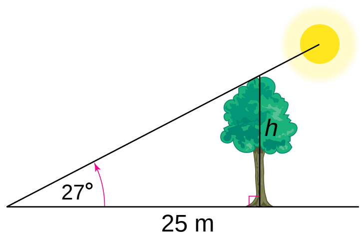
$$x^2 + y^2 \leq 81$$
$$x^2 + y^2 \geq 64$$

49. **Navigation** A submarine sonar is tracking a ship. The path of the ship is recorded as $r \cos\left(\theta - \frac{\pi}{2}\right) + 5 = 0$. Find the linear equation of the path of the ship. *(Lesson 9-4)*

50. Graph the line whose parametric equations are $x = 4t$, and $y = 3 + 2t$. *(Lesson 8-6)*

51. Find the area of the sector of a circle of radius 8 feet, given its central angle is 98°. Round your answer to the nearest tenth. *(Lesson 6-1)*

52. When the angle of elevation of the sun is 27°, the shadow of a tree is 25 meters long. How tall is the tree? Round your answer to the nearest tenth. *(Lesson 5-4)*



53. **Photography** A photographer has a frame that is 3 feet by 4 feet. She wants to mat a group photo such that there is a uniform width of mat surrounding the photo. If the area of the photo is 6 square feet, find the width of the mat. *(Lesson 4-2)*

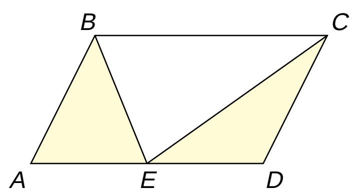
54. Find the value(s) of x at which $f(x) = \frac{5}{x^2 - 4}$ is discontinuous. Use the continuity test to justify your answer. *(Lesson 3-5)*

55. **SAT/ACT Practice** In parallelogram $ABCD$, the ratio of the shaded area to the unshaded area is

A 1:2 **B** 1:1

C 4:3 **D** 2:1

E It cannot be determined from the information given.



Extra Practice See p. A52.

CONTENTS

13-6

The Binomial Theorem and Probability

OBJECTIVE
- Find the probability of an event by using the Binomial Theorem.

Real World Application

LANDSCAPING Managers at the Eco-Landscaping Company know that a mahogany tree they plant has a survival rate of about 90% if cared for properly. If 10 trees are planted in the last phase of a landscaping project, what is the probability that 7 of the trees will survive? *This problem will be solved in Example 3.*

We can examine a simpler form of this problem. Suppose that there are only 5 trees to be planted. What is the probability that 4 will survive? The number of ways that this can happen is $C(5, 4)$ or 5.

Let S represent the probability of a tree surviving.
Let D represent the probability of a tree dying.

Look Back
Refer to Lesson 12-6 to review binomial expansions and the Binomial Theorem.

Since this situation has two outcomes, we can represent it using the binomial expansion of $(S + D)^5$. The terms of the expansion can be used to find the probabilities of each combination of the survival and death of the trees.

$$(S + D)^5 = 1S^5 + 5S^4D + 10S^3D^2 + 10S^2D^3 + 5SD^4 + 1D^5$$

coefficient	term	meaning
$C(5, 5) = 1$	$1S^5$	1 way to have all 5 trees survive
$C(5, 4) = 5$	$5S^4D$	5 ways to have 4 trees survive and 1 die
$C(5, 3) = 10$	$10S^3D^2$	10 ways to have 3 trees survive and 2 die
$C(5, 2) = 10$	$10S^2D^3$	10 ways to have 2 trees survive and 3 die
$C(5, 1) = 5$	$5SD^4$	5 ways to have 1 tree survive and 4 die
$C(5, 0) = 1$	$1D^5$	1 way to have all 5 trees die

The probability of a tree surviving is 0.9. So, the probability of a tree not surviving is $1 - 0.9$ or 0.1. The probability of having 4 trees survive out of 5 can be determined as follows.

Use $5S^4D$ since this term represents 4 trees surviving and 1 tree dying.

$5S^4D = 5(0.9)^4(0.1)$ *Substitute 0.9 for S and 0.1 for D*
$5S^4D = 5(0.6561)0.1$
$5S^4D = 0.3281$ or about $\frac{1}{3}$

Thus, the probability of having 4 trees survive is about $\frac{1}{3}$.

Other probabilities can be determined from the expansion of $(S + D)^5$. For example, what is the probability that at least 2 trees out of the 5 trees planted will die?

Example 1 Real World Application

LANDSCAPING **Refer to the application at the beginning of the lesson. Five mahogany trees are planted. What is the probability that at least 2 trees die?**

The third, forth, fifth, and sixth terms represent the conditions that two or more trees die. So, the probability of this happening is the sum of the probabilities of those terms.

$$\begin{aligned} P(\text{at least 2 trees die}) & \\ &= 10S^3D^2 + 10S^2D^3 + 5SD^4 + 1D^5 \\ &= 10(0.9)^3(0.1)^2 + 10(0.9)^2(0.1)^3 + 5(0.9)(0.1)^4 + (0.1)^5 \\ &= 10(0.729)(0.01) + 10(0.81)(0.001) + 5(0.9)(0.0001) + (0.00001) \\ &= 0.0729 + 0.0081 + 0.00045 + 0.00001 \\ &= 0.0815 \end{aligned}$$

The probability that at least 2 trees die is about 8%.

Problems that can be solved using the binomial expansion are called **binomial experiments.**

Conditions of a Binomial Experiment

A binomial experiment exists if and only if these conditions occur.

- Each trial has exactly two outcomes, or outcomes that can be reduced to two outcomes.
- There must be a fixed number of trials.
- The outcomes of each trial must be independent.
- The probabilities in each trial are the same.

Example 2 **Eight out of every 10 persons who contract a certain viral infection can recover. If a group of 7 people become infected, what is the probability that exactly 3 people will recover from the infection?**

There are 7 people involved, and there are only 2 possible outcomes, recovery R or not recovery N. These events are independent, so this is a binomial experiment.

When $(R + N)^7$ is expanded, the term R^3N^4 represents 3 people recovering and 4 people not recovering from the infection. The coefficient of R^3N^4 is $C(7, 3)$ or 35.

$$\begin{aligned} P(\text{exactly 3 people recovering}) &= 35(0.8)^3(0.2)^4 && R = 0.8,\ N = 1 - 0.8 \text{ or } 0.2 \\ &= 35(0.512)(0.0016) \\ &= 0.028672 \end{aligned}$$

The probability that exactly 3 of the 7 people will recover from the infection is 2.9%.

The Binomial Theorem can be used to find the probability when the number of trials makes working with the binomial expansion unrealistic.

Example 3 Real World Application

LANDSCAPING **Refer to the application at the beginning of the lesson. What is the probability that 7 of the 10 trees planted will survive?**

Let S be the probability that a tree will survive.
Let D be the probability that a tree will die.

Since there are 10 trees, we can use the Binomial Theorem to find any term in the expression $(S + D)^{10}$.

$$(S + D)^{10} = \sum_{r=0}^{10} \frac{10!}{r!(10 - r)!} S^{10-r}D^{r}$$

Look Back
Refer to Lesson 12-5 to review sigma notation.

Having 7 trees survive means that 3 will die. So the probability can be found using the term where $r = 3$, the fourth term.

$$\begin{aligned}\frac{10!}{3!\,(10 - 3)!} S^{7}D^{3} &= 120S^{7}D^{3}\\ &= 120(0.9)^{7}(0.1)^{3}\\ &= 120(0.4782969)(0.001) \text{ or } 0.057395628\end{aligned}$$

The probability of exactly 7 trees surviving is about 5.7%.

So far, the probabilities we have found have been **theoretical probabilities.** These are determined using mathematical methods and provide an idea of what to expect in a given situation. **Experimental probability** is determined by performing experiments and observing and interpreting the outcomes. One method for finding experimental probability is a **simulation.** In a simulation, a device such as a graphing calculator is used to model the event.

GRAPHING CALCULATOR EXPLORATION

You can use a graphing calculator to simulate a binomial experiment. Consider the following situation.

Robby wins 2 out of every 3 chess matches he plays with Marlene. What is the probability that he wins exactly 5 of the next 6 matches?

TRY THIS

To simulate this situation, enter **int(3*rand)** and press ENTER. Note: **int(** and **rand** can be found in the menus accessed by pressing MATH. This will randomly generate the numbers 0, 1, or 2. Robby wins if the outcome is 0 or 1. Robby loses if 2 comes up.

In the simulation, one repetition of the complete binomial experiment consists of six trials or six presses of the ENTER key. Try 40 repetitions.

WHAT DO YOU THINK?

1. What is the sample space?
2. What is P(Robby wins)?
3. In the simulation, with what probability did Robby win exactly 5 times?
4. Using the formula for computing binomial probabilities, what is the probability of Robby winning exactly five games?
5. Why do you think there is a difference between the simulation (experimental probability) and the probability computed using the formula (theoretical probability)?
6. What would you do to have the experimental probability approximate the theoretical probability?

Check for Understanding

Communicating Mathematics

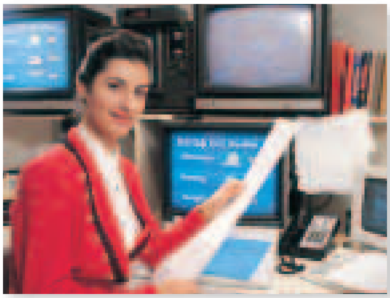
Read and study the lesson to answer each question.

1. **Explain** whether or not each situation represents a binomial experiment.
   a. the probability of winning in a game where a number cube is tossed, and if 1, 2, or 3 comes up you win.
   b. the probability of drawing two red marbles from a jar containing 10 red, 30 blue, and 5 yellow marbles.
   c. the probability of drawing a jack from a standard deck of cards, knowing that the card is red.
2. **Write** an explanation of experimental probability. Give a real-world example that uses experimental probability.
3. **Describe** how to find the probability of getting exactly 2 correct answers on a true/false quiz that has 5 questions.

Guided Practice

Find each probability if a number cube is tossed five times.

4. *P*(only one 4)
5. *P*(no more than two 4s)
6. *P*(at least three 4s)
7. *P*(exactly five 4s)



Jasmine Myers, a weather reporter for Channel 6, is forecasting a 30% chance of rain for today and the next four days. Find each probability.

8. *P*(not having rain on any day)
9. *P*(having rain on exactly one day)
10. *P*(having rain no more than three days)

11. **Cooking** In cooking class, 1 out of 5 soufflés that Sabrina makes will collapse. She is preparing 6 soufflés to serve at a party for her parents. What is the probability that exactly 4 of them do not collapse?
12. **Finance** A stock broker is researching 13 independent stocks. An investment in each will either make or lose money. The probability that each stock will make money is $\frac{5}{8}$. What is the probability that exactly 10 of the stocks will make money?

Exercises

Practice

Isabelle carries lipstick tubes in a bag in her purse. The probability of pulling out the color she wants is $\frac{2}{3}$. Suppose she uses her lipstick 4 times in a day. Find each probability.

13. *P*(never the correct color)
14. *P*(correct at least 3 times)
15. *P*(no more than 3 times correct)
16. *P*(correct exactly 2 times)

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Maura guesses at all 10 questions on a true/false test. Find each probability.

17. P(7 correct)

18. P(at least 6 correct)

19. P(all correct)

20. P(at least half correct)

The probability of tossing a head on a bent coin is $\frac{1}{3}$. Find each probability if the coin is tossed 4 times.

21. P(4 heads)

22. P(3 heads)

23. P(at least 2 heads)

Kyle guesses at all of the 10 questions on his multiple choice test. Find each probability if each question has 4 choices.

24. P(6 correct answers)

25. P(half answers correct)

26. P(from 3 to 5 correct answers)

If a thumbtack is dropped, the probability of its landing point up is $\frac{2}{5}$. Mrs. Davenport drops 10 tacks while putting up the weekly assignment sheet on the bulletin board. Find each probability.

27. P(all point up)

28. P(exactly 3 point up)

29. P(exactly 5 point up)

30. P(at least 6 point up)

Find each probability if three coins are tossed.

31. P(3 heads or 3 tails)

32. P(at least 2 heads)

33. P(exactly 2 tails)

Graphing Calculator

34. Enter the expression **6 nCr X** into the **Y=** menu. The **nCr** command is found in the probability section of the **MATH** menu. Use the **TABLE** feature to observe the results.

a. How do these results compare with the expansion of $(a + b)^6$?

b. How would you change the expression to find the expansion of $(a + b)^8$?

35. Sports A football team is scheduled to play 16 games in its next season. If there is a 70% probability the team will win each game, what is the probability that the team will win at least 12 of its games? (*Hint*: Use the information from Exercise 34.)

Applications and Problem Solving

Real World Application

36. Military Science During the Gulf War in 1990–1991, SCUD missiles hit 20% of their targets. In one incident, six missiles were fired at a fuel storage installation.

a. Describe what success means in this case, and state the number of trials and the probability of success on each trial.

b. Find the probability that between 2 and 6 missiles hit the target.

37. Critical Thinking Door prizes are given at a party through a drawing. Four out of 10 tickets are given to men who will attend, and 6 out of 10 tickets are distributed to women. Each person will receive only one ticket. Ten tickets will be drawn at random with replacement. What is the probability that all winners will be the same sex?

38. Medicine Ten percent of African-Americans are carriers of the genetic disease sickle-cell anemia. Find each probability for a random sample of 20 African-Americans.

a. P(all carry the disease)

b. P(exactly half have the disease)

39. **Airlines** A commuter airline has found that 4% of the people making reservations for a flight will not show up. As a result, the airline decides to sell 75 seats on a plane that has 73 seats (overbooking). What is the probability that for every person who shows up for the flight there will be a seat available?

40. **Sales** Luis is an insurance agent. On average, he sells 1 policy for every 2 prospective clients he meets. On a particular day, he calls on 4 clients. He knows that he will not receive a bonus if the sales are less than or equal to three policies. What is the probability that he will not get a bonus?

41. **Critical Thinking** Trina is waiting for her friend who is late. To pass the time, she takes a walk using the following rules. She tosses a fair coin. If it falls heads, she walks 10 meters north. If it falls tails, she walks 10 meters south. She repeats this process every 10 meters and thus executes what is called a random walk. What is the probability that after 100 meters of walking she will be at one of the following points?
   a. *P*(back at her starting point)
   b. *P*(within 10 meters of the starting point)
   c. *P*(exactly 20 meters from the starting point)

Mixed Review

42. A pair of number cubes is thrown. Find the probability that their sum is less than 9 if both cubes show the same number. *(Lesson 13-5)*

43. A letter is picked at random from the alphabet. Find the probability that the letter is contained in the word *house* or in the word *phone*. *(Lesson 13-4)*

44. **Physical Science** Dry air expands as it moves upward into the atmosphere. For each 1000 feet that it moves upward, the air cools 5° F. Suppose the temperature at ground level is 80° F. *(Lesson 12-1)*
   a. Write a sequence representing the temperature decrease per 1000 feet.
   b. If n is the height of the air in thousands of feet, write a formula for the temperature T in terms of n.
   c. What is the ground level temperature if the air at 40,000 feet is $-125°$?

45. Solve $3^{x-1} = 6^{-x}$ using logarithms. Round to the nearest hundredth. *(Lesson 11-6)*

46. Name the coordinates of the center, foci, and vertices of the ellipse with the equation $\frac{x^2}{49} + \frac{(y+3)^2}{25} = 1$. *(Lesson 10-3)*

47. Express $\sqrt{2}\left(\cos -\frac{\pi}{2} + \boldsymbol{i} \sin -\frac{\pi}{2}\right)$ in rectangular form. *(Lesson 9-6)*

48. Find the ordered pair that represents $\overrightarrow{WX}$ if $W(8, -3)$ and $X(6, 5)$. Then find the magnitude of $\overrightarrow{WX}$. *(Lesson 8-2)*

49. **Geometry** The sides of a parallelogram are 55 cm and 71 cm long. Find the length of each diagonal if the larger angle measures 106°. *(Lesson 5-8)*

50. Use the Remainder Theorem to find the remainder when $x^4 + 12x^3 + 21x^2 - 62x - 72$ is divided by $x + 4$. State whether the binomial is a factor of the polynomial. *(Lesson 4-3)*

51. **SAT Practice** **Grid-In** A word processor uses a sheet of paper that is 9 inches wide by 12 inches long. It leaves a 1-inch margin on each side and a 1.5-inch margin on the top and bottom. What fraction of the page is used for text?

Extra Practice See p. A52.

STUDY GUIDE AND ASSESSMENT

VOCABULARY

Basic Counting Principle (p. 837)
binomial experiments (p. 876)
circular permutation (p. 847)
combination (p. 841)
combinatorics (p. 837)
complements (p. 853)
conditional probability (p. 868)
dependent event (p. 837)
experimental probability (p. 877)
failure (p. 852)
inclusive event (p. 863)
independent event (p. 837)
mutually exclusive (p. 862)
odds (p. 854)
permutation (p. 838)
permutation with repetition (p. 846)
probability (p. 852)
reduced sample space (p. 862)
sample space (p. 852)
simulation (p. 877)
success (p. 852)
theoretical probability (p. 877)
tree diagram (p. 837)

UNDERSTANDING AND USING THE VOCABULARY

Choose the correct term to best complete each sentence.

1. Events that do not affect each other are called (dependent, independent) events.
2. In probability, any outcome other than the desired outcome is called a (failure, success).
3. The sum of the probability of an event and the probability of the complement of the event is always (0, 1).
4. The (odds, probability) of an event occurring is the ratio of the number of ways the event can succeed to the sum of the number of ways the event can succeed and the number of ways the event can fail.
5. The arrangement of objects in a certain order is called a (combination, permutation).
6. A (permutation with repetitions, circular permutation) specifically deals with situations in which some objects that are alike.
7. Two (inclusive, mutually exclusive) events cannot happen at the same time.
8. A (sample space, Venn diagram) is the set of all possible outcomes of an event.
9. The probability of an event A given that event B has occurred is called a (conditional, inclusive) probability.
10. The branch of mathematics that studies different possibilities for the arrangement of objects is called (statistics, combinatorics).

interNET CONNECTION For additional review and practice for each lesson, visit: **www.amc.glencoe.com**

SKILLS AND CONCEPTS

OBJECTIVES AND EXAMPLES

Lesson 13-1 Solve problems related to the Basic Counting Principle.

- How many possible ways can a group of eight students line up to buy tickets to a play?

  There are eight choices for the first spot in line, seven choices for the second spot, six for the third spot, and so on.

  $8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 40{,}320$

  There are 40,320 ways for the students to line up.

REVIEW EXERCISES

11. How many different ways can three books be arranged in a row on a shelf?

12. How many different ways can the digits 1, 2, 3, 4, and 5 be arranged to create a password?

13. How many ways can six teachers be assigned to teach six different classes, if each teacher can teach any of the classes?

Lesson 13-1 Solve problems involving permutations and combinations.

- From a choice of 3 meat toppings and 4 vegetable toppings, how many 5-topping pizzas are possible?

  Since order is not important, the selection is a combination of 7 objects taken 5 at a time, or $C(7, 5)$.

  $C(7, 5) = \frac{7!}{(7-5)!\,5!} = 21$

  There are 21 possible 5-topping pizzas.

Find each value.

14. $P(6, 3)$

15. $P(8, 6)$

16. $C(5, 3)$

17. $C(11, 8)$

18. $\frac{P(6, 3)}{P(5, 3)}$

19. $C(5, 5) \cdot C(3, 2)$

20. How many ways can 6 different books be placed on a shelf if the only dictionary must be on an end?

21. From a group of 3 men and 7 women, how many committees of 2 men and 2 women can be formed?

Lesson 13-2 Solve problems involving permutations with repetitions.

- How many ways can the letters of *Tallahassee* be arranged?

  There are 3 *a*'s, 2 *l*'s, 2 *s*'s, and 2 *e*'s. So the number of possible arrangements is $\frac{11!}{3!\,2!\,2!\,2!}$ or 831,600 ways.

How many different ways can the letters of each word be arranged?

22. *level*

23. *Cincinnati*

24. *graduate*

25. *banana*

26. How many different 9-digit Social Security numbers can have the digits 2, 9, 5, 5, 0, 7, 0, 5, and 8.

OBJECTIVES AND EXAMPLES

Lesson 13-3 Find the probability of an event.

Find the probability of randomly selecting 3 red pencils from a box containing 5 red, 3 blue, and 4 green pencils.

There are $C(5, 3)$ ways to select 3 out of 5 red pencils and $C(12, 3)$ ways to select 3 out of 12 pencils.

$$P(\text{3 red pencils}) = \frac{C(5, 3)}{C(12, 3)}$$

$$= \frac{\frac{5!}{2!\,3!}}{\frac{12!}{9!\,3!}}$$

$$= \frac{12}{220} \text{ or } \frac{1}{22}$$

REVIEW EXERCISES

A bag containing 7 pennies, 4 nickels, and 5 dimes. Three coins are drawn at random. Find each probability.

27. P(3 pennies)

28. P(2 pennies and 1 nickel)

29. P(3 nickels)

30. P(1 nickel and 2 dimes)

Lesson 13-3 Find the odds for the success and failure of an event.

Find the odds of randomly selecting 3 red pencils from a box containing 5 red, 3 blue, and 4 green pencils.

$$P(\text{3 red pencils}) = P(s) = \frac{1}{22}$$

$$P(\text{not 3 red pencils}) = P(f) = 1 - \frac{1}{22} \text{ or } \frac{21}{22}$$

$$\text{Odds} = \frac{P(s)}{P(f)} = \frac{\frac{1}{22}}{\frac{21}{22}}$$

$$= \frac{1}{21} \text{ or } 1{:}21$$

Refer to the bag of coins used for Exercises 27-30. Find the odds of each event occurring.

31. 3 pennies

32. 2 pennies and 1 nickel

33. 3 nickels

34. 1 nickel and 2 dimes

Lesson 13-4 Find the probability of independent and dependent events.

Three yellow and 5 black marbles are placed in a bag. What is the probability of drawing a black marble, replacing it, and then drawing a yellow marble?

$$P(\text{black}) = \frac{5}{8} \qquad P(\text{yellow}) = \frac{3}{8}$$

$$P(\text{black and yellow}) = P(\text{black}) \cdot P(\text{yellow})$$

$$= \frac{5}{8} \cdot \frac{3}{8} = \frac{15}{64}$$

Determine if each event is *independent* or *dependent*. Then determine the probability.

35. the probability of rolling a sum of 2 on the first toss of two number cubes and a sum of 6 on the second toss

36. the probability of randomly selecting two yellow markers from a box that contains 4 yellow and 6 pink markers

SKILLS AND CONCEPTS

OBJECTIVES AND EXAMPLES

Lesson 13-4 Find the probability of mutually exclusive and inclusive events.

- On a school board, 2 of the 4 female members are over 40 years of age, and 5 of the 6 male members are over 40. If one person did not attend the meeting, what is the probability that the person was a male or a member over 40?

$$P(\text{male or over } 40) = P(\text{male}) + P(\text{over } 40) - P(\text{male \& over } 40)$$
$$= \frac{6}{10} + \frac{7}{10} - \frac{5}{10} \text{ or } \frac{4}{5}$$

REVIEW EXERCISES

A box contains slips of paper numbered from 1 to 14. One slip of paper is drawn at random. Find each probability.

37. *P*(selecting a prime number or a multiple of 4)

38. *P*(selecting a multiple of 2 or a multiple of 3)

39. *P*(selecting a 3 or a 4)

40. *P*(selecting an 8 or a number less than 8)

OBJECTIVES AND EXAMPLES

Lesson 13-5 Find the probability of an event given the occurrence of another event.

- A coin is tossed 3 times. What is the probability that at the most 2 heads are tossed given that at least 1 head has been tossed?

Let event *A* be that at most 2 heads are tossed.
Let event *B* be that there is at least 1 head.

$$P(A \mid B) = \frac{P(A \text{ and } B)}{P(B)}$$
$$= \frac{\frac{6}{8}}{\frac{7}{8}} \text{ or } \frac{6}{7}$$

REVIEW EXERCISES

Two number cubes are tossed.

41. What is the probability that the sum of the numbers shown on the cubes is less than 5 if exactly one cube shows a 1?

42. What is the probability that the numbers shown on the cubes are different given that their sum is 8?

43. What is the probability that the numbers shown on the cubes match given that their sum is greater than or equal to 5?

OBJECTIVES AND EXAMPLES

Lesson 13-6 Find the probability of an event by using the Binomial Theorem.

- If you guess the answers on all 8 questions of a true/false quiz, what is the probability that exactly 5 of your answers will be correct?

$$(p + q)^8 = \sum_{r=0}^{8} \frac{8!}{r!(8-r)!} p^{8-r}q^r$$
$$= \frac{8!}{5!(8-5)!}\left(\frac{1}{2}\right)^5\left(\frac{1}{2}\right)^3$$
$$= \frac{56}{256} \text{ or } \frac{7}{32}$$

REVIEW EXERCISES

Find each probability if a coin is tossed 4 times.

44. *P*(exactly 1 head)

45. *P*(no heads)

46. *P*(2 heads and 2 tails)

47. *P*(at least 3 tails)

APPLICATIONS AND PROBLEM SOLVING

48. Travel Five people, including the driver, can be seated in Nate's car. Nate and 6 of his friends want to go to a movie. How many different groups of friends can ride in Nate's car on the first trip if the car is full? *(Lesson 13-1)*

49. Sommer has 7 different keys. How many ways can she place these keys on the key ring shown below? *(Lesson 13-2)*

50. Quality Control A collection of 15 memory chips contains 3 chips that are defective. If 2 memory chips are selected at random, what is the probability that at least one of them is good? *(Lesson 13-3)*

51. Gift Exchange The Burnette family is drawing names from a bag for a gift exchange. There are 7 males and 8 females in the family. If someone draws their own name, then they must draw again before replacing their name. *(Lesson 13-4)*

a. Reba draws the first name. What is the probability that Reba will draw a female's name that is not her own?

b. What is the probability that Reba will draw her own name, not replace it, and then draw a male's name?

ALTERNATIVE ASSESSMENT

OPEN-ENDED ASSESSMENT

1. The probability of two independent events occurring is $\frac{1}{12}$. If the probability of one of the events occurring is $\frac{1}{2}$, is it possible to find the probability of the other event? If so, find the probability and give an example of a situation for which this probability could apply. If not, explain why not.

2. Perry says "A permutation is the same as a combination." How would you explain to Perry that his statement is incorrect?

PORTFOLIO

Choose one of the types of probability you studied in this chapter. Describe a situation in which this type of probability would be used. Explain why no other type of probability should be used in this situation.

Unit 4 interNET Project

THE UNITED STATES CENSUS BUREAU

Radically Random!

- Use the Internet to find the population of the United States by age groups or ethnic background for the most recent census.
- Make a table or spreadsheet of the data.
- Suppose that a person was selected at random from all the people in the United States to answer some survey questions. Find the probability that the person was from each one of the age or ethnic groups you used for your table or spreadsheet.
- Write a summary describing how you calculated the probabilities. Include a graph with your summary. Discuss why someone might be interested in your findings.

Additional Assessment See p. A68 for Chapter 13 practice test.

Probability and Combination Problems

Both the ACT and SAT contain probability problems. You'll need to know these concepts:

Combinations	Permutations	Tree Diagram
Outcomes	Probability	

Memorize the definition of the probability of an event:

$$P(\text{event}) = \frac{\text{number of favorable outcomes}}{\text{total number of possible outcomes}}$$

THE PRINCETON REVIEW

TEST-TAKING TIP

For problems involving combinations, either use the formula or make a list.

Example:

$$C(5, 3) = \frac{5 \cdot 4 \cdot 3}{3!} = 10$$

SAT EXAMPLE

1. A bag contains 4 red balls, 10 green balls, and 6 yellow balls. If three balls are removed at random and no ball is returned to the bag after removal, what is the probability that all three balls will be green?

   A $\frac{1}{2}$ **B** $\frac{1}{8}$ **C** $\frac{3}{20}$ **D** $\frac{2}{19}$ **E** $\frac{3}{8}$

HINT Calculate the probability of two independent events by multiplying the probability of the first event by the probability of the second event.

Solution Use the definition of the probability of an event. Calculate the probability of getting a green ball each time a ball is removed. The first time a ball is removed there are a total of 20 balls and 10 of them are green. So the probability of removing a green ball as the first ball is $\frac{10}{20}$ or $\frac{1}{2}$. Now there are just 19 balls and 9 of them are green. The probability of removing a green ball is $\frac{9}{19}$. When the third ball is removed, there are 18 balls and 8 of them are green, so the probability of removing a green ball is $\frac{8}{18}$ or $\frac{4}{9}$. To find the probability of removing green balls as the first *and* the second *and* the third balls chosen, multiply the three probabilities.

$$\frac{1}{2} \times \frac{9}{19} \times \frac{4}{19} = \frac{2}{19}$$

The answer is choice **D**.

ACT EXAMPLE

2. If you toss 3 fair coins, what is the probability of getting exactly 2 heads?

   A $\frac{1}{3}$ **B** $\frac{3}{8}$

   C $\frac{1}{2}$ **D** $\frac{2}{3}$

   E $\frac{7}{8}$

HINT Start by listing all the possible outcomes. You can do this since the numbers are small.

Solution Make a list and then count the possible outcomes.

HHH, HHT, HTH, HTT,

TTH, THT, THH, TTT

There are 8 possible outcomes for 3 coins. Since the coins are fair, these are equally likely outcomes.

The favorable outcomes are those that include exactly 2 heads: HHT, HTH, THH. There are 3 favorable outcomes. Give the answer.

$$P(A) = \frac{\text{number of successful outcomes}}{\text{total number of outcomes}}$$

$$P(\text{exactly 2 heads}) = \frac{3}{8}$$

The answer is choice **B**.

CONTENTS

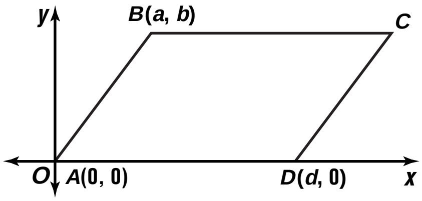
After you work each problem, record your answer on the answer sheet provided or on a piece of paper.

Multiple Choice

1. A coin was flipped 20 times and came up heads 10 times and tails 10 times. If the first and the last flips were both heads, what is the greatest number of consecutive heads that could have occurred?

   A 1 **B** 2 **C** 8
   D 9 **E** 10

2. In the figure below, *ABCD* is a parallelogram. What must be the coordinates of Point *C*?



   A (x, y) **B** $(d + a, y)$ **C** $(d - a, b)$
   D $(d + x, b)$ **E** $(d + a, b)$

3. In a plastic jar there are 5 red marbles, 7 blue marbles, and 3 green marbles. How many green marbles need to be added to the jar in order to double the probability of selecting a green marble?

   A 2 **B** 3 **C** 5 **D** 6 **E** 7

4. The average of 5 numbers is 20. If one of the numbers is 18, then what is the sum of the other four numbers?

   A 2 **B** 20.5 **C** 82
   D 90 **E** 100

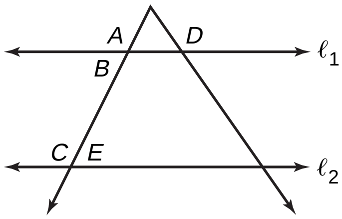
5. If the sum of *x* and *y* is an even number, and the sum of *x* and *z* is an even number, and *z* is an odd number, then which of the following must be true?
   I. *y* is an even number
   II. $y + z$ is an even number
   III. *y* is an odd number

   A I only **B** II only **C** III only
   D I and II **E** II and III

6. A bag contains only white and blue marbles. The probability of selecting a blue marble is $\frac{1}{5}$. The bag contains 200 marbles. If 100 white marbles are added to the bag, what is the probability of selecting a white marble?

   A $\frac{2}{15}$ **B** $\frac{7}{15}$ **C** $\frac{8}{15}$ **D** $\frac{4}{5}$ **E** $\frac{13}{15}$

7. In the figure below, $\ell_1 \parallel \ell_2$. Which of the labeled angles must be equal to each other?



   A *A* and *C* **B** *D* and *E* **C** *A* and *B*
   D *D* and *B* **E** *C* and *B*

8. What is the probability of drawing a diamond from a well-shuffled standard deck of playing cards?

   A $\frac{1}{52}$ **B** $\frac{1}{13}$ **C** $\frac{1}{4}$
   D $\frac{4}{13}$ **E** 1

9. A caterer offers 7 different entrees. A customer may choose any 3 of the entrees for a dinner. How many different combinations of entrees can a customer choose?
   A 6
   B 35
   C 84
   D 210
   E 840

10. **Grid-In** Six cards are numbered 0 through 5. Two are selected without replacement. What is the probability that the sum of the cards is 4?

internet CONNECTION **SAT/ACT Practice** For additional test practice questions, visit: **www.amc.glencoe.com**