4.2

Multiplying Matrices

What *you should learn*

GOAL 1 Multiply two matrices.

GOAL 2 Use matrix multiplication in **real-life** situations, such as finding the number of calories burned in **Ex. 40**.

Why *you should learn it*

▼ To solve **real-life** problems, such as calculating the cost of softball equipment in **Example 5**.

GOAL 1 MULTIPLYING TWO MATRICES

The product of two matrices A and B is defined provided the number of columns in A is equal to the number of rows in B.

If A is an $m \times n$ matrix and B is an $n \times p$ matrix, then the product AB is an $m \times p$ matrix.

$$\begin{array}{ccccc} A & \cdot & B & = & AB \\ m \times n & & n \times p & & m \times p \end{array}$$

equal

dimensions of AB

EXAMPLE 1 *Describing Matrix Products*

State whether the product AB is defined. If so, give the dimensions of AB.

a. A: 2×3, B: 3×4

b. A: 3×2, B: 3×4

SOLUTION

a. Because A is a 2×3 matrix and B is a 3×4 matrix, the product AB is defined and is a 2×4 matrix.

b. Because the number of columns in A (two) does not equal the number of rows in B (three), the product AB is not defined.

EXAMPLE 2 *Finding the Product of Two Matrices*

Find AB if $A = \begin{bmatrix} -2 & 3 \\ 1 & -4 \\ 6 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 3 \\ -2 & 4 \end{bmatrix}$.

SOLUTION

Because A is a 3×2 matrix and B is a 2×2 matrix, the product AB is defined and is a 3×2 matrix. To write the entry in the first row and first column of AB, multiply corresponding entries in the first row of A and the first column of B. Then add. Use a similar procedure to write the other entries of the product.

$$AB = \begin{bmatrix} -2 & 3 \\ 1 & -4 \\ 6 & 0 \end{bmatrix} \begin{bmatrix} -1 & 3 \\ -2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} (-2)(-1) + (3)(-2) & (-2)(3) + (3)(4) \\ (1)(-1) + (-4)(-2) & (1)(3) + (-4)(4) \\ (6)(-1) + (0)(-2) & (6)(3) + (0)(4) \end{bmatrix}$$

$$= \begin{bmatrix} -4 & 6 \\ 7 & -13 \\ -6 & 18 \end{bmatrix}$$

EXAMPLE 3 *Finding the Product of Two Matrices*

If $A = \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -4 \\ 2 & 1 \end{bmatrix}$, find each product.

a. AB **b.** BA

STUDENT HELP

INTERNET

HOMEWORK HELP
Visit our Web site
www.mcdougallittell.com
for extra examples.

SOLUTION

a. $AB = \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1 & -4 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 7 & -10 \\ -1 & 4 \end{bmatrix}$

b. $BA = \begin{bmatrix} 1 & -4 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 7 & 2 \\ 5 & 4 \end{bmatrix}$

……

Notice in Example 3 that $AB \neq BA$. Matrix multiplication is not, in general, commutative.

EXAMPLE 4 *Using Matrix Operations*

If $A = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} -2 & 0 \\ 4 & 2 \end{bmatrix}$, and $C = \begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix}$, simplify each expression.

a. $A(B + C)$ **b.** $AB + AC$

SOLUTION

a. $A(B + C) = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}\left(\begin{bmatrix} -2 & 0 \\ 4 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix}\right)$

$= \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 7 & 4 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 22 & 11 \end{bmatrix}$

b. $AB + AC = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} -2 & 0 \\ 4 & 2 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix}$

$= \begin{bmatrix} 0 & 2 \\ 14 & 6 \end{bmatrix} + \begin{bmatrix} 5 & 4 \\ 8 & 5 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 22 & 11 \end{bmatrix}$

……

Notice in Example 4 that $A(B + C) = AB + AC$, which is true in general. This and other properties of matrix multiplication are summarized below.

CONCEPT SUMMARY

PROPERTIES OF MATRIX MULTIPLICATION

Let A, B, and C be matrices and let c be a scalar.

ASSOCIATIVE PROPERTY OF MATRIX MULTIPLICATION	$A(BC) = (AB)C$
LEFT DISTRIBUTIVE PROPERTY	$A(B + C) = AB + AC$
RIGHT DISTRIBUTIVE PROPERTY	$(A + B)C = AC + BC$
ASSOCIATIVE PROPERTY OF SCALAR MULTIPLICATION	$c(AB) = (cA)B = A(cB)$

FOCUS ON PEOPLE

DOT RICHARDSON helped lead the United States to the first women's softball gold medal in the 1996 Olympics by playing shortstop.

APPLICATION LINK
www.mcdougallittell.com

GOAL 2 USING MATRIX MULTIPLICATION IN REAL LIFE

Matrix multiplication is useful in business applications because an *inventory* matrix, when multiplied by a *cost per item* matrix, results in a *total cost* matrix.

$$\underset{m \times n}{\begin{bmatrix}\text{Inventory}\\ \text{matrix}\end{bmatrix}} \cdot \underset{n \times p}{\begin{bmatrix}\text{Cost per item}\\ \text{matrix}\end{bmatrix}} = \underset{m \times p}{\begin{bmatrix}\text{Total cost}\\ \text{matrix}\end{bmatrix}}$$

For the total cost matrix to be meaningful, the column labels for the inventory matrix must match the row labels for the cost per item matrix.

EXAMPLE 5 *Using Matrices to Calculate the Total Cost*

SPORTS Two softball teams submit equipment lists for the season.

Women's team

12 bats
45 balls
15 uniforms

Men's team

15 bats
38 balls
17 uniforms

Each bat costs \$21, each ball costs \$4, and each uniform costs \$30. Use matrix multiplication to find the total cost of equipment for each team.

SOLUTION

To begin, write the equipment lists and the costs per item in matrix form. Because you want to use matrix multiplication to find the total cost, set up the matrices so that the columns of the equipment matrix match the rows of the cost matrix.

$$\begin{array}{r} \\ \text{Women's team} \\ \text{Men's team} \end{array}\!\!\begin{array}{c} \textbf{EQUIPMENT} \\ \begin{array}{ccc}\text{Bats} & \text{Balls} & \text{Uniforms}\end{array} \\ \begin{bmatrix} 12 & 45 & 15 \\ 15 & 38 & 17 \end{bmatrix} \end{array} \qquad \begin{array}{r} \\ \text{Bats} \\ \text{Balls} \\ \text{Uniforms} \end{array}\!\!\begin{array}{c} \textbf{COST} \\ \text{Dollars} \\ \begin{bmatrix} 21 \\ 4 \\ 30 \end{bmatrix} \end{array}$$

The total cost of equipment for each team can now be obtained by multiplying the equipment matrix by the cost per item matrix. The equipment matrix is 2×3 and the cost per item matrix is 3×1, so their product is a 2×1 matrix.

$$\begin{bmatrix} 12 & 45 & 15 \\ 15 & 38 & 17 \end{bmatrix}\begin{bmatrix} 21 \\ 4 \\ 30 \end{bmatrix} = \begin{bmatrix} 12(21) + 45(4) + 15(30) \\ 15(21) + 38(4) + 17(30) \end{bmatrix} = \begin{bmatrix} 882 \\ 977 \end{bmatrix}$$

The labels for the product matrix are as follows.

$$\begin{array}{r} \\ \text{Women's team} \\ \text{Men's team} \end{array}\!\!\begin{array}{c} \textbf{TOTAL COST} \\ \text{Dollars} \\ \begin{bmatrix} 882 \\ 977 \end{bmatrix} \end{array}$$

▶ The total cost of equipment for the women's team is \$882, and the total cost of equipment for the men's team is \$977.

GUIDED PRACTICE

Vocabulary Check ✓

1. Complete this statement: The product of matrices A and B is defined provided the number of __?__ in A is equal to the number of __?__ in B.

Concept Check ✓

2. Matrix A is 6×1. Matrix B is 1×2. Which of the products is defined, AB or BA? Explain.

3. Tell whether the matrix equation is *true* or *false*. Explain.

$$\begin{bmatrix}5&3\\-3&5\end{bmatrix}\begin{bmatrix}2&0\\0&1\end{bmatrix}=\begin{bmatrix}2&0\\0&1\end{bmatrix}\begin{bmatrix}5&3\\-3&5\end{bmatrix}$$

Skill Check ✓

State whether the product *AB* is defined. If so, give the dimensions of *AB*.

4. A: 3×2, B: 2×3

5. A: 3×3, B: 3×3

6. A: 3×2, B: 3×2

Find the product.

7. $\begin{bmatrix}1&0\\-2&-1\end{bmatrix}\begin{bmatrix}2&0\\1&3\end{bmatrix}$

8. $\begin{bmatrix}4&4\end{bmatrix}\begin{bmatrix}-2\\-3\end{bmatrix}$

9. $\begin{bmatrix}-3&3\\3&-2\\0&-1\end{bmatrix}\begin{bmatrix}1&0\\-2&-1\end{bmatrix}$

10. **SOFTBALL EQUIPMENT** Use matrix multiplication to find the total cost of equipment in Example 5 if the women's team needs 16 bats, 42 balls, and 16 uniforms and the men's team needs 14 bats, 43 balls, and 15 uniforms.

PRACTICE AND APPLICATIONS

STUDENT HELP

Extra Practice to help you master skills is on p. 944.

MATRIX PRODUCTS **State whether the product *AB* is defined. If so, give the dimensions of *AB*.**

11. A: 1×3, B: 3×2

12. A: 2×4, B: 4×3

13. A: 4×2, B: 3×5

14. A: 5×5, B: 5×4

15. A: 3×4, B: 4×1

16. A: 3×3, B: 2×4

FINDING MATRIX PRODUCTS **Find the product. If it is not defined, state the reason.**

17. $\begin{bmatrix}-\frac{1}{6}&\frac{1}{2}&-\frac{1}{3}\end{bmatrix}\begin{bmatrix}12\\0\\-12\end{bmatrix}$

18. $\begin{bmatrix}7.3&1.5\\1.8&0\\2.9&3.2\end{bmatrix}\begin{bmatrix}-4.2&2.6&-8.7\end{bmatrix}$

19. $\begin{bmatrix}1&-4\\3&-2\end{bmatrix}\begin{bmatrix}4&-1\\0&-3\end{bmatrix}$

20. $\begin{bmatrix}-6&-2\\0&3\end{bmatrix}\begin{bmatrix}-1&4\\-5&3\end{bmatrix}$

21. $\begin{bmatrix}2&-8&1\\0&-5&2\end{bmatrix}\begin{bmatrix}0&1&-2\\8&-2&-5\end{bmatrix}$

22. $\begin{bmatrix}6.0&0\\-0.2&0.2\\2.9&0.3\end{bmatrix}\begin{bmatrix}1&0\\1.5&-0.5\end{bmatrix}$

23. $\begin{bmatrix}-1&-0.5&1.25\\1&-1.5&-0.25\end{bmatrix}\begin{bmatrix}1.2\\0.2\\0\end{bmatrix}$

24. $\begin{bmatrix}-6&1&1\\-2&3&8\\0.1&7&1\end{bmatrix}\begin{bmatrix}0&-1&3\\-7&-2&4\\-1&3&4\end{bmatrix}$

25. $\begin{bmatrix}6&-2\\1&4\\0&5\end{bmatrix}\begin{bmatrix}-4&-2&5\\4&-6&-1\end{bmatrix}$

26. $\begin{bmatrix}0&1&0\\6&-3&-1\\-2&5&3\end{bmatrix}\begin{bmatrix}5&-7&4\\3&12&6\\-4&-5&-12\end{bmatrix}$

STUDENT HELP

HOMEWORK HELP

Example 1: Exs. 11–16

Examples 2, 3: Exs. 17–26

Example 4: Exs. 27–32

Example 5: Exs. 35–40

SIMPLIFYING EXPRESSIONS **Using the given matrices, simplify the expression.**

$$A = \begin{bmatrix} 4 & -2 \\ 6 & -1 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ -2 & 4 \end{bmatrix}, C = \begin{bmatrix} -1 & 3 \\ -2 & 1 \end{bmatrix}, D = \begin{bmatrix} 3 & -2 & 1 \\ -1 & 2 & 4 \\ -2 & -3 & 3 \end{bmatrix}, E = \begin{bmatrix} -2 & 5 & 6 \\ -1 & 4 & 2 \\ 3 & 1 & -4 \end{bmatrix}$$

27. $2AB$ **28.** $AB + AC$ **29.** $D(D + E)$

30. $(E + D)E$ **31.** $-3(AC)$ **32.** $0.5(AB) + 2AC$

SOLVING MATRIX EQUATIONS **Solve for *x* and *y*.**

33. $\begin{bmatrix} -2 & 1 & 2 \\ 3 & 2 & 4 \\ 0 & -2 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ x \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ 19 \\ y \end{bmatrix}$

34. $\begin{bmatrix} 4 & 1 & 3 \\ -2 & x & 1 \end{bmatrix} \begin{bmatrix} 9 & -2 \\ 2 & 1 \\ -1 & 4 \end{bmatrix} = \begin{bmatrix} y & 5 \\ -13 & 11 \end{bmatrix}$

STUDENT HELP

DATA UPDATE
Visit our Web site
www.mcdougallittell.com

AGRICULTURE **In Exercises 35 and 36, use the following information.**

The percents of the total 1997 world production of wheat, rice, and maize are shown in the matrix for the four countries that grow the most grain: China, India, the Commonwealth of Independent States (formerly the Soviet Union), and the United States. The total 1997 world production (in thousands of metric tons) of wheat, rice, and maize is 608,846, 570,906, and 586,923, respectively.

Source: Food and Agriculture Organization of the United Nations

GRAIN PRODUCTION

$$\begin{array}{r} \\ \textbf{China} \\ \textbf{India} \\ \textbf{C.I.S.} \\ \textbf{U.S.} \end{array} \begin{array}{c} \begin{array}{ccc} \textbf{Wheat} & \textbf{Rice} & \textbf{Maize} \end{array} \\ \begin{bmatrix} 20.1\% & 34.8\% & 18\% \\ 22\% & 21.5\% & 1.7\% \\ 7.3\% & 0.1\% & 0.5\% \\ 11.3\% & 1.4\% & 40.5\% \end{bmatrix} \end{array}$$

35. Rewrite the matrix to give the percents as decimals.

36. Show how matrix multiplication can be used to determine how many metric tons of all three grains were produced in each of the four countries.

FOCUS ON APPLICATIONS

EXERCISE
A 120 pound person walking at a moderate pace of 3 mi/h would burn about 64 Cal in 20 min. At a brisk pace of 4.5 mi/h, the person would burn about 82 Cal in 20 min.

CLASS DEBATE **In Exercises 37–39, use the following information.**

Three teams participated in a debating competition. The final score for each team is based on how many students ranked first, second, and third in a debate. The results of 12 debates are shown in matrix *A*.

MATRIX *A*

$$\begin{array}{r} \\ \textbf{Team 1} \\ \textbf{Team 2} \\ \textbf{Team 3} \end{array} \begin{array}{c} \begin{array}{ccc} \textbf{1st} & \textbf{2nd} & \textbf{3rd} \end{array} \\ \begin{bmatrix} 3 & 5 & 4 \\ 5 & 2 & 5 \\ 4 & 6 & 2 \end{bmatrix} \end{array}$$

37. Teams earn 6 points for each first place, 5 points for each second place, and 4 points for each third place. Organize this information into a matrix *B*.

38. Find the product *AB*.

39. **LOGICAL REASONING** Which team won the competition? How many points did the winning team score?

40. **EXERCISE** The numbers of calories burned by people of different weights doing different activities for 20 minutes are shown in the matrix. Show how matrix multiplication can be used to write the total number of calories burned by a 120 pound person and a 150 pound person who each bicycled for 40 minutes, jogged for 10 minutes, and then walked for 60 minutes.

Source: *Medicine and Science in Sports and Exercise*

CALORIES BURNED

$$\begin{array}{r} \\ \textbf{Bicycling} \\ \textbf{Jogging} \\ \textbf{Walking} \end{array} \begin{array}{c} \begin{array}{cc} \textbf{120 lb person} & \textbf{150 lb person} \end{array} \\ \begin{bmatrix} 109 & 136 \\ 127 & 159 \\ 64 & 79 \end{bmatrix} \end{array}$$

41. *Writing* Describe the process you use when multiplying any two matrices.

Test Preparation

42. MULTIPLE CHOICE What is the product of $\begin{bmatrix} 0 & -1 \\ -4 & -2 \end{bmatrix}$ and $\begin{bmatrix} 7 & -2 \\ -1 & 0 \end{bmatrix}$?

Ⓐ $\begin{bmatrix} 2 & 0 \\ -26 & -24 \end{bmatrix}$ Ⓑ $\begin{bmatrix} 8 & 0 \\ -30 & 6 \end{bmatrix}$ Ⓒ $\begin{bmatrix} 1 & 0 \\ -26 & 8 \end{bmatrix}$ Ⓓ $\begin{bmatrix} 1 & 0 \\ -30 & 8 \end{bmatrix}$

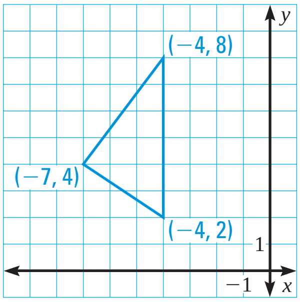
43. MULTIPLE CHOICE If A is a 2×3 matrix and B is a 3×2 matrix, what are the dimensions of BA?

Ⓐ 2×2 Ⓑ 3×3 Ⓒ 3×2 Ⓓ 2×3 Ⓔ BA not defined

★ Challenge

44. ROTATIONAL MATRIX Matrix A is a 90° rotational matrix. Matrix B contains the coordinates of the triangle's vertices shown in the graph.

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} -7 & -4 & -4 \\ 4 & 8 & 2 \end{bmatrix}$$



a. Calculate AB. Graph the coordinates of the vertices given by AB. What rotation does AB represent in the graph?

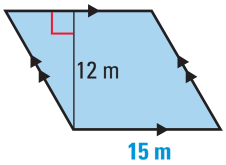
b. Find the 180° and 270° rotations of the original triangle by using repeated multiplication of the 90° rotational matrix. What are the coordinates of the vertices of the rotated triangles?

EXTRA CHALLENGE
www.mcdougallittell.com

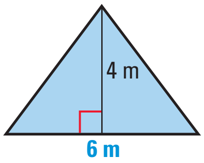
MIXED REVIEW

CALCULATING AREA **Find the area of the figure.** **(Skills Review, p. 914)**

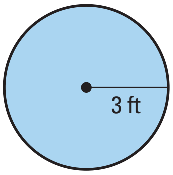
45.



46.



47.



WRITING EQUATIONS **Write an equation of the line with the given properties.** **(Review 2.4)**

48. slope: $\frac{1}{2}$, passes through (1, 8)

49. slope: $-\frac{1}{4}$, passes through (0, 4)

50. passes through (3, −6) and (2, 10)

51. passes through (1, 5) and (4, 14)

52. x-intercept: −7, y-intercept: −5

53. x-intercept: 4, y-intercept: −6

SOLVING SYSTEMS **Solve the system of linear equations using any algebraic method.** **(Review 3.2 for 4.3)**

54. $4x + y = 6$
$-3x - 2y = 8$

55. $2x + y = -9$
$3x + 5y = 4$

56. $-9x + 5y = 1$
$3x - 2y = 2$

57. $2x - 2y = 8$
$x - y = 1$

58. $-3x + 4y = -1$
$6x + 2y = 7$

59. $5x + 2y = -10$
$-3x - 8y = 40$

60. $7x + 3y = 11$
$-2x + 5y = 32$

61. $5x - 4y = -1$
$2x - 9y = 10$

62. $-x + 7y = -49$
$12x + y = -24$