Lesson 7-3 Sum and Difference Identities

**Real World Application**

Consider two angles \( \alpha \) and \( \beta \) in standard position. Let the terminal side of \( \alpha \) intersect the unit circle at point \( A(\cos \alpha, \sin \alpha) \). Let the terminal side of \( \beta \) intersect the unit circle at \( B(\cos \beta, \sin \beta) \). We will calculate \( AB^2 \) in two different ways.

First use the Law of Cosines.

\[
(AB)^2 = (OA)^2 + (OB)^2 - 2(OA)(OB) \cos (\alpha - \beta)
\]

\[
(AB)^2 = 1^2 + 1^2 - 2(1)(1) \cos (\alpha - \beta) \quad OA = OB = 1
\]

\[
(AB)^2 = 2 - 2 \cos (\alpha - \beta)
\]

Simplify.

Now use the distance formula.

\[
(AB)^2 = (\cos \alpha - \cos \beta)^2 + (\sin \alpha - \sin \beta)^2
\]

\[
(AB)^2 = \cos^2 \alpha - 2 \cos \alpha \cos \beta + \cos^2 \beta + \sin^2 \alpha - 2 \sin \alpha \sin \beta + \sin^2 \beta
\]

\[
(AB)^2 = (\cos^2 \alpha + \sin^2 \alpha) + (\cos^2 \beta + \sin^2 \beta) - 2(\cos \alpha \cos \beta + \sin \alpha \sin \beta)
\]

\[
(AB)^2 = 1 + 1 - 2(\cos \alpha \cos \beta + \sin \alpha \sin \beta) \quad \cos^2 \alpha + \sin^2 \alpha = 1
\]

\[
(AB)^2 = 2 - 2(\cos \alpha \cos \beta + \sin \alpha \sin \beta) \quad \text{Simplify.}
\]

Set the two expressions for \( (AB)^2 \) equal to each other.

\[
2 - 2 \cos (\alpha - \beta) = 2 - 2(\cos \alpha \cos \beta + \sin \alpha \sin \beta)
\]

\[
-2 \cos (\alpha - \beta) = -2(\cos \alpha \cos \beta + \sin \alpha \sin \beta) \quad \text{Subtract 2 from each side.}
\]

\[
\cos (\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta \quad \text{Divide each side by } -2.
\]

This equation is known as the **difference identity for cosine**.
The sum identity for cosine can be derived by substituting \(-\beta\) for \(\beta\) in the difference identity.

\[
\cos (\alpha + \beta) = \cos (\alpha - (-\beta))
\]

\[
= \cos \alpha \cos (-\beta) + \sin \alpha \sin (-\beta)
\]

\[
= \cos \alpha \cos \beta + \sin \alpha (-\sin \beta)\quad \cos (-\beta) = \cos \beta; \sin (-\beta) = -\sin \beta
\]

\[
= \cos \alpha \cos \beta - \sin \alpha \sin \beta
\]

### Example 1

a. Show by producing a counterexample that \(\cos (x + y) \neq \cos x + \cos y\).

b. Show that the sum identity for cosine is true for the values used in part a.

a. Let \(x = \frac{\pi}{3}\) and \(y = \frac{\pi}{6}\). First find \(\cos (x + y)\) for \(x = \frac{\pi}{3}\) and \(y = \frac{\pi}{6}\).

\[
\cos (x + y) = \cos \left(\frac{\pi}{3} + \frac{\pi}{6}\right)\quad \text{Replace } x \text{ with } \frac{\pi}{3} \text{ and } y \text{ with } \frac{\pi}{6}.
\]

\[
= \cos \frac{\pi}{2}\quad \frac{\pi}{3} + \frac{\pi}{6} = \frac{\pi}{2}
\]

\[
= 0
\]

Now find \(\cos x + \cos y\).

\[
\cos x + \cos y = \cos \frac{\pi}{3} + \cos \frac{\pi}{6}\quad \text{Replace } x \text{ with } \frac{\pi}{3} \text{ and } y \text{ with } \frac{\pi}{6}.
\]

\[
= \frac{1}{2} + \frac{\sqrt{3}}{2}\quad \text{or}\quad \frac{1 + \sqrt{3}}{2}
\]

So, \(\cos (x + y) \neq \cos x + \cos y\).

b. Show that \(\cos (x + y) = \cos x \cos y - \sin x \sin y\) for \(x = \frac{\pi}{3}\) and \(y = \frac{\pi}{6}\).

First find \(\cos (x + y)\). From part a, we know that \(\cos \left(\frac{\pi}{3} + \frac{\pi}{6}\right) = 0\).

Now find \(\cos x \cos y - \sin x \sin y\).

\[
\cos x \cos y - \sin x \sin y = \cos \frac{\pi}{3} \cos \frac{\pi}{6} - \sin \frac{\pi}{3} \sin \frac{\pi}{6}\quad \text{Substitute for } x \text{ and } y.
\]

\[
= \left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right) - \left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right)
\]

\[
= 0
\]

Thus, the sum identity for cosine is true for \(x = \frac{\pi}{3}\) and \(y = \frac{\pi}{6}\).
**Example 2** Use the sum or difference identity for cosine to find the exact value of \( \cos 735° \).

\[
735° = 2(360°) + 15° \\
\cos 735° = \cos 15° \\
\cos 15° = \cos (45° - 30°) \quad 45° and 30° are two common angles that differ by 15°. \\
= \cos 45° \cos 30° + \sin 45° \sin 30° \quad \text{Difference identity for cosine} \\
= \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \cdot \frac{1}{2} \\
= \frac{\sqrt{6} + \sqrt{2}}{4} \\
\text{Therefore, } \cos 735° = \frac{\sqrt{6} + \sqrt{2}}{4}.
\]

We can derive sum and difference identities for the sine function from those for the cosine function. Replace \( \alpha \) with \( \frac{\pi}{2} \) and \( \beta \) with \( s \) in the identities for \( \cos (\alpha \pm \beta) \). The following equations result.

\[
\cos \left( \frac{\pi}{2} + s \right) = -\sin s \quad \cos \left( \frac{\pi}{2} - s \right) = \sin s
\]

Replace \( s \) with \( \frac{\pi}{2} + s \) in the equation for \( \cos \left( \frac{\pi}{2} + s \right) \) and with \( \frac{\pi}{2} - s \) in the equation for \( \cos \left( \frac{\pi}{2} - s \right) \) to obtain the following equations.

\[
\cos s = \sin \left( \frac{\pi}{2} + s \right) \quad \cos s = \sin \left( \frac{\pi}{2} - s \right)
\]

Replace \( s \) with \( (\alpha + \beta) \) in the equation for \( \cos \left( \frac{\pi}{2} - s \right) \) to derive an identity for the sine of the sum of two real numbers.

\[
\cos \left[ \frac{\pi}{2} - (\alpha + \beta) \right] = \sin (\alpha + \beta)
\]

\[
\cos \left[ \frac{\pi}{2} - (\alpha - \beta) \right] = \sin (\alpha + \beta)
\]

\[
\cos \left( \frac{\pi}{2} - \alpha \right) \cos \beta + \sin \left( \frac{\pi}{2} - \alpha \right) \sin \beta = \sin (\alpha + \beta) \quad \text{Identity for } \cos (\alpha - \beta) \\
\sin \alpha \cos \beta + \cos \alpha \sin \beta = \sin (\alpha + \beta) \quad \text{Substitute.}
\]

This equation is known as the **sum identity for sine**.

The **difference identity for sine** can be derived by replacing \( \beta \) with \((-\beta)\) in the sum identity for sine.

\[
\sin [\alpha + (-\beta)] = \sin \alpha \cos (-\beta) + \cos \alpha \sin (-\beta) \\
\sin (\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta
\]

---

<table>
<thead>
<tr>
<th>Sum and Difference Identities for the Sine Function</th>
</tr>
</thead>
<tbody>
<tr>
<td>If ( \alpha ) and ( \beta ) represent the measures of two angles, then the following identities hold for all values of ( \alpha ) and ( \beta ).</td>
</tr>
</tbody>
</table>
| \[
\sin (\alpha \pm \beta) = \sin \alpha \cos \beta \pm \cos \alpha \sin \beta
\] |
Find the value of \( \sin \left( \frac{x}{y} \right) \) if \( 0 < x < \frac{\pi}{2}, \ 0 < y < \frac{\pi}{2} \), \( \sin x = \frac{9}{41} \), and \( \sin y = \frac{7}{25} \).

In order to use the difference identity for sine, we need to know \( \cos x \) and \( \cos y \). We can use a Pythagorean identity to determine the necessary values.

\[
\sin^2 \alpha + \cos^2 \alpha = 1 \implies \cos^2 \alpha = 1 - \sin^2 \alpha \quad \text{Pythagorean identity}
\]

Since we are given that the angles are in Quadrant I, the values of sine and cosine are positive. Therefore, \( \cos \alpha = \sqrt{1 - \sin^2 \alpha} \).

\[
\cos x = \sqrt{1 - \left( \frac{9}{41} \right)^2} = \sqrt{1600} \text{ or } \frac{40}{41}
\]

\[
\cos y = \sqrt{1 - \left( \frac{7}{25} \right)^2} = \sqrt{576} \text{ or } \frac{24}{25}
\]

Now substitute these values into the difference identity for sine.

\[
\sin \left( \frac{x}{y} \right) = \sin x \cos y - \cos x \sin y \\
= \left( \frac{9}{41} \right) \left( \frac{24}{25} \right) - \left( \frac{40}{41} \right) \left( \frac{7}{25} \right) \\
= -\frac{64}{1025} \text{ or about } -0.0624
\]

**Broadcasting** Refer to the application at the beginning of the lesson. What type of interference results when signals modeled by the equations \( y = 20 \sin (3t + 45^\circ) \) and \( y = 20 \sin (3t + 225^\circ) \) are combined?

Add the two sine functions together and simplify.

\[
20 \sin (3t + 45^\circ) + 20 \sin (3t + 225^\circ) \\
= 20(\sin 3t \cos 45^\circ + \cos 3t \sin 45^\circ) + 20(\sin 3t \cos 225^\circ + \cos 3t \sin 225^\circ) \\
= 20 \left( \sin 3t \left( \frac{\sqrt{2}}{2} \right) + \cos 3t \left( \frac{\sqrt{2}}{2} \right) \right) + 20 \left( \sin 3t \left( -\frac{\sqrt{2}}{2} \right) + \cos 3t \left( -\frac{\sqrt{2}}{2} \right) \right) \\
= 10\sqrt{2} \sin 3t + 10\sqrt{2} \cos 3t - 10\sqrt{2} \sin 3t - 10\sqrt{2} \cos 3t \\
= 0
\]

The interference is destructive. The signals cancel each other completely.

You can use the sum and difference identities for the cosine and sine functions to find sum and difference identities for the tangent function.
You will be asked to derive these identities in Exercise 47.

**Example 5**

Use the sum or difference identity for tangent to find the exact value of \( \tan 285^\circ \).

\[
\tan 285^\circ = \tan (240^\circ + 45^\circ) \quad 240^\circ \text{ and } 45^\circ \text{ are common angles whose sum is } 285^\circ.
\]

\[
= \frac{\tan 240^\circ + \tan 45^\circ}{1 - \tan 240^\circ \tan 45^\circ} \quad \text{Sum identity for tangent}
\]

\[
= \frac{\sqrt{3} + 1}{1 - (\sqrt{3})(1)} \quad \text{Multiply by } \frac{1 + \sqrt{3}}{1 + \sqrt{3}} \text{ to simplify.}
\]

\[
= -2 - \sqrt{3}
\]

You can use sum and difference identities to verify other identities.

**Example 6**

Verify that \( \csc \left( \frac{3\pi}{2} + A \right) = -\sec A \) is an identity.

Transform the left side since it is more complicated.

\[
\csc \left( \frac{3\pi}{2} + A \right) \quad \text{isomorphic}
\]

\[
= \frac{1}{\sin \left( \frac{3\pi}{2} + A \right)} \quad -\sec A \quad \text{Reciprocal identity: } \csc x = \frac{1}{\sin x}
\]

\[
= \frac{1}{\sin \frac{3\pi}{2} \cos A + \cos \frac{3\pi}{2} \sin A} \quad -\sec A \quad \text{Sum identity for sine}
\]

\[
= \frac{1}{(-1) \cos A + (0) \sin A} \quad -\sec A \quad \sin \frac{3\pi}{2} = -1; \cos \frac{3\pi}{2} = 0
\]

\[
= \frac{1}{-\cos A} \quad -\sec A \quad \text{Simplify.}
\]

\[
-\sec A = -\sec A \quad \text{Reciprocal identity}
\]

**Check for Understanding**

Read and study the lesson to answer each question.

1. **Describe** how you would convince a friend that \( \sin (x + y) \neq \sin x + \sin y \).

2. **Explain** how to use the sum and difference identities to find values for the secant, cosecant, and cotangent functions of a sum or difference.
Guided Practice

Use sum or difference identities to find the exact value of each trigonometric function.

5. \( \cos 165^\circ \)  
6. \( \tan \frac{\pi}{12} \)  
7. \( \sec 795^\circ \)

Find each exact value if \( 0 < x < \frac{\pi}{2} \) and \( 0 < y < \frac{\pi}{2} \).

8. \( \sin (x - y) \) if \( \sin x = \frac{4}{9} \) and \( \sin y = \frac{1}{4} \)

9. \( \tan (x + y) \) if \( \csc x = \frac{5}{3} \) and \( \cos y = \frac{5}{13} \)

Verify that each equation is an identity.

10. \( \sin (90^\circ + A) = \cos A \)  
11. \( \tan \left( \theta + \frac{\pi}{2} \right) = - \cot \theta \)

12. \( \sin (x - y) = \frac{1 - \cot x \tan y}{\csc x \sec y} \)

Electrical Engineering

Analysis of the voltage in certain types of circuits involves terms of the form \( \sin (n\omega_0 t - 90^\circ) \), where \( n \) is a positive integer, \( \omega_0 \) is the frequency of the voltage, and \( t \) is time. Use an identity to simplify this expression.

Exercises

Practice

Use sum or difference identities to find the exact value of each trigonometric function.

14. \( \cos 105^\circ \)  
15. \( \sin 165^\circ \)  
16. \( \cos \frac{7\pi}{12} \)

17. \( \sin \frac{\pi}{12} \)  
18. \( \tan 195^\circ \)  
19. \( \cos \left( -\frac{\pi}{12} \right) \)

20. \( \tan 165^\circ \)  
21. \( \tan \frac{23\pi}{12} \)  
22. \( \sin 735^\circ \)

23. \( \sec 1275^\circ \)  
24. \( \csc \frac{5\pi}{12} \)  
25. \( \cot \frac{113\pi}{12} \)
Find each exact value if \(0 < x < \frac{\pi}{2}\) and \(0 < y < \frac{\pi}{2}\).

26. \(\sin (x + y)\) if \(\cos x = \frac{8}{17}\) and \(\sin y = \frac{12}{37}\)

27. \(\cos (x - y)\) if \(\cos x = \frac{3}{5}\) and \(\cos y = \frac{4}{5}\)

28. \(\tan (x - y)\) if \(\sin x = \frac{8}{17}\) and \(\cos y = \frac{3}{5}\)

29. \(\cos (x + y)\) if \(\tan x = \frac{5}{3}\) and \(\sin y = \frac{1}{3}\)

30. \(\tan (x + y)\) if \(\cot x = \frac{6}{5}\) and \(\sec y = \frac{3}{2}\)

31. \(\sec (x - y)\) if \(\csc x = \frac{5}{3}\) and \(\tan y = \frac{12}{5}\)

32. If \(\alpha\) and \(\beta\) are two angles in Quadrant I such that \(\sin \alpha = \frac{1}{5}\) and \(\cos \beta = \frac{2}{7}\), find \(\sin (\alpha - \beta)\).

33. If \(x\) and \(y\) are acute angles such that \(\cos x = \frac{1}{3}\) and \(\cos y = \frac{3}{4}\), what is the value of \(\cos (x + y)\)?

Verify that each equation is an identity.

34. \(\cos \left(\frac{\pi}{2} + x\right) = -\sin x\)

35. \(\cos (60° + A) = \sin (30° - A)\)

36. \(\sin (A + \pi) = -\sin A\)

37. \(\cos (180° + x) = -\cos x\)

38. \(\tan (x + 45°) = \frac{1 + \tan x}{1 - \tan x}\)

39. \(\sin (A + B) = \frac{\tan A + \tan B}{\sec A \sec B}\)

40. \(\cos (A + B) = \frac{1 - \tan A \tan B}{\sec A \sec B}\)

41. \(\sec (A - B) = \frac{\sec A \sec B}{1 + \tan A \tan B}\)

42. \(\sin (x + y) \sin (x - y) = \sin^2 x - \sin^2 y\)

43. **Electronics**  In an electric circuit containing a capacitor, inductor, and resistor the voltage drop across the inductor is given by \(V_L = I_0 \omega L \cos (\omega t + \frac{\pi}{2})\), where \(I_0\) is the peak current, \(\omega\) is the frequency, \(L\) is the inductance, and \(t\) is time. Use the sum identity for cosine to express \(V_L\) as a function of \(\sin \omega t\).

44. **Optics**  The index of refraction for a medium through which light is passing is the ratio of the velocity of light in free space to the velocity of light in the medium. For light passing symmetrically through a glass prism, the index of refraction \(n\) is given by the equation \(n = \sqrt[n]{\frac{1}{\sin \left(\frac{1}{2}(\alpha + \beta)\right)}}\), where \(\alpha\) is the deviation angle and \(\beta\) is the angle of the apex of the prism as shown in the diagram. If \(\beta = 60°\), show that \(n = \sqrt{3} \sin \frac{\alpha}{2} \cos \frac{\alpha}{2}\).
45. **Critical Thinking** Simplify the following expression without expanding any of the sums or differences.

\[
\sin \left( \frac{\pi}{3} - A \right) \cos \left( \frac{\pi}{3} + A \right) - \cos \left( \frac{\pi}{3} - A \right) \sin \left( \frac{\pi}{3} + A \right)
\]

46. **Calculus** In calculus, you will explore the difference quotient \( \frac{f(x + h) - f(x)}{h} \).
   a. Let \( f(x) = \sin x \). Write and expand an expression for the difference quotient.
   b. Set your answer from part a equal to \( y \). Let \( h = 0.1 \) and graph.
   c. What function has a graph similar to the graph in part b?

47. **Critical Thinking** Derive the sum and difference identities for the tangent function.

48. **Critical Thinking** Consider the following theorem.
   If \( A, B, \) and \( C \) are the angles of a nonright triangle, then
   \[
   \tan A + \tan B + \tan C = \tan A \tan B \tan C.
   \]
   a. Choose values for \( A, B, \) and \( C \). Verify that the conclusion is true for your specific values.
   b. Prove the theorem.

**Mixed Review**

49. Verify the identity \( \sec^2 x = \frac{1 - \cos^2 x}{1 - \sin^2 x} + \csc^2 x - \cot^2 x \). (Lesson 7-2)

50. If \( \sin \theta = -\frac{1}{8} \) and \( \pi < \theta < \frac{3\pi}{2} \), find \( \tan \theta \). (Lesson 7-1)

51. Find \( \sin \left[ \arctan \sqrt{3} \right] \). (Lesson 6-8)

52. Find the values of \( \theta \) for which \( \csc \theta \) is undefined. (Lesson 6-7)

53. **Weather** The average seasonal high temperatures for Greensboro, North Carolina, are given in the table. Write a sinusoidal function that models the temperatures, using \( t = 1 \) to represent winter. (Lesson 6-6)

<table>
<thead>
<tr>
<th>Winter</th>
<th>Spring</th>
<th>Summer</th>
<th>Fall</th>
</tr>
</thead>
<tbody>
<tr>
<td>50°</td>
<td>70°</td>
<td>86°</td>
<td>71°</td>
</tr>
</tbody>
</table>

Source: Rand McNally & Company

54. State the amplitude, period, and phase shift for the function \( y = 8 \cos (\theta - 30°) \). (Lesson 6-5)

55. Find the value of \( \sin (-540°) \). (Lesson 6-3)

56. **Geometry** A sector has arc length of 18 feet and a central angle measuring 2.9 radians. Find the radius and the area of the sector. (Lesson 6-1)

57. **Navigation** A ship at sea is 70 miles from one radio transmitter and 130 miles from another. The angle formed by the rays from the ship to the transmitters measures 130°. How far apart are the transmitters? (Lesson 5-8)

58. Determine the number of possible solutions for a triangle if \( A = 120°, b = 12, \) and \( a = 4 \). (Lesson 5-7)
59. **Photography**  A photographer observes a 35-foot totem pole that stands vertically on a uniformly-sloped hillside and the shadow cast by it at different times of day. At a time when the angle of elevation of the sun is $37°12’$, the shadow of the pole extends directly down the slope. This is the effect that the photographer is seeking. If the hillside has an angle of inclination of $6°40’$, find the length of the shadow. *(Lesson 5-6)*

60. Find the roots of the equation $4x^3 + 3x^2 - x = 0$. *(Lesson 4-1)*

61. Solve $|x + 1| > 4$. *(Lesson 3-3)*

62. Find the value of the determinant \[
\begin{vmatrix}
-1 & -2 \\
3 & -6
\end{vmatrix}.
\] *(Lesson 2-5)*

63. If $f(x) = 3x^2 - 4$ and $g(x) = 5x + 1$, find $f \circ g(4)$ and $g \circ f(4)$. *(Lesson 1-2)*

64. **SAT Practice**  What is the value of $(-8)^{62} + 8^{62}$?

A 1  
B 0  
C $-1$  
D $-8$  
E $-62$

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**MID-CHAPTER QUIZ**

**Use the given information to determine the exact trigonometric value.** *(Lesson 7-1)*

1. $\sin \theta = \frac{2}{3}, 0 < \theta < \frac{\pi}{2}$; $\cot \theta$

2. $\tan \theta = \frac{4}{3}, 90° < \theta < 180°$; $\cos \theta$

3. Express $\cos \frac{19\pi}{4}$ as a trigonometric function of an angle in Quadrant I. *(Lesson 7-1)*

**Verify that each equation is an identity.** *(Lessons 7-2 and 7-3)*

4. $\frac{1}{1 + \tan^2 x} + \frac{1}{1 + \cot^2 x} = 1$

5. $\frac{\csc^2 \theta + \sec^2 \theta}{\sec^2 \theta} = \csc^2 \theta$

6. $\cot x \sec x \sin x = 2 - \tan x \cos x \csc x$

7. $\tan (\alpha - \beta) = \frac{1 - \cot \alpha \tan \beta}{\cot \alpha + \tan \beta}$

8. Use a sum or difference identity to find the exact value of $\cos 75°$. *(Lesson 7-3)*

**Find each exact value if $0 < x < \frac{\pi}{2}$ and $0 < y < \frac{\pi}{2}$.** *(Lesson 7-3)*

9. $\cos (x + y)$ if $\sin x = \frac{2}{3}$ and $\sin y = \frac{3}{4}$

10. $\tan (x - y)$ if $\tan x = \frac{5}{4}$ and $\sec y = 2$
7-3B Reduction Identities

An Extension of Lesson 7-3

In Chapter 5, you learned that any trigonometric function of an acute angle is equal to the cofunction of the complement of the angle. For example, \( \sin \theta = \cos (90° - \theta) \). This is a part of a large family of identities called the reduction identities. These identities involve adding and subtracting the quadrant angles, 90°, 180°, and 270°, from the angle measure to find equivalent values of the trigonometric function. You can use your knowledge of phase shifts and reflections to find the components of these identities.

Find the values of the sine and cosine functions for \( \alpha - 90°, \alpha - 180°, \) and \( \alpha - 270° \) that are equivalent to \( \sin \alpha \).

\( \alpha - 90° \)
Graph \( y = \sin \alpha \), \( y = \sin (\alpha - 90°) \), and \( y = \cos (\alpha - 90°) \), letting X in degree mode represent \( \alpha \). You can select different display formats to help you distinguish the three graphs.

Note that the graph of \( y = \cos (X - 90) \) is the same as the graph of \( y = \sin X \). This suggests that \( \sin \alpha = \cos (\alpha - 90°) \). Remember that an identity must be proved algebraically. A graph does not prove an identity.

\( \alpha - 180° \)
Graph \( y = \sin \alpha \), \( y = \sin (\alpha - 180°) \), and \( y = \cos (\alpha - 180°) \) using X to represent \( \alpha \).

Discount \( y = \cos (\alpha - 180°) \) as a possible equivalence because it would involve a phase shift, which would change the actual value of the angle being considered.

Note that the graph of \( \sin (\alpha - 180°) \) is a mirror reflection of \( \sin \alpha \). Remember that a reflection over the x-axis results in the mapping \( (x, y) \rightarrow (x, -y) \). So to obtain a graph that is identical to \( y = \sin \alpha \), we need the reflection of \( y = \sin (\alpha - 180°) \) over the x-axis, or \( y = -\sin (\alpha - 180°) \). Thus, \( \sin \alpha = -\sin (\alpha - 180°) \). Graph the two equations to investigate this equality.
TRY THESE

Copy and complete each statement with the proper trigonometric functions.

1. \( \cos \alpha = \ ? \ (\alpha - 90^\circ) = \ ? \ (\alpha - 180^\circ) = \ ? \ (\alpha - 270^\circ) \)

2. \( \tan \alpha = \ ? \ (\alpha - 90^\circ) = \ ? \ (\alpha - 180^\circ) = \ ? \ (\alpha - 270^\circ) \)

3. \( \cot \alpha = \ ? \ (\alpha - 90^\circ) = \ ? \ (\alpha - 180^\circ) = \ ? \ (\alpha - 270^\circ) \)

4. \( \sec \alpha = \ ? \ (\alpha - 90^\circ) = \ ? \ (\alpha - 180^\circ) = \ ? \ (\alpha - 270^\circ) \)

5. \( \csc \alpha = \ ? \ (\alpha - 90^\circ) = \ ? \ (\alpha - 180^\circ) = \ ? \ (\alpha - 270^\circ) \)

WHAT DO YOU THINK?

6. Suppose the expressions involving subtraction in Exercises 1-5 were changed to sums.

a. Copy and complete each statement with the proper trigonometric functions.

   (1) \( \sin \alpha = \ ? \ (\alpha + 90^\circ) = \ ? \ (\alpha + 180^\circ) = \ ? \ (\alpha + 270^\circ) \)

   (2) \( \cos \alpha = \ ? \ (\alpha + 90^\circ) = \ ? \ (\alpha + 180^\circ) = \ ? \ (\alpha + 270^\circ) \)

   (3) \( \tan \alpha = \ ? \ (\alpha + 90^\circ) = \ ? \ (\alpha + 180^\circ) = \ ? \ (\alpha + 270^\circ) \)

   (4) \( \cot \alpha = \ ? \ (\alpha + 90^\circ) = \ ? \ (\alpha + 180^\circ) = \ ? \ (\alpha + 270^\circ) \)

   (5) \( \sec \alpha = \ ? \ (\alpha + 90^\circ) = \ ? \ (\alpha + 180^\circ) = \ ? \ (\alpha + 270^\circ) \)

   (6) \( \csc \alpha = \ ? \ (\alpha + 90^\circ) = \ ? \ (\alpha + 180^\circ) = \ ? \ (\alpha + 270^\circ) \)

b. How do the identities in part a compare to those in Exercises 1-5?

7. a. Copy and complete each statement with the proper trigonometric functions.

   (1) \( \sin \alpha = \ ? \ (90^\circ - \alpha) = \ ? \ (180^\circ - \alpha) = \ ? \ (270^\circ - \alpha) \)

   (2) \( \cos \alpha = \ ? \ (90^\circ - \alpha) = \ ? \ (180^\circ - \alpha) = \ ? \ (270^\circ - \alpha) \)

   (3) \( \tan \alpha = \ ? \ (90^\circ - \alpha) = \ ? \ (180^\circ - \alpha) = \ ? \ (270^\circ - \alpha) \)

   (4) \( \cot \alpha = \ ? \ (90^\circ - \alpha) = \ ? \ (180^\circ - \alpha) = \ ? \ (270^\circ - \alpha) \)

   (5) \( \sec \alpha = \ ? \ (90^\circ - \alpha) = \ ? \ (180^\circ - \alpha) = \ ? \ (270^\circ - \alpha) \)

   (6) \( \csc \alpha = \ ? \ (90^\circ - \alpha) = \ ? \ (180^\circ - \alpha) = \ ? \ (270^\circ - \alpha) \)

b. How do the identities in part a compare to those in Exercise 6a?

8. a. How did reduction identities get their name?

b. If you needed one of these identities, but could not remember it, what other type(s) of identities could you use to derive it?